Online Appendix to "Carpool Services for Ride-sharing Platforms: Price and Welfare Implications"

The following lemma establishes the convexity of the cost function $C(\cdot)$ and is, therefore, useful throughout the proof of our technical statements.

LEMMA 3. *Assume that $G(\cdot)$ satisfies the log-concave property. Then $C(y) := yG^{-1}(y)$ is convexly increasing in $0 \le y \le 1$.*

Proof. Let $h(x) := \log G(x)$. Since $h(x)$ is concave, we have $h''(x) = \frac{G''(x)\cdot G(x) - (G'(x))^2}{(G(x))^2} \le 0$, which implies

$$G''(x)\cdot G(x) \le (G'(x))^2 \tag{13}$$

To show $C(y)$ is convexly increasing in y, it suffices to show that $C'(y) \ge 0$ and $C''(y) \ge 0$. Since $G(\cdot)$ is non-decreasing and by the inverse function theorem, we have $C'(y) = G^{-1}(y) + y\cdot(G^{-1})'(y) = G^{-1}(y) + \frac{y}{G'(G^{-1}(y))} \ge 0$. It then follows that

$$\begin{aligned} C''(y) &= (G^{-1})'(y) + \frac{G'(G^{-1}(y)) - y\cdot[G''(G^{-1}(y))\cdot(G^{-1})'(y)]}{(G'(G^{-1}(y)))^2} \\ &= \frac{1}{G'(G^{-1}(y))} + \frac{G'(G^{-1}(y)) - y\cdot\frac{G''(G^{-1}(y))}{G'(G^{-1}(y))}}{(G'(G^{-1}(y)))^2} = \frac{2(G'(G^{-1}(y)))^2 - y\cdot G''(G^{-1}(y))}{(G'(G^{-1}(y)))^3} \ge 0 \end{aligned}$$

where the last inequality follows from $y = G(x)$ and (13). *Q.E.D.*

Proof of Proposition 1.

We write $\Pi_n(s,k) = \bar\lambda v_n(1-s_n)s_n - KC(k/K)$. Clearly, $\Pi_n(s,k)$ is decreasing in k, so $\bar\lambda T_n \tilde s_n^* = \rho_{\max}\tilde k_n^*$. Plugging this into $\Pi_n(s,k)$, we have that it suffices to solve the optimization problem:

$$\tilde s_n^* = \arg\max_s f(s) := \bar\lambda v_n(1-s)s - KC\left(\frac{\bar\lambda T_n s}{\rho_{\max}K}\right)$$

subject to the constraints $s\in[0,1]$ and $\frac{\bar\lambda T_n s}{\rho_{\max}} \le K$.

When $\bar\lambda$ increases: 1(a) $\tilde s_n^*$ is decreasing in $\bar\lambda$: We have $f'(s) = \bar\lambda v_n(1-2s) - \frac{\bar\lambda T_n}{\rho_{\max}}C'\left(\frac{\bar\lambda T_n s}{\rho_{\max}K}\right)$. Let s^* satisfy $f'(s^*) = 0$, which is unique. We have $\tilde s_n^* = \min\{s^*, (K\rho_{\max})/(\bar\lambda T_n)\}$. It is easy to check that s^* and $(K\rho_{\max})/(\bar\lambda T_n)$ are both decreasing in $\bar\lambda$. Hence, $\tilde s_n^*$ is decreasing in $\bar\lambda$.

1(b) $\bar\lambda\tilde s_n^*$ is increasing in $\bar\lambda$: Let $\bar\lambda s := \lambda$. We have $f(s) = g(\lambda) = v_n\left(1-\frac{\lambda}{\bar\lambda}\right)\lambda - KC\left(\frac{\lambda T_n}{\rho_{\max}K}\right)$. Thus, we have $g'(\lambda) = v_n\left(1-\frac{2\lambda}{\bar\lambda}\right) - \frac{T_n}{\rho_{\max}}C'\left(\frac{T_n\lambda}{\rho_{\max}K}\right)$. Let λ^* satisfies $g'(\lambda^*) = 0$, so $\bar\lambda\tilde s_n^* = \min\{\lambda^*, (K\rho_{\max})/T_n\}$. Since $g'(\lambda^*) = 0$ implies that $\lambda^* \le 0.5\bar\lambda$, λ^* is increasing in $\bar\lambda$. Thus, $\bar\lambda\tilde s_n^*$ is also increasing in $\bar\lambda$.

1(c) $\tilde p_n^*$ is increasing in $\bar\lambda$: It follows immediately that $\tilde p_n^* = (1-\tilde s_n^*)v_n/d_n$ is increasing in $\bar\lambda$.

1(d) $\tilde k_n^*$ is increasing in $\bar\lambda$: Note that $\tilde k_n^* = \bar\lambda\tilde s_n^* T_n/\rho_{\max}$. By (b), $\tilde k_n^*$ is increasing in $\bar\lambda$.

1(e) $\tilde w_n^*$ is increasing in $\bar\lambda$: Note that $\tilde w_n^* = \tilde k_n^*/(\bar\lambda\tilde s_n^* d_n)G^{-1}(\frac{\tilde k_n^*}{K}) = T_n/(\rho_{\max}d_n)G^{-1}(\tilde k_n^*/K)$. Since $\tilde k_n^*$ is increasing in $\bar\lambda$, $\tilde w_n^*$ is also increasing in $\bar\lambda$.

1(f) $\tilde\Pi_n^*$ is increasing in $\bar\lambda$: By the envelope theorem, $\tilde\Pi_n^* = \max\Pi_n(s,k)$ is continuously differentiable in $\bar\lambda$ with $\frac{\partial\tilde\Pi_n^*}{\partial\bar\lambda} = v_n(1-s_b^*)s_b^* > 0$. Thus, $\tilde\Pi_n^*$ is increasing in $\bar\lambda$.

When K increases: 2(a) $\tilde s_n^*$ is increasing in K: As shown in part 1(a), $\tilde s_n^* = \min\{s^*, (K\rho_{\max})/(\bar\lambda T_n)\}$, where s^* satisfies $f'(s^*) = 0$. It is easy to check that s^* and $(K\rho_{\max})/(\bar\lambda T_n)$ are both increasing in K. Hence, $\tilde s_n^*$ is also increasing in K.

2(b) $\tilde{p}_n^*$ is decreasing in K: By part 2(a), it follows immediately that $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$ decreases in K.

2(c) $\tilde{k}_n^*$ is increasing in K: Note that $\tilde{k}_n^* = \bar{\lambda}\tilde{s}_n^* T_n/\rho_{\max}$. By part 2(a), $\tilde{k}_n^*$ is increasing in K.

2(d) $\tilde{k}_n^*/K$ is decreasing in K: Let $z := k/K$. We have $f(s) = h(z) = \bar{\lambda}v_n\left(1 - \frac{\rho_{\max}Kz}{\bar{\lambda}T_n}\right)\frac{\rho_{\max}Kz}{\bar{\lambda}T_n} - KC(z)$. Thus, we have $h'(z) = \bar{\lambda}v_n\left(\frac{\rho_{\max}K}{\bar{\lambda}T_n} - 2\left(\frac{\rho_{\max}K}{\bar{\lambda}T_n}\right)^2 z\right) - KC'(z)$. Let z^* satisfies $h'(z^*) = 0$. By $\frac{\tilde{k}_n^*}{K} = \frac{\bar{\lambda}T_n\tilde{s}_n^*}{\rho_{\max}K}$ and $\tilde{s}_n^* \le 1$, we then have $\tilde{k}_n^*/K = \min\{z^*, 1, \frac{\bar{\lambda}T_n}{\rho_{\max}K}\}$. It is easy to check that if K increases, z^* will decrease. Since $\frac{\bar{\lambda}T_n}{\rho_{\max}K}$ is also decreasing in K, $\tilde{k}_n^*/K$ is decreasing in K.

