Online Appendices to "Dynamic Pricing and Inventory Management in the Presence of Online Reviews"

Nan Yang and Renyu Zhang

We use ∂ to denote the derivative operator of a single variable function, ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. We use $\epsilon_1 \overset{d}{=} \epsilon_2$ to denote that two random variables ϵ_1 and ϵ_2 follow the same distribution.

Appendix A: Table of Notations

Table 1 Summary of Notations

T :	planning horizon length	t:	period-index, labeled backwards
N_t :	aggregate net rating of period t	η :	discount factor for reviews
V :	intrinsic valuation of the customers	$\bar{V}_t$:	maximum potential demand without reviews
$\gamma(\cdot)$:	impact of online ratings on demand	p_t:	price of period t
$D_t(\cdot,\cdot)$:	demand function of the product in period t	θ:	net rating contribution ratio of demand
σ :	impact of inventory availability on net rating	α:	discount factor of the firm
c:	inventory purchasing cost	b:	backlogging cost of inventory
h:	holding cost of inventory	$R_t(\cdot,\cdot)$:	adjusted revenue of period t
β:	effective benefit of ordering inventory	I_t :	inventory level of period t
x_t:	base-stock level of period t	Δ_t :	safety-stock of period t
ξ_t:	demand perturbation in period t	ϵ_t:	perturbation of net rating in period t
n_t:	net rating increase with paid reviews in period t	$y_t(\cdot,\cdot) = \mathbb{E}[D_t(\cdot,\cdot)]$:	expected demand in period t
$c_n(n_t)$:	cost of generating n_t additional net rating	r_t^+ (resp. r_t^-):	number of positive (resp. negative) reviews

Appendix B: Auxiliary Results

We first give some auxiliary results serving as building blocks of our subsequent analysis. The proofs of these results are given in Appendix C. The following lemma (adapted from Lemma 4 in Yang and Zhang 2022) is essential in proving various comparative statics results in our model. This comparative statics analysis technique has been proposed by Yang and Zhang (2022).

LEMMA 2. *Let $F_i(z, Z)$ be a continuously differentiable and jointly concave function in (z, Z) for $i = 1, 2$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^n$. For $i = 1, 2$, let $(z_i, Z_i) := \arg\max_{(z,Z)} F_i(z, Z)$ be the optimizers of $F_i(\cdot,\cdot)$. If $z_1 < z_2$, we have: $\partial_z F_1(z_1, Z_1) \leq \partial_z F_2(z_2, Z_2)$.*

Next, we develop the preliminary concavity and differentiability properties of the value and objective functions in the following lemma, which serves as a stepping stone for our subsequent analysis.

LEMMA 3. *For each $t = T, T-1, \cdots, 1$, the following statements hold:*

(a) $\Psi_t(\cdot,\cdot)$ is jointly concave and continuously differentiable in (x, y). Moreover, $\Psi_t(x, y)$ is decreasing in x and increasing in y.

(b) $J_t(\cdot,\cdot,\cdot)$ is jointly concave and continuously differentiable in (x_t, p_t, N_t).

(b) $v_t(\cdot,\cdot)$ is jointly concave and continuously differentiable in (I_t, N_t). Moreover, $v_t(I_t, N_t)$ is increasing in N_t, and $v_t(I_t, N_t) - cI_t$ is decreasing in I_t.

Lemma 3 proves that, in each period t, the objective and value functions are concave and continuously differentiable. Moreover, after normalized with the value of inventory, the profit-to-go, $v_t(I_t, N_t) - cI_t$, is decreasing in the inventory level I_t and increasing in the network size N_t. Results similar to lemma 3 have also been established in other joint pricing and inventory management settings (see, e.g., Theorem 1 in Federgruen and Heching 1999).

In the following, we give the derivation for the objective functions in each period t.