2(e) $\tilde{w}_n^*$ is decreasing in K. Note that $\tilde{w}_n^* = \tilde{k}_n^*/(\bar{\lambda}\tilde{s}_n^* d_n)G^{-1}(\frac{\tilde{k}_n^*}{K}) = T_n/(\rho_{\max}d_n)G^{-1}(\tilde{k}_n^*/K)$. Since $\tilde{k}_n^*/K$ is decreasing in K, $\tilde{w}_n^*$ is also decreasing in K.

2(f) $\tilde{\Pi}_n^*$ is increasing in K. Since $G^{-1}\left(\frac{k}{K}\right)$ is decreasing K, $\Pi_n(s,k) = \bar{\lambda}v_n(1-s)s - kG^{-1}\left(\frac{k}{K}\right)$ is increasing in K. Furthermore, the constraint $k \le K$ is less tight as K increases. Thus, $\tilde{\Pi}_n^* = \max \Pi_n(s,k)$ is increasing in K as well. *Q.E.D.*

Proof of Lemma 2. We prove joint concavity by showing that the Hessian matrix of $f_p(\cdot)$ is negative semidefinite, or alternatively, its leading principal minors have alternate signs. Taking derivatives and by $v_n = v_p + \Delta$, we have

$$\frac{\partial f_p(s_n,s_p)}{\partial s_n} = \bar{\lambda}[-2v_n s_n - 2s_p v_p + v_n] - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right),$$
$$\frac{\partial f_p(s_n,s_p)}{\partial s_p} = \bar{\lambda}[-2v_p s_n - 2s_p v_p + v_p] - \frac{\bar{\lambda}T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right).$$

It then follows that $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_n^2} = -2v_n\bar{\lambda} - \frac{\bar{\lambda}^2T_n^2}{\rho_{\max}^2K}C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right) \le 0$ because $C(\cdot)$ is convexly increasing. Similarly, we have $\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p^2} = -2v_p\bar{\lambda} - \frac{\bar{\lambda}^2T_p^2}{m^2\rho_{\max}^2K}C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right) \le 0$. It remains to show

$$\frac{\partial^2 f_p(s_n,s_p)}{\partial s_n^2}\cdot\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p^2} \ge \left(\frac{\partial^2 f_p(s_n,s_p)}{\partial s_p \partial s_n}\right)^2. \tag{14}$$

It is straightforward to check that (14) holds if and only if

$$2\bar{\lambda}^2v_pv_n + \frac{\bar{\lambda}^3T_p^2\alpha v_n}{m^2\rho_{\max}^2K} + \frac{\bar{\lambda}^3T_n^2\alpha v_p}{\rho_{\max}^2K} \ge 2\bar{\lambda}^2v_p^2 + \frac{2\bar{\lambda}^3T_pT_n\alpha v_p}{m\rho_{\max}^2K}, \tag{15}$$

where $\alpha := C''\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right)$. Since $v_n \ge v_p$ and $\alpha \ge 0$, a sufficient condition for (15) to hold is $T_p^2v_n + m^2T_n^2v_p \ge 2mT_pT_nv_p$, which is clearly true since $v_n \ge v_p$ and $(T_p - mT_n)^2 \ge 0$. *Q.E.D.*

Proof of Proposition 2. We first show that if $\Delta = 0$, $s_n^* = 0$. If $\Delta = 0$,

$$f_p(s_n,s_p) = \bar{\lambda}\left[(1-s_n-s_p)(s_n+s_p)v_n\right] - KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p + s_nT_n)}{\rho_{\max}K}\right).$$

Assume to the contrary that $s_n^* > 0$. Let $\epsilon > 0$ be small enough such that $s_n' = s_n^* - \epsilon \ge 0$, $s_p' = s_p^* + \epsilon$. Since $T_n > \frac{T_p}{m}$, we have

$$\frac{\bar{\lambda}(\frac{1}{m}s_p'T_p + s_n'T_n)}{\rho_{\max}K} = \frac{\bar{\lambda}(\frac{1}{m}(s_p^*+\epsilon)T_p + (s_n^*-\epsilon)T_n)}{\rho_{\max}K} = \frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K} + \frac{\bar{\lambda}(\frac{1}{m}T_p - T_n)\epsilon}{\rho_{\max}K} < \frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}.$$

Thus,

$$C\left(\frac{\bar{\lambda}(\frac{1}{m}s_p'T_p + s_n'T_n)}{\rho_{\max}K}\right) < C\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K}\right).$$

In addition, $(1-s_n'-s_p')(s_n'+s_p')v_n=(1-s_n^*-s_p^*)(s_n^*+s_p^*)v_n$. Hence,

$$\begin{aligned} f_p(s_n',s_p')=&\bar{\lambda}\left[(1-s_n'-s_p')(s_n'+s_p')v_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_p'T_p+s_n'T_n)}{\rho_{\max}K}\right)\\ >&\bar{\lambda}\left[(1-s_n^*-s_p^*)(s_n^*+s_p^*)v_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)=f_p(s_n^*,s_p^*). \end{aligned}$$

Therefore, $s_n^*=0$ if $\Delta=0$.

We now show that if $\Delta=v_n$, $s_p^*=0$. If $\Delta=v_n$, we have $f_p(s_n,s_p):=\bar{\lambda}\left[(1-s_n)v_ns_n\right]-KC\left(\frac{\bar{\lambda}(\frac{1}{m}s_pT_p+s_nT_n)}{\rho_{\max}K}\right)$. Since $C(\cdot)$ is convexly increasing, $f_p(s_n,s_p)$ is decreasing in s_p for all s_n. Therefore, $s_p^*=0$ if $\Delta=v_n$.

Next, we show that s_n^* is increasing in Δ. Assume $\hat{\Delta}>\Delta$, $\hat{f}_p(\cdot,\cdot)$ is the profit function associated with $\hat{\Delta}$, and $(\hat{s}_n^*,\hat{s}_p^*)$ is the maximizer of $\hat{f}_p(\cdot,\cdot)$. Assume to the contrary that $\hat{s}_n^*<s_n^*$. Then we have $\partial_{s_n}\hat{f}_p(\hat{s}_n^*,\hat{s}_p^*)\le 0\le\partial_{s_n}f_p(s_n^*,s_p^*)$. Therefore,

$$-2\bar{\lambda}v_ns_n^*-2\bar{\lambda}(v_n-\Delta)s_p^*+\bar{\lambda}v_n-\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\ge-2\bar{\lambda}v_n\hat{s}_n^*-2\bar{\lambda}(v_n-\hat{\Delta})\hat{s}_p^*+\bar{\lambda}v_n-\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p+\hat{s}_n^*T_n)}{\rho_{\max}K}\right),$$

which implies that

$$y^*-\hat{y}^*\le 2v_n(\hat{s}_n^*-s_n^*)+2(v_n-\hat{\Delta})\hat{s}_p^*-2(v_n-\Delta)s_p^*, \tag{16}$$

where

$$y^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\text{ and }\hat{y}^*:=\frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p+\hat{s}_n^*T_n)}{\rho_{\max}K}\right).$$

If $\hat{s}_p^*\le s_p^*$, the convexity of $C(\cdot)$ suggests that $y^*-\hat{y}^*>0$. Since $\hat{s}_n^*<s_n^*$, $\hat{\Delta}>\Delta$, and $\hat{s}_p^*\le s_p^*$, we have $2v_n(\hat{s}_n^*-s_n^*)+2(v_n-\hat{\Delta})\hat{s}_p^*-2(v_n-\Delta)s_p^*<0$. This forms a contradiction. Thus, we have $\hat{s}_p^*>s_p^*$. It then follows that $\partial_{s_p}\hat{f}_p(\hat{s}_n^*,\hat{s}_p^*)\ge 0\ge\partial_{s_p}f_p(s_n^*,s_p^*)$. Therefore, we have $(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*)-\frac{T_p}{mT_n}\hat{y}^*\ge(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{mT_n}y^*$. It then follows that

$$\begin{aligned} y^*-\hat{y}^*\ge&\frac{mT_n}{T_p}((v_n-\Delta)(1-2s_n^*-2s_p^*)-(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*))\\ \ge&(v_n-\Delta)(1-2s_n^*-2s_p^*)-(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*)\\ =&(\hat{\Delta}-\Delta)+2(v_n-\hat{\Delta})\hat{s}_n^*-2(v_n-\Delta)s_n^*+2(v_n-\hat{\Delta})\hat{s}_p^*-2(v_n-\Delta)s_p^*\\ >&2(v_n-\Delta)(\hat{s}_n^*-s_n^*)+2(v_n-\hat{\Delta})\hat{s}_p^*-2(v_n-\Delta)s_p^*\\ >&2v_n(\hat{s}_n^*-s_n^*)+2(v_n-\hat{\Delta})\hat{s}_p^*-2(v_n-\Delta)s_p^*, \end{aligned} \tag{17}$$

where the second inequality follows from $T_n>\frac{1}{m}T_p$ and $s_n^*+s_p^*\le 0.5$ (which will be shown later in (19)), the third inequality follows from $\hat{s}_n^*<s_n^*$, and the last inequality follows from the assumption that $\hat{s}_n^*<s_n^*$. Inequality (16) contradicts with inequality (17). Therefore, $\hat{s}_n^*\ge s_n^*$ if $\hat{\Delta}>\Delta$.

Next, we show that $\hat{s}_p^*\le s_p^*$ if $\hat{\Delta}>\Delta$. Assume to the contrary that $\hat{s}_p^*>s_p^*$. Then we have $\partial_{s_p}\hat{f}_p(\hat{s}_n^*,\hat{s}_p^*)\ge 0\ge\partial_{s_p}f_p(s_n^*,s_p^*)$, and therefore

$$(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*)-\frac{T_p}{mT_n}\hat{y}^*\ge(v_n-\Delta)(1-2s_n^*-2s_p^*)-\frac{T_p}{mT_n}y^*. \tag{18}$$

We have shown that $\hat{s}_n^*\ge s_n^*$. Thus, $\hat{s}_n^*+\hat{s}_p^*>s_n^*+s_p^*$, $(v_n-\hat{\Delta})(1-2\hat{s}_n^*-2\hat{s}_p^*)<(v_n-\Delta)(1-2s_n^*-2s_p^*)$, and

$$\hat{y}^*=\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p+\hat{s}_n^*T_n)}{\rho_{\max}K}\right)>\frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)=y^*.$$

Therefore,

$$(v_n - \hat{\Delta})(1 - 2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* < (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{mT_n}y^*.$$

The above inequality contradicts with (18) and hence implies that $\hat{s}_p^* \le s_p^*$ if $\hat{\Delta} > \Delta$.

Next, we show the existence of $\underline{\Delta}$ and $\bar{\Delta}$. Note that if $\Delta = 0$ we have $s_p^* > 0$, and if $\Delta = v_n$ we have $s_n^* > 0$. Since $f_p(s_n, s_p|\Delta)$ is continuously differentiable with respect to (s_n, s_p, Δ), by the maximum theorem, the maximizer $(s_n^*(\Delta), s_p^*(\Delta))$ is continuous in Δ. Therefore, the monotonicity and continuity of s_n^* and s_p^* with respect to Δ yields that there exists $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s_n^* \begin{cases} = 0, & \text{if } \Delta \in [0, \underline{\Delta}], \\ > 0, & \text{if } \Delta \in (\underline{\Delta}, v_n]; \end{cases} \quad \text{and} \quad s_p^* \begin{cases} > 0, & \text{if } \Delta \in [0, \bar{\Delta}), \\ = 0, & \text{if } \Delta \in [\bar{\Delta}, v_n]. \end{cases}$$

To show $\bar{\Delta} > \underline{\Delta}$, observe that $s_n^* = s_p^* = 0$ is never optimal for any $\Delta \in [0, v_n]$, which immediately implies that $\underline{\Delta} < \bar{\Delta}$. In the remainder of the proof, we show that

$$s_p^* + s_n^* \le 0.5, \text{ and} \tag{19}$$

$$\bar{\Delta} = v_n \left(1 - \frac{T_p}{mT_n}\right). \tag{20}$$

We first show (19). Assume to the contrary that $s_p^* + s_n^* > 0.5$. We have

$$\partial_{s_p} f_p(s_n^*, s_p^*) = \bar{\lambda}\left[(v_n - \Delta)(1 - 2s_n^* - s_p^*) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right)\right] < 0,$$

so we must have $s_p^* = 0$, and thus $s_n^* > 0.5$. Therefore,

$$\partial_{s_n} f_p(s_n^*, s_p^*) = \bar{\lambda}\left[v_n(1 - 2s_n^*) - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right)\right] < 0,$$

which implies that $s_n^* = 0$, contradicting with $s_n^* > 0.5$. We next show (20). It suffices to show that if $\Delta > v_n(1 - \frac{T_p}{mT_n})$ (resp. $\Delta < v_n(1 - \frac{T_p}{mT_n})$), $s_p^* = 0$ (resp. $s_p^* > 0$). If $\Delta > v_n(1 - \frac{T_p}{mT_n})$ and $s_p^* > 0$, the First Order Condition (FOC) with respect to s_p implies that

$$(v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \frac{T_p}{m}\mu_2^*,$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n) \le \rho_{\max} K$. By $\Delta > v_n(1 - \frac{T_p}{mT_n})$, we have $\frac{v_n - \Delta}{v_n} < \frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned}
\partial_{s_n} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right)\right) \\
=& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - \frac{m(v_n - \Delta)(1 - 2s_n^* - 2s_p^*)T_n}{T_p} + T_n\mu_2^*\right) \\
>& \bar{\lambda}\left(-2v_n s_n^* - 2(v_n - \Delta)s_p^* + v_n - v_n(1 - 2s_n^* - 2s_p^*) + T_n\mu_2^*\right) = 2\bar{\lambda}\Delta s_p^* + \bar{\lambda}T_n\mu_2^* > \bar{\lambda}T_n\mu_2^*,
\end{aligned}$$

where the first inequality follows from $\frac{v_n - \Delta}{v_n} < \frac{T_p}{mT_n}$. Therefore we have $\partial_{s_n} f_p(s_n^*, s_p^*) - \bar{\lambda}T_n\mu_2^* > 0$, which contradicts the FOC that $\partial_{s_n} f_p(s_n^*, s_p^*) - \bar{\lambda}T_n\mu_2^* = 0$. If then follows that $s_p^* = 0$ if $\Delta > v_n(1 - \frac{T_p}{mT_n})$.