Derivation of $J_t(x_t, p_t, N_t)$:

$$\begin{aligned}
J_t(x_t,p_t,N_t) &= -cI_t + \mathbb{E}\{p_tD_t(p_t,N_t) - c(x_t - I_t) - h(x_t - D_t(p_t,N_t))^+ - b(x_t - D_t(p_t,N_t))^- \\
&\quad + \alpha v_{t-1}(x_t - D_t(p_t,N_t), \eta N_t + \theta D_t(p_t,N_t) - \sigma(D_t(p_t,N_t) - x_t)^+ + \epsilon_t)|N_t\}, \\
&= (p_t - \alpha c - b)y_t(p_t,N_t) + (b - (1-\alpha)c)x_t + \mathbb{E}\{-(h+b)(x_t - y_t(p_t,N_t) - \xi_t)^+ \\
&\quad + \alpha[v_{t-1}(x_t - y_t(p_t,N_t) - \xi_t, \theta(y_t(p_t,N_t)+\xi_t) + \eta N_t - \sigma(y_t(p_t,N_t)+\xi_t - x_t)^+ + \epsilon_t) \\
&\quad - c(x_t - y_t(p_t,N_t) - \xi_t)]|N_t\} \\
&= R_t(p_t,N_t) + \beta x_t + \Lambda(x_t - y_t(p_t,N_t)) + \\
&\quad + \mathbb{E}[\Psi_t(x_t - y_t(p_t,N_t) - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(y_t(p_t,N_t)+\xi_t - x_t)^+)]. \qquad (11)
\end{aligned}$$

Derivation of $O_t(\cdot,\cdot,\cdot)$:

$$\begin{aligned}
O_t(\Delta_t,p_t,N_t) :=& J_t(\Delta_t + y_t(p_t,N_t), p_t, N_t) \\
=& R_t(p_t,N_t) + \beta(\Delta_t + y_t(p_t,N_t)) + \Lambda(\Delta_t) \\
&+ \mathbb{E}[\Psi_t(\Delta_t - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(-\Delta_t+\xi_t)^+)] \qquad (12) \\
=& Q_t(p_t,N_t) + \beta\Delta_t + \Lambda(\Delta_t) + \mathbb{E}[\Psi_t(\Delta_t - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(-\Delta_t+\xi_t)^+)].
\end{aligned}$$

The following lemma paves our way to reduce the original dynamic program (4), which has a two-dimension state space, to one with a single-dimension state space.

LEMMA 4. *The sequence of functions $\{\pi_t(\cdot) : t = T, T-1, \cdots, 1\}$ satisfy the following statements:*

(i) $\pi_t(\cdot)$ is concavely increasing and continuously differentiable in N_t with $\pi_t(N_t) := \max\{O_t(\Delta_t, p_t, N_t) : \Delta_t \in \mathbb{R}, p_t \in [\underline{p}, \bar{p}]\}$ for any $N_t \geq 0$;

(ii) $v_t(I_t, N_t) = cI_t + \pi_t(N_t)$ for all $N_t \geq 0$ and $I_t \leq x_t(N_t)$;

(iii) For all $N_t \geq 0$ and $\Delta_t \leq \Delta_t(N_t)$,

$$O_t(\Delta_t,p_t,N_t) = Q_t(p_t,N_t) + \beta\Delta_t + \Lambda(\Delta_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(-\Delta_t+\xi_t)^+)], \qquad (13)$$

where $G_t(y) := \alpha\mathbb{E}[\pi_{t-1}(y + \epsilon_t)]$;

(iv) $(\Delta_t(N_t), p_t(N_t))$ maximizes the right-hand side of equation (13) over the feasible set $\mathbb{R} \times [\underline{p}, \bar{p}]$.

More specifically, it follows immediately from Lemma 4 that the optimal rating-dependent safety-stock level and list-price in each period t, $(\Delta_t(N_t), p_t(N_t))$, can be recursively determined by solving the dynamic program (9).

For the model with the paid-review strategy (Section 6.1), we have the following lemma that characterizes the optimal policy in the presence of paid reviews.

LEMMA 5. *Iteratively define a sequence of functions* $\{\pi_t^e(N_t) : t = T, T-1, \cdots, 1\}$ *and a sequence of pricing and inventory policies* $\{(x_t^e(N_t), p_t^e(N_t), n_t(N_t)) : t = T, T-1, \cdots, 1\}$ *as follows:*

$$\begin{aligned} \pi_t^e(N_t) &= \max_{(\Delta_t, p_t, n_t) \in \mathcal{F}_e} O_t^e(\Delta_t, p_t, n_t, N_t), \qquad (14)\\ \textit{where } O_t^e(x_t, p_t, n_t, N_t) &= Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) - c_n(n_t) \\ &\quad + \mathbb{E}[G_t^e(\eta N_t + \theta(y_t(p_t, N_t) + \xi_t) - \sigma(-\Delta_t + \xi_t)^+ + n_t)], \\ \textit{with } G_t^e(y) &:= \alpha\mathbb{E}[\pi_{t-1}^e(y + \epsilon_t)], \;\; \pi_0^e(\cdot) \equiv 0, \\ (\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t)) &:= \arg\max_{(\Delta_t, p_t, n_t) \in \mathcal{F}_e} O_t^e(x_t, p_t, n_t, N_t), \;\; \textit{and } \mathcal{F}_e := \mathbb{R} \times [\underline{p}, \bar{p}] \times \mathbb{R}_+. \end{aligned}$$

Also define $x_t^e(N_t) := \Delta_t^e(N_t) + y_t(p_t^e(N_t), N_t)$.