If $\Delta < v_n(1 - \frac{T_p}{mT_n})$ and $s_p^* = 0$, we have that $s_n^* > 0$ since both of s_n^* and s_p^* being equal to zero is clearly suboptimal. The FOC with respect to s_n implies that

$$v_n - 2v_n s_n^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}s_n^* T_n}{\rho_{\max} K}\right) = T_n\mu_2^*,$$

and by $\Delta < v_n(1-\frac{T_p}{mT_n})$ we have $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. It then follows that

$$\begin{aligned}\partial_{s_p} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left((v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\ >& \bar{\lambda}\left(v_n(1-2s_n^*)\frac{T_p}{mT_n} - \frac{T_p}{m\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\ =& \frac{\bar{\lambda}T_p}{mT_n}\left(v_n(1-2s_n^*) - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{2}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right) = \frac{\bar{\lambda}T_p}{m}\mu_2^*,\end{aligned}$$

where the inequality follows from $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$ and the assumption $s_p^* = 0$. Thus, $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* > 0$, which contradicts with $\partial_{s_p} f_p(s_n^*, s_p^*) - \frac{\bar{\lambda}T_p}{m}\mu_2^* = 0$. Therefore, we have $s_p^* > 0$ if $\Delta < v_n(1-\frac{T_p}{mT_n})$. *Q.E.D.*

Proof of Theorem 1. Let $\hat{\Delta} > \Delta$. We need to show that $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \leq s^* = s_n^* + s_p^*$. Notice that $\hat{s}_n^* \geq s_n^*$ by Proposition 2. If $\hat{s}_n^* = s_n^*$, then we have $\hat{s}^* = \hat{s}_n^* + \hat{s}_p^* \leq s^* = s_n^* + s_p^*$ since $\hat{s}_p^* \leq s_p^*$ by Proposition 2. Therefore, it remains to consider the case where $\hat{s}_n^* > s_n^*$.

If $\hat{s}_n^* > s_n^*$, we have $\partial_{s_n}\hat{f}_p(\hat{s}_n^*, \hat{s}_p^*) \geq 0 \geq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e.,

$$-2v_n\hat{s}_n^* - 2(v_n-\hat{\Delta})\hat{s}_p^* - \hat{y}^* \geq -2v_n s_n^* - 2(v_n-\Delta)s_p^* - y^*,$$

where $y^* = \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)$ and $\hat{y}^* = \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}\hat{s}_p^*T_p+\hat{s}_n^*T_n)}{\rho_{\max}K}\right)$. It then follows that

$$2(v_n-\hat{\Delta})(\hat{s}^*-s^*) \leq 2\hat{\Delta}(s_n^*-\hat{s}_n^*) + 2(\Delta-\hat{\Delta})s_p^* + y^* - \hat{y}^*.$$

If $y^* \leq \hat{y}^*$, then $s^* > \hat{s}^*$ immediately follows from $s_n^* < \hat{s}_n^*$ and $\Delta < \hat{\Delta}$. If $y^* > \hat{y}^*$, the convexity of $C(\cdot)$ implies that $\frac{1}{m}s_p^*T_p + s_n^*T_n > \frac{1}{m}\hat{s}_p^*T_p + \hat{s}_n^*T_n$. Since $(T_p/m) < T_n$, it then follows that $s_p^* - \hat{s}_p^* > \hat{s}_n^* - s_n^*$, or equivalently, $s^* = s_n^* + s_p^* > \hat{s}_n^* + \hat{s}_p^* = \hat{s}^*$. *Q.E.D.*

Proof of Theorem 2. We first show $\underline{p_p^* \leq \tilde{p}_n^* \text{ for all } \Delta \in [0, v_n]}$. Note that $p_p^* = (1-s_p^*-s_n^*)(v_n-\Delta)/d_p$ and $\tilde{p}_n^* = (1-\tilde{s}_n^*)v_n/d_n$. By Theorem 1, we have $\tilde{s}_n^* \leq s_p^* + s_n^*$ ($\tilde{s}_n^*$ corresponds to $s_n^* + s_p^*$ in the case with $\Delta = v_n$), and $p_p^* \leq \tilde{p}_n^*$ follows immediately from $\Delta \geq 0$ and $d_p \geq d_n$. Next, we show that $\underline{p_n^* \leq \tilde{p}_n^* \text{ for all } \Delta \in [0, \bar{\Delta})}$. We proceed in two steps. First, we show that $p_n^* \leq \tilde{p}_n^*$ when $\Delta \in [\underline{\Delta}, \bar{\Delta})$. Then we show that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$, which would complete the proof.

First, consider the case where $\Delta \in (\underline{\Delta}, \bar{\Delta})$ (i.e., $s_n^* > 0$ and $s_p^* > 0$). Assume, to the contrary, that $p_n^* > \tilde{p}_n^*$, i.e., $(1-s_n^*)v_n - s_p^*(v_n-\Delta) > (1-\tilde{s}_n^*)v_n$. Rearranging terms, we get

$$\tilde{s}_n^* > s_n^* + \frac{v_n-\Delta}{v_n}s_p^* > s_n^* + \frac{T_p}{mT_n}s_p^*,$$

where the second inequality holds because $\Delta < \bar{\Delta}$ implies $\frac{v_n-\Delta}{v_n} > \frac{T_p}{mT_n}$. Note that

$$\begin{aligned}\partial_{s_n} f_p(s_n^*, s_p^*) =& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p+s_n^*T_n)}{\rho_{\max}K}\right)\right)\\ >& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2(v_n-\Delta)s_p^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right)\\ >& \bar{\lambda}\left(v_n - 2v_n s_n^* - 2v_n(\tilde{s}_n^*-s_n^*) - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right)\\ =& \bar{\lambda}\left(v_n - 2v_n\tilde{s}_n^* - \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}\tilde{s}_n^*T_n}{\rho_{\max}K}\right)\right) = f_n'(\tilde{s}_n^*) \geq 0,\end{aligned} \tag{21}$$

where $f_n(\tilde{s}_n) := \bar{\lambda}v_n(1-\tilde{s}_n)\tilde{s}_n - KC\left(\frac{\bar{\lambda}\tilde{s}_nT_n}{\rho_{\max}K}\right)$ is the profit of the platform which only offers the normal service. In (21), the first inequality follows from $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n}s_p^*$, the second inequality follows from $\tilde{s}_n^* > s_n^* + \frac{v_n-\Delta}{v_n}s_p^*$, and the last inequality follows from $\tilde{s}_n^* > 0$. In addition, it is straightforward to check that $\tilde{s}_n^* > s_n^* + \frac{T_p}{mT_n}s_p^*$ implies $\frac{\bar{\lambda}(\frac{1}{m}s_p^*T_p + s_n^*T_n)}{\rho_{\max}K} < 1$. It then follows from (21) that $\partial_{s_n} f_p(s_n^*, s_p^*) > 0$, which contradicts with (s_n^*, s_p^*) being the optimal solution. Therefore, we have $p_n^* \le \tilde{p}_n^*$ when $\Delta \in (\underline{\Delta}, \bar{\Delta})$. Finally, we show p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$. When $\Delta \in [0, \underline{\Delta}]$, we have $s_n^* = 0$ and

$$p_n^* = ((1-s_n^*)\Delta + (1-s_n^*-s_p^*)(v_n-\Delta))/d_n = (\Delta + (1-s_p^*)(v_n-\Delta))/d_n = (v_n - (v_n-\Delta)s_p^*)/d_n.$$

By Proposition 2, s_p^* is decreasing in Δ. Therefore, $(v_n-\Delta)s_p^*$ is decreasing in Δ and it follows that p_n^* is increasing in Δ on $\Delta \in [0, \underline{\Delta}]$.