(a) If $I_t \le x_t^e(N_t)$, $(x_t^{e*}(I_t, N_t), p_t^{e*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$ *and* $v_t^e(I_t, N_t) = cI_t + \pi_t^e(N_t)$. *Otherwise,* $I_t > x_t^e(N_t)$, $x_t^{e*}(I_t, N_t) = I_t$.

(b) If $I_T \le x_T^e(N_T)$, $(x_t^{e*}(I_t, N_t), p_t^{e*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$ *for all t with probability 1.*

Appendix C: Proofs of Statements

For completeness, we also give the proof of Lemma 2, which is identical to Lemma 4 in Yang and Zhang (2022).

Proof of Lemma 2: Since $z_1 < z_2$, we have $\underline{z} \le z_1 < z_2 \le \bar{z}$. Hence, $\partial_z F_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \le 0 & \text{if } z_1 = \underline{z}; \end{cases}$ and $\partial_z F_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \ge 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z F_1(z_1, Z_1) \le 0 \le \partial_z F_2(z_2, Z_2)$. *Q.E.D.*

Proof of Lemma 3: We prove parts (a) - (c) together by backward induction.

We first show, by backward induction that if $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$ is jointly concave in (I_{t-1}, N_{t-1}), decreasing in I_{t-1}, and increasing in N_{t-1}, (i) $\Psi_t(\cdot,\cdot)$ is jointly concave in (x, y), decreasing in x, and increasing in y; (ii) $J_t(\cdot,\cdot,\cdot)$ is jointly concave in (x_t, p_t, N_t); and (iii) $v_t(I_t, N_t) - cI_t$ is jointly concave in (I_t, N_t), decreasing in I_t, and increasing in N_t. It is clear that $v_0(I_0, N_0) - cI_0 = 0$ is jointly concave, decreasing in I_0, and increasing in N_0. Hence, the initial condition holds.

Assume that $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$ is jointly concave in (I_{t-1}, N_{t-1}), decreasing in I_{t-1}, and increasing in N_{t-1}. Since concavity and monotonicity are preserved under expectation, $\Psi_t(\cdot,\cdot)$ is jointly concave in (x, y), decreasing in x, and increasing in y. Analogously, $\Lambda(x)$ is concavely decreasing in x. We now verify that

$$\Phi_t(x_t, p_t, N_t) := \mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+)]$$

is jointly concave in (x_t, p_t, N_t) and increasing in N_t. Since $\gamma(\cdot)$ is increasing in N_t, $\sigma \le \theta$, and $\Psi_t(x, y)$ is decreasing in x and increasing in y, $\Phi_t(x_t, p_t, N_t)$ is increasing in N_t. Let $\lambda \in [0, 1]$, $x_* = \lambda x_t + (1-\lambda)\hat{x}_t$, $p_* = \lambda p_t + (1-\lambda)\hat{p}_t$, and $N_* = \lambda N_t + (1-\lambda)\hat{N}_t$, we have:

$$\begin{aligned} &\lambda\Phi_t(x_t, p_t, N_t) + (1-\lambda)\Phi_t(\hat{x}_t, \hat{p}_t, \hat{N}_t) \qquad (15)\\ &= \lambda\mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+ + \eta N_t)] \\ &\quad + (1-\lambda)\mathbb{E}[\Psi_t(\hat{x}_t - \bar{V}_t + \hat{p}_t - \gamma(\hat{N}_t) - \xi_t, \theta(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t) - \sigma(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t - \hat{x}_t + \xi_t)^+ + \eta\hat{N}_t)] \\ &\le \mathbb{E}[\Psi_t(A_t - \xi_t, \eta N_* + \theta B_t + \sigma C_t)] \end{aligned}$$