Proof of Proposition 3. First, it follows immediately from

$$\Pi_p^* = \max\left\{\bar{\lambda}[((1-s_n-s_p)(v_n-\Delta) + (1-s_n)\Delta)s_n + (1-s_n-s_p)(v_n-\Delta)s_p] - KC\left(\frac{\bar{\lambda}(\frac{s_p}{\gamma} + s_nT_n)}{\rho_{\max}K}\right)\right\}$$

that Π_p^* is increasing in γ (as $C(\cdot)$ is decreasing in y). If $\Delta \le \underline{\Delta}$, as shown in Proposition 2, $s_n^* = 0$. It can be easily checked that $\partial_{s_p} f_p(0, s_p) = (v_n-\Delta)(1-2s_p) - \frac{\bar{\lambda}}{\rho_{\max}\gamma}C'\left(\frac{\bar{\lambda}s_p}{\gamma}\right)$ is increasing in s_p, so $f_p(0, s_p)$ is supermodular in (s_p, γ). Hence, s_p^* is increasing in γ. Since $s_n^* = 0$, $s^* = s_n^* + s_p^* = s_p^*$ is increasing in γ, whereas $p_n^* = ((1-s^*)v_p + (1-s_n^*)\Delta)/d_n = ((1-s^*)v_p + \Delta)/d_n$ and $p_p^* = (1-s^*)v_p/d_p$ are decreasing in s^* and thus in γ as well.

We now consider the case $\Delta > \underline{\Delta}$, in which case $s_p^* > 0$ and $s_n^* > 0$. Assume that $\hat{\gamma} > \gamma$, $\hat{f}_p(\cdot,\cdot)$ is the profit function associated with $\hat{\gamma}$, and $(\hat{s}_n^*, \hat{s}_n^*)$ is the optimizer of $\hat{f}_p(\cdot,\cdot)$. We first show $\hat{s}_p^* \ge s_p^*$. Assume to the contrary that $\hat{s}_p^* < s_p^*$. Then we have $\partial_{s_p}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \le 0 \le \partial_{s_p} f_p(s_n^*, s_p^*)$, or alternatively, $(v_n-\Delta)(1-2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{\hat{y}^*}{\hat{\gamma}T_n} \le (v_n-\Delta)(1-2s_n^*-2s_p^*) - \frac{y^*}{\gamma T_n}$, where $\hat{y}^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + \hat{s}_n^*T_n)}{\rho_{\max}K}\right)$ and $y^* := \frac{T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^*T_n)}{\rho_{\max}K}\right)$. Equivalently,

$$\frac{\hat{y}^*}{\hat{\gamma}T_n} - \frac{y^*}{\gamma T_n} \ge 2(v_n-\Delta)(s_n^* - \hat{s}_n^*) + 2(v_n-\Delta)(s_p^* - \hat{s}_p^*). \tag{22}$$

If in addition we have $\hat{s}_n^* \le s_n^*$, the convexity of $C(\cdot)$ suggests that $\hat{y}^* < y^*$. However, (22) implies that $\hat{y}^* > y^*$, which forms a contradiction. Hence, we must have $\hat{s}_n^* > s_n^*$. Thus, $\partial_{s_n}\hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \ge 0 \ge \partial_{s_n} f_p(s_n^*, s_p^*)$, or alternatively, $-2(v_n-\Delta)\hat{s}_p^* + v_n(1-2\hat{s}_n^*) - \hat{y}^* \ge -2(v_n-\Delta)s_p^* + v_n(1-2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \le 2(v_n-\Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*). \tag{23}$$

By (22) and $\hat{\gamma}T_n > \gamma T_n > 1$, $\hat{y}^* - y^* > 2(v_n-\Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*)$, which contradicts (23). We have thus shown that $\hat{s}_p^* \ge s_p^*$.

Next, we show that $\hat{s}_p^* + \hat{s}_n^* \ge s_p^* + s_n^*$. If $\hat{s}_p^* = s_p^*$, then $\frac{\hat{s}_p^*}{\hat{\gamma}} \le \frac{s_p^*}{\gamma}$. We have

$$\begin{aligned}\partial_{s_n}\hat{f}_p(s_n^*, \hat{s}_p^*) &= \bar{\lambda}(v_n - 2v_ns_n^* - 2(v_n-\Delta)\hat{s}_p^*) - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{\hat{s}_p^*}{\hat{\gamma}} + s_n^*T_n)}{\rho_{\max}K}\right) \\ &\ge \bar{\lambda}(v_n - 2v_ns_n^* - 2(v_n-\Delta)\hat{s}_p^*) - \frac{\bar{\lambda}T_n}{\rho_{\max}}C'\left(\frac{\bar{\lambda}(\frac{s_p^*}{\gamma} + s_n^*T_n)}{\rho_{\max}K}\right) = \partial_{s_n} f_p(s_n^*, s_p^*) \ge 0.\end{aligned}$$

Therefore, we have $\hat{s}_n^* \geq s_n^*$ and hence, $\hat{s}_p^* + \hat{s}_n^* \geq s_p^* + s_n^*$.

Now we consider the case $\hat{s}_p^* > s_p^*$. If $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$, we must have $\hat{s}_n^* < s_n^*$. Thus, $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1 - 2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n - \Delta)s_p^* + v_n(1 - 2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{24}$$

where the last inequality follows from $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Since $C(\cdot)$ is convex, (24) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \hat{s}_p^* - s_p^*$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n > 1$. Thus, $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$, contradicting with $\hat{s}_n^* + \hat{s}_p^* \leq s_n^* + s_p^*$. Therefore, we must have $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Next, we show that $\underline{\hat{p}_p^* \leq p_p^*}$. Note that $\hat{p}_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$ and $p_p^* = (v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*)/d_p$, $\hat{p}_p^* \leq p_p^*$ follows immediately from $\hat{s}_n^* + \hat{s}_p^* \geq s_n^* + s_p^*$.