where

$$A_t = x_* - \bar{V}_t + p_* - \lambda\gamma(N_t) - (1-\lambda)\gamma(\hat{N}_t),$$

$$B_t = \lambda(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + (1-\lambda)(\bar{V}_t - \hat{p}_t + (1-\lambda)\gamma(\hat{N}_t) + \xi_t),$$

$$C_t = -\lambda(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+ - (1-\lambda)(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t) + \xi_t - \hat{x}_t)^+,$$

and the inequality follows from the joint concavity of $\Psi_t(\cdot,\cdot)$. Since $\gamma(\cdot)$ is concave, $A_t \geq x_* - \bar{V}_t + p_* - \gamma(N_*)$. Since $\gamma(\cdot)$ is concave, $\theta \geq \sigma$, and $-(\cdot)^+$ is concavely decreasing, $\theta B_t + \sigma C_t \leq \theta(\bar{V}_t - p_* + \gamma(N_*) + \xi_t) - \sigma(\bar{V}_t - p_* + \gamma(N_*) + \xi_t - x_*)^+$. Therefore, since $\Psi_t(x,y)$ is decreasing in x and increasing in y,

$$\begin{aligned}&\mathbb{E}[\Psi_t(A_t - \xi_t, \eta N_* + \theta B_t + \sigma C_t)]\\ \leq& \mathbb{E}[\Psi_t(x_* - \bar{V}_t + p_* - \gamma(N_*) - \xi_t, \eta N_* + \theta(\bar{V}_t - p_* + \gamma(N_*) + \xi_t) - \sigma(\bar{V}_t - p_* + \gamma(N_*) + \xi_t - x_*)^+)],\end{aligned}$$

i.e., $\Psi_t(\cdot,\cdot,\cdot)$ is jointly concave in (x_t, p_t, N_t).

Since $\Lambda(x) = \mathbb{E}\{-(h+b)(x-\xi_t)^+\}$ is concavely decreasing in x, similar argument to the case of $\Phi_t(x_t, p_t, N_t)$ implies that $\Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t))$ is jointly concave in (x_t, p_t, N_t) and increasing in N_t. By Assumption 1, $R_t(p_t, N_t)$ is jointly concave in (p_t, N_t). Moreover, since $\gamma(\cdot)$ is increasing in N_t, $R_t(p_t, N_t)$ is increasing in N_t as well. Hence,

$$\begin{aligned}J_t(x_t, p_t, N_t) =& R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t))\\ &+ \mathbb{E}[\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) - \sigma(\bar{V}_t - p_t + \gamma(N_t) + \xi_t - x_t)^+)]\end{aligned}$$

is jointly concave in (x_t, p_t, N_t) and increasing in N_t.

Since concavity is preserved under maximization (e.g., Boyd and Vanderberghe 2004), the joint concavity of $v_t(\cdot,\cdot)$ follows directly from that of $J_t(\cdot,\cdot,\cdot)$. Note that for any $\hat{I}_t > I_t$, $\hat{\mathcal{F}}(\hat{I}_t) \subseteq \hat{\mathcal{F}}(I_t)$. Thus,

$$\begin{aligned}v_t(\hat{I}_t, N_t) - c\hat{I}_t =& \max_{(x_t,p_t)\in\hat{\mathcal{F}}(\hat{I}_t)} J_t(x_t, p_t, N_t)\\ \leq& \max_{(x_t,p_t)\in\hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t)\\ =& v_t(I_t, N_t) - cI_t.\end{aligned}$$

Hence, $v_t(I_t, N_t) - cI_t$ is decreasing in I_t. Since $J_t(x_t, p_t, N_t)$ is increasing in N_t for any (x_t, p_t, N_t), for any $\hat{N}_t > N_t$,

$$\begin{aligned}v_t(I_t, \hat{N}_t) - cI_t =& \max_{(x_t,p_t)\in\hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, \hat{N}_t)\\ \geq& \max_{(x_t,p_t)\in\hat{\mathcal{F}}(I_t)} J_t(x_t, p_t, N_t)\\ =& v_t(I_t, N_t) - cI_t.\end{aligned}$$

Thus, $v_t(I_t, N_t) - cI_t$ is increasing in N_t.

Second, we show, by backward induction, that if $v_{t-1}(\cdot,\cdot)$ is continuously differentiable, $\Psi_t(\cdot,\cdot)$, $J_t(\cdot,\cdot,\cdot)$, and $v_t(\cdot,\cdot)$ are continuously differentiable as well. For $t=0$, $v_0(I_0, N_0) = cI_0$ is clearly continuously differentiable. Thus, the initial condition holds.