Finally, we show that $\underline{\hat{p}_n^* \leq p_n^*}$. Assume to the contrary that $\hat{p}_n^* > p_n^*$, i.e., $(v_n - \Delta)(1 - \hat{s}_n^* - \hat{s}_p^*) + \Delta(1 - \hat{s}_n^*) > (v_n - \Delta)(1 - s_n^* - s_p^*) + \Delta(1 - s_n^*)$. Hence, $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$, where the second inequality follows from that $\hat{s}_p^* > s_p^*$. The inequality $\hat{s}_n^* < s_n^*$ implies that $\partial_{s_n} \hat{f}_p(\hat{s}_p^*, \hat{s}_n^*) \leq 0 \leq \partial_{s_n} f_p(s_n^*, s_p^*)$, i.e., $-2(v_n - \Delta)\hat{s}_p^* + v_n(1 - 2\hat{s}_n^*) - \hat{y}^* \leq -2(v_n - \Delta)s_p^* + v_n(1 - 2s_n^*) - y^*$. Equivalently,

$$\hat{y}^* - y^* \geq 2(v_n - \Delta)(s_p^* - \hat{s}_p^*) + 2v_n(s_n^* - \hat{s}_n^*) > 0, \tag{25}$$

where the last inequality follows from $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*) > 0$. Since $C(\cdot)$ is convex, (25) implies that $\frac{\hat{s}_p^*}{\hat{\gamma}} + T_n \hat{s}_n^* > \frac{s_p^*}{\gamma} + T_n s_n^*$, which is equivalent to that $s_n^* - \hat{s}_n^* < \frac{\hat{s}_p^*}{\hat{\gamma} T_n} - \frac{s_p^*}{\gamma T_n} < \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n}$, where the inequality follows from that $\hat{\gamma} T_n > \gamma T_n$. Since $\Delta < \bar{\Delta}$, $(v_n - \Delta)/v_n > 1/(\hat{\gamma} T_n)$. So we have $\frac{(v_n - \Delta)(\hat{s}_p^* - s_p^*)}{v_n} > \frac{\hat{s}_p^* - s_p^*}{\hat{\gamma} T_n} > s_n^* - \hat{s}_n^*$. This inequality contradicts that $v_n(s_n^* - \hat{s}_n^*) > (v_n - \Delta)(\hat{s}_p^* - s_p^*)$. Therefore, we must have $\hat{p}_n^* \leq p_n^*$. *Q.E.D.*

Proof of Proposition 4. We use $\lambda_p^* := \bar{\lambda} s_p^*$ and $\lambda_n^* := \bar{\lambda} s_n^*$. Notice that when $(v_n - \Delta)/v_n > T_p/(mT_n)$, we have $\Delta < \bar{\Delta}$ and hence $s_p^* > 0$. By the KKT condition (which is both necessary and sufficient for optimality by the joint concavity of $f_p(\cdot)$ and compactness of the feasible region of (s_n, s_p)), we have

$$\bar{\lambda}\left[-2v_p s_n^* - 2\Delta s_n^* - 2s_p^* v_p + v_p + \Delta\right] - \frac{\bar{\lambda} T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \bar{\lambda} T_n \mu_2^* - \eta_1^*, \tag{26}$$

$$\bar{\lambda}\left[-2v_p s_n^* - 2s_p^* v_p + v_p\right] - \frac{\bar{\lambda} T_p}{m\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m} s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right) = \mu_1^* + \frac{1}{m}\bar{\lambda} T_p \mu_2^* - \eta_2^*, \tag{27}$$

$$\mu_1^*(1 - s_n^* - s_p^*) = 0, \tag{28}$$

$$\mu_2^*\left(\rho_{\max} K - \bar{\lambda}\left(\frac{1}{m} s_p^* T_p + s_n^* T_n\right)\right) = 0, \tag{29}$$

$$\eta_1^* s_n^* = 0, \quad \eta_2^* s_p^* = 0, \tag{30}$$

$$\mu_1^*, \mu_2^*, \eta_1^*, \eta_2^* \geq 0, \tag{31}$$

where $\mu_1^*, \mu_2^*, \eta_1^*$, and η_2^* are the Lagrangian multipliers with respect to the constraints $s_n^* + s_p^* \leq 1$, $\bar{\lambda}(\frac{1}{m} s_n^* T_n + s_p^* T_p) \leq \rho_{\max} K$, $s_n^* \geq 0$ and $s_p^* \geq 0$, respectively. Notice that by (19), $s_n^* + s_p^* < 1$ and hence by complementary slackness condition, we have $\mu_1^* = 0$.

(a) s_n^* is decreasing in $\bar{\lambda}$. Consider $\hat{\bar{\lambda}}$ and $\bar{\lambda}$ with $\hat{\bar{\lambda}} > \bar{\lambda}$. Notice that $(v_n - \Delta)/v_n > T_p/(mT_n)$ and hence $\Delta < \bar{\Delta}$, we have $s_p^* > 0$ and $\hat{s}_p^* > 0$. By (30), $\eta_2^* = \hat{\eta}_2 = 0$. We first consider the case where $\hat{s}_n^*, s_n^* > 0$, and therefore $\eta_1^* = \hat{\eta}_1 = 0$. Then the KKT conditions (26) and (27) imply that:

$$\begin{aligned} &v_n - 2v_n s_n^* - 2(v_n - \Delta)s_p^* - y^* - \mu_2^* T_n = 0,\\ &(v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{mT_n} y^* - \mu_2^* \frac{T_p}{m} = 0,\\ &v_n - 2v_n \hat{s}_n^* - 2(v_n - \Delta)\hat{s}_p^* - \hat{y}^* - \hat{\mu}_2^* T_n = 0,\\ &(v_n - \Delta)(1 - 2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{T_p}{mT_n} \hat{y}^* - \hat{\mu}_2^* \frac{T_p}{m} = 0, \end{aligned} \tag{32}$$

where $y^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\bar{\lambda}(\frac{1}{m}s_p^* T_p + s_n^* T_n)}{\rho_{\max} K}\right)$ and $\hat{y}^* := \frac{T_n}{\rho_{\max}} C'\left(\frac{\hat{\bar{\lambda}}(\frac{1}{m}\hat{s}_p^* T_p + \hat{s}_n^* T_n)}{\rho_{\max} K}\right)$. Observe that both (s_n^*, s_p^*) and $(\hat{s}_n^*, \hat{s}_p^*)$ are located on the line

$$\frac{v_n - 2v_n s_n - 2(v_n - \Delta)s_p}{(v_n - \Delta)(1 - 2s_n - 2s_p)} = \frac{mT_n}{T_p}. \tag{33}$$

If $\hat{s}_n^* - s_n^* = \delta > 0$, then it is easy to check by (33) that $s_p^* > \hat{s}_p^*$ and $s_p^* - \hat{s}_p^* < \delta$. Thus, we have $\hat{s}_n^* + \hat{s}_p^* > s_n^* + s_p^*$ and $\hat{\bar{\lambda}}(\hat{s}_n^* T_n + \frac{1}{m} T_p \hat{s}_p^*) > \bar{\lambda}(s_n^* T_n + \frac{1}{m} s_p^* T_p)$. Hence, $\hat{y}^* > y^*$. Moreover, by the complementary slackness condition (29), $\hat{\mu}_2^* \geq \mu_2^*$. Therefore,

$$(v_n - \Delta)(1 - 2\hat{s}_n^* - 2\hat{s}_p^*) - \frac{T_p}{mT_n}\hat{y}^* - \hat{\mu}_2^* \frac{T_p}{m} < (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{mT_n} y^* - \mu_2^* \frac{T_p}{m},$$

which contradicts with (32). Hence, in the range of $s_n^* > 0$, s_n^* is decreasing in $\bar{\lambda}$. By the continuity of s_n^*, it is clear that s_n^* is decreasing in $\bar{\lambda}$ for all $\bar{\lambda}$.

(b) There exists a λ_0 such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$. Note that $\Delta < \bar{\Delta}$ is equivalent to $\frac{v_n - \Delta}{v_n} > \frac{T_p}{mT_n}$. We use $\lambda_p := \bar{\lambda} s_p$ and $\lambda_n := \bar{\lambda} s_n$ as the decision variables. The platform is then to maximize

$$f_p(\lambda_n, \lambda_p) = \left(\left(1 - \frac{\lambda_p}{\bar{\lambda}} - \frac{\lambda_n}{\bar{\lambda}}\right)(v_n - \Delta) + \left(1 - \frac{\lambda_n}{\bar{\lambda}}\right)\Delta\right)\lambda_n + \left(1 - \frac{\lambda_n}{\bar{\lambda}} - \frac{\lambda_p}{\bar{\lambda}}\right)(v_n - \Delta)\lambda_p - KC\left(\frac{\frac{1}{m}\lambda_p T_p + \lambda_n T_n}{\rho_{\max} K}\right), \tag{34}$$

subject to the constraint $0 \leq \lambda_n + \lambda_p \leq \bar{\lambda}$ and $\lambda_n T_n + \lambda_p \frac{T_p}{m} \leq \rho_{\max} K$. We have

$$\begin{aligned} \partial_{\lambda_n} f_p(\lambda_n^*, \lambda_p^*) =& v_n - 2v_n \frac{\lambda_n^*}{\bar{\lambda}} - 2(v_n - \Delta)\frac{\lambda_p^*}{\bar{\lambda}} - \frac{T_n}{\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right)\\ =& v_n - 2v_n s_n^* - 2(v_n - \Delta)s_p^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right), \end{aligned}$$

and

$$\begin{aligned} \partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) =& (v_n - \Delta)\left(1 - \frac{2\lambda_n^*}{\bar{\lambda}} - \frac{2\lambda_p^*}{\bar{\lambda}}\right) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right)\\ =& (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right). \end{aligned}$$

Since $\lambda_n^* T_n + \lambda_p^* \frac{T_p}{m} \leq \rho_{\max} K$, it follows that $s_n^* = \frac{\lambda_n^*}{\bar{\lambda}} \leq \frac{\rho_{\max} K}{T_n \bar{\lambda}}$ and $s_p^* = \frac{\lambda_p^*}{\bar{\lambda}} \leq \frac{m\rho_{\max} K}{T_n \bar{\lambda}}$. Therefore, we have $s_n^* \to 0$ and $s_p^* \to 0$ as $\bar{\lambda} \to +\infty$. Because $\Delta < v_n\left(1 - \frac{T_p}{mT_n}\right)$, we have

$$v_n - \Delta - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right) > \frac{T_p}{mT_n}\left(v_n - \frac{T_n}{\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right)\right).$$

Therefore, when $\bar{\lambda}$ is sufficiently large (where $s_n^* \to 0$ and $s_p^* \to 0$), we have

$$\begin{aligned} \partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\mu_2^* =& (v_n - \Delta)(1 - 2s_n^* - 2s_p^*) - \frac{T_p}{m\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right) - \frac{T_p}{m}\mu_2^*\\ >& \frac{T_p}{mT_n}\left(v_n - 2v_n s_n^* - 2(v_n - \Delta)s_p^* - \frac{T_n}{\rho_{\max}} C'\left(\frac{\frac{1}{m}\lambda_p^* T_p + \lambda_n^* T_n}{\rho_{\max} K}\right) - T_n \mu_2^*\right)\\ =& \frac{T_p}{mT_n}(\partial_{\lambda_n} f_p(\lambda_n^*, \lambda_p^*) - T_n \mu_2^*), \end{aligned} \tag{35}$$

where μ_2^* is the Lagrangian multiplier with respect to the constraint $\lambda_n T_n + \lambda_p \frac{T_p}{m} \leq \rho_{\max} K$. Since $s_p^* > 0$ and thus $\lambda_p^* > 0$, the first-order condition $\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\mu_2^* = 0$ when $\bar{\lambda}$ is sufficiently large. In this case, (35) implies that $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < \frac{mT_n}{T_p}\left(\partial_{\lambda_p} f_p(\lambda_n^*, \lambda_p^*) - \frac{T_p}{m}\right) = 0$. It is straightforward to check that by the KKT condition of optimization problem (34), $\partial_{\lambda_n} f_n(\lambda_n^*, \lambda_p^*) - T_n \mu_2^* < 0$ implies that $\lambda_n^* = 0$. It then follows that $s_n^* = 0$ when $\bar{\lambda}$ is sufficiently large, or, there exists a threshold λ_0, such that $s_n^* = 0$ for $\bar{\lambda} \geq \lambda_0$.

(c) s_p^* is increasing (resp. decreasing) in $\bar{\lambda}$ for $\bar{\lambda} < \lambda_0$ (resp. $\bar{\lambda} > \lambda_0$). Recall that $\lambda_0 := \min\{\bar{\lambda} : s_n^* = 0\}$. If $\bar{\lambda} < \lambda_0$, (s_n^*, s_p^*) satisfies (33). Since s_n^* is decreasing in $\bar{\lambda}$, it is straightforward to check that s_p^* is decreasing in s_n^*, thus increasing in $\bar{\lambda}$ as well. If $\bar{\lambda} > \lambda_0$, then we have $s_n^* = 0$. By Proposition 1, s_p^* is decreasing in $\bar{\lambda}$.

(d) p_n^* and p_p^* are increasing in $\bar{\lambda}$, and $p_n^* d_n - p_p^* d_p$ is increasing in $\bar{\lambda}$. Note that $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$. If $\bar{\lambda} < \lambda_0$, $s_n^* > 0$ and (s_n^*, s_p^*) satisfies (33). Since s_n^* is decreasing in $\bar{\lambda}$, it is easy to check, by (33), that $s_n^* + s_p^*$ is decreasing in $\bar{\lambda}$. Thus, $p_p^* = (v_n - \Delta)(1 - s_n^* - s_p^*)/d_p$ is increasing in $\bar{\lambda}$. Furthermore, $p_n^* d_n - p_p^* d_p = (1 - s_n^*)\Delta$ is decreasing in s_n^*, thus increasing $\bar{\lambda}$. Hence, $p_n^* = (p_p^* d_p + (1 - s_n^*)\Delta)/d_n$ is also increasing in $\bar{\lambda}$. *Q.E.D.*

Proof of Proposition 5. It follows from (10) that if $\Delta = 0$, $RS_p^* = \frac{1}{2}\bar{\lambda}(s_p^*)^2$. $\tilde{RS}_n^* = RS_p^*(\bar{\Delta}) = \frac{1}{2}\bar{\lambda}(s_n^*)^2$. We now show that $s_p^*(0) > s_n^*(\bar{\Delta})$. By Theorem 1, $s_p^*(0) + s_n^*(0) > s_p^*(\bar{\Delta}) + s_n^*(\bar{\Delta})$. By Proposition 2, $s_n^*(0) = s_p^*(\bar{\Delta}) = 0$, we have $s_p^*(0) > s_n^*(\bar{\Delta})$, which implies that $RS_p^*(0) > RS_p^*(\bar{\Delta})$. The existence of $\underline{\Delta}_r$ then follows directly from $RS_p^*(\Delta)$ being continuous in Δ.

For the ease of exposition, we normalize $K = 1$, $T_n = 1$, and $v_n = 1$. We also define $\gamma = m/T_p$ and $\eta = v_n - \Delta$. Then, we have the constraints $\gamma > 1$, $\eta < 1$, and $\eta\gamma > 1$. If $G(r) = r$, we first compare $RS_p^*(\Delta)$ with $\tilde{RS}_n^*$ for $\Delta \in (\underline{\Delta}, \bar{\Delta})$. In this case, $s_n^*(\Delta) > 0$. Then, It is straightforward to calculate that

$$\begin{cases} s_n^*(\Delta) = & \frac{1}{2}\left(1 - \frac{\eta\bar{\lambda}(1/\gamma - 1)}{-\eta\bar{\lambda} + (\eta-1)\eta\rho_{\max}^2 + 2\eta\bar{\lambda}/\gamma - \lambda/\gamma^2}\right) \\ s_p^*(\Delta) = & \frac{\bar{\lambda}(1/\gamma - \eta)}{-2\eta\bar{\lambda} + 2(\eta-1)\eta\rho_{\max}^2 + 4\eta\bar{\lambda}/\gamma - 2\bar{\lambda}/\gamma^2} \\ \tilde{s}_n^* = & \frac{\rho_{\max}^2}{2(\bar{\lambda} + \rho_{\max}^2)} \end{cases}$$

Then, we can calculate the difference between the setting with carpool services and that without:

$$RS_p^*(\Delta) - \tilde{RS}_n^* = -\frac{\bar{\lambda}^2(\eta - 1/\gamma)^2(\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2)}{4(\lambda + \rho_{\max}^2)^2(\eta\lambda - (\eta-1)\eta\rho_{\max}^2 - 2\eta\bar{\lambda}/\gamma + \bar{\lambda}/\gamma^2)^2}$$

Hence, it suffices to show that

$$\eta(-\bar{\lambda}^2 + 2(\eta-2)\bar{\lambda}\rho_{\max}^2 + 3(\eta-1)\rho_{\max}^4) + 2\eta\bar{\lambda}(\bar{\lambda} + 2\rho_{\max}^2)/\gamma - \lambda(\lambda + 2\rho_{\max}^2)/\gamma^2 < 0.$$

Rearranging the terms, it suffices to show that

$$\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{36}$$

$$2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 0, \tag{37}$$

$$3(1-\eta)\eta\rho_{\max}^4 > 0. \tag{38}$$

To show (36), observe that $\bar{\lambda}^2(\eta - 2\eta/\gamma + 1/\gamma^2) > \bar{\lambda}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = \bar{\lambda}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from $\eta < 1$ and the second from $\eta\gamma > 1$. To show (37), observe that $2\bar{\lambda}\rho_{\max}^2((2-\eta)\eta - 2\eta/\gamma + 1/\gamma^2) > 2\bar{\lambda}\rho_{\max}^2(\eta^2 - 2\eta/\gamma + 1/\gamma^2) = 2\bar{\lambda}\rho_{\max}^2(\eta - 1/\gamma)^2 > 0$, where the first inequality follows from

$(2-\eta)\eta > \eta^2$ for $\eta \in (0,1)$, and the second from $\eta > 1/\gamma$. This proves that if $\Delta \in (\underline{\Delta}, \bar{\Delta})$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Inequality (38) follows immediately from $0 < \eta < 1$. Putting everything together, we have that $RS_p^*(\Delta) > \tilde{RS}_p^*$ for $\Delta \in [\underline{\Delta}, \bar{\Delta})$.

Finally we show that for the case $\Delta \le \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. By continuity, if $\Delta = \underline{\Delta}$, $RS_p^*(\Delta) > \tilde{RS}_n^*$. Furthermore, $s_p^*(\Delta)$ is decreasing in Δ (by Proposition 2). Therefore, $RS_p^*(\Delta) = \frac{1}{2}(v_n - \Delta)(s_p^*(\Delta))^2$ is decreasing in Δ. Hence, $RS_p^*(\Delta) > RS_p^*(\underline{\Delta})$ for all $\Delta < \underline{\Delta}$. This concludes the proof of Proposition 5. *Q.E.D.*

Proof of Proposition 6. It is clear from (11) and (12) that the driver surplus is strictly increasing in the number of active drivers k^* in equilibrium, and hence it boils down to analyzing the impact of carpool services on k^* (which is also equivalent to analyzing the impact of carpool services on the per-unit-time wage for the drivers in equilibrium, since $w^* = k^* G^{-1}(k^*/K)$ and G^{-1} is a monotonically increasing function). When $\Delta \in (\underline{\Delta}, \bar{\Delta})$, it follows from Proposition 2 that $s_n^* > 0$ and $s_p^* > 0$. Then by first order conditions $\partial_{s_n} f_p(s_n^*, s_p^*) = 0$ and $\partial_{s_p} f_p(s_n^*, s_p^*) = 0$, it is straightforward to derive that

$$\begin{cases} s_n^* = \dfrac{(\Delta^2 K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)T_p v_n - \Delta m(\bar{\lambda} T_n T_p + K m \rho_{\max}^2 v_n))}{(2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)}, \\ s_p^* = \dfrac{\bar{\lambda} m T_n (\Delta m T_n - (mT_n - T_p) v_n)}{2\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n(mT_n - 2T_p)) - 2(\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n}. \end{cases}$$

Similarly, the first order condition $\partial_{\tilde{s}_n} f_b(\tilde{s}_n^* | \bar{\lambda}) = 0$ implies that

$$\tilde{s}_n^* = \frac{K \rho_{\max}^2 v_n}{2\bar{\lambda} T_n^2 + 2K \rho_{\max}^2 v_n}.$$

Note that

$$\tilde{k}_n^* - k^* = \frac{\bar{\lambda} T_n (\tilde{s}_n^* - s_n^* - (T_p s_p^*)/(mT_n))}{\rho_{\max}}.$$

Therefore, $\tilde{k}_n^* > k^*$ is equivalent to $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$. We next compute $\tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right)$ as follows:

$$\begin{aligned} \tilde{s}_n^* - \left(s_n^* + \frac{T_p s_p^*}{mT_n}\right) &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p) v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)(-\Delta m(\Delta K m \rho_{\max}^2 + \bar{\lambda} T_n (mT_n - 2T_p)) + (\Delta K m^2 \rho_{\max}^2 + \bar{\lambda}(mT_n - T_p)^2) v_n)} \\ &= \frac{K \bar{\lambda} \rho_{\max}^2 (\Delta m T_n - (mT_n - T_p) v_n)^2}{2(\bar{\lambda} T_n^2 + K \rho_{\max}^2 v_n)[\Delta(v_n - \Delta) K m^2 \rho_{\max}^2 + \bar{\lambda}((mT_n - T_p)^2 (v_n - \Delta) + T_p^2 \Delta))]} \\ &> 0, \end{aligned}$$

where the inequality follows from $v_n > \Delta \ge 0$. Therefore, $\tilde{s}_n^* > s_n^* + \frac{T_p s_p^*}{mT_n}$. It then follows that $\tilde{k}_n^* > k^*$, which implies $DS_p^* < \tilde{DS}_n^*$ in view of (11) and (12).

Finally, we show $w^* < \tilde{w}^*$. Note that $w^* = k^* G^{-1}(k^*/K)$ and $\tilde{w}^* = \tilde{k}_n^* G^{-1}(\tilde{k}_n^*/K)$. It then immediately follows from $\tilde{k}_n^* > k^*$ that $\tilde{w}^* > w^*$. *Q.E.D.*