If $v_{t-1}(\cdot,\cdot)$ is continuously differentiable, $\Psi_t(\cdot,\cdot)$ is continuously differentiable with partial derivatives given by

$$\partial_x \Psi_t(x,y) = \mathbb{E}\{\alpha[\partial_{I_t} v_{t-1}(x, y+\epsilon_t) - c]\}, \tag{16}$$

$$\partial_y \Psi_t(x,y) = \alpha\mathbb{E}\{\partial_{N_{t-1}} v_{t-1}(x, y+\epsilon_t)\}, \tag{17}$$

where the exchangeability of differentiation and expectation is easily justified using the canonical argument (see, e.g., Theorem A.5.1 in Durrett 2010, the condition of which can be easily verified observing the continuity of the partial derivatives of $v_{t-1}(\cdot,\cdot)$, and that the distributions of ξ_t and ϵ_t are continuous.). Moreover, since ξ_t is continuously distributed, $\Lambda(\cdot)$ and $\Phi_t(x_t,p_t,N_t)$ are continuously differentiable with

$$\begin{aligned}\partial_{x_t}\Phi_t(x_t,p_t,N_t)=&\mathbb{E}[\partial_x\Psi_t(x_t-y_t(p_t,N_t)-\xi_t,\theta(y_t(p_t,N_t)+\xi_t)-\sigma(y_t(p_t,N_t)+\xi_t-x_t)^++\eta N_t)]\\&+\sigma\mathbb{E}[\partial_y\Psi_t(x_t-y_t(p_t,N_t)-\xi_t,\theta(y_t(p_t,N_t)+\xi_t)-\sigma(y_t(p_t,N_t)+\xi_t-x_t)^++\eta N_t)1_{\{\xi_t\ge x_t-y_t(p_t,N_t)\}}]\\ \partial_{p_t}\Phi_t(x_t,p_t,N_t)=&\mathbb{E}[\partial_x\Psi_t(x_t-y_t(p_t,N_t)-\xi_t,\theta(y_t(p_t,N_t)+\xi_t)-\sigma(y_t(p_t,N_t)+\xi_t-x_t)^++\eta N_t)]\\&-\theta\mathbb{E}[\partial_y\Psi_t(x_t-y_t(p_t,N_t)-\xi_t,\theta(y_t(p_t,N_t)+\xi_t)-\sigma(y_t(p_t,N_t)+\xi_t-x_t)^++\eta N_t)]\\&+\sigma\mathbb{E}[\partial_y\Psi_t(x_t-y_t(p_t,N_t)-\xi_t,\theta(y_t(p_t,N_t)+\xi_t)-\sigma(y_t(p_t,N_t)+\xi_t-x_t)^++\eta N_t)1_{\{\xi_t\ge x_t-y_t(p_t,N_t)\}}]\end{aligned}$$

By the continuous differentiability of $\gamma(\cdot)$, $R_t(\cdot,\cdot)$ is continuously differentiable. Therefore, $J_t(\cdot,\cdot,\cdot)$ is continuously differentiable in (x_t,p_t,N_t). If $I_t\neq x_t(N_t)$, the continuous differentiability of $v_t(\cdot,\cdot)$ follows immediately from that of $J_t(\cdot,\cdot,\cdot)$ and the envelope theorem. To complete the proof, it suffices to check that, for all $N_t\ge 0$, the left and right partial derivatives of the first variable at $(x_t(N_t),N_t)$, $\partial_{I_t}v_t(x_t(N_t)-,N_t)$ and $\partial_{I_t}v_t(x_t(N_t)+,N_t)$ are equal. By the envelope theorem,

$$\begin{cases}\partial_{I_t}v_t(x_t(N_t)-,N_t)= & c,\\ \partial_{I_t}v_t(x_t(N_t)+,N_t)= & c+\beta+\partial_x\Lambda(x_t(N_t)-y_t(p_t(N_t),N_t))+\partial_x\Phi_t(x_t(N_t),p_t(N_t),N_t).\end{cases}$$

The first-order condition with respect to x_t implies that

$$\beta+\partial_x\Lambda(x_t(N_t)-y_t(p_t(N_t),N_t))+\partial_x\Phi_t(x_t(N_t),p_t(N_t),N_t)=0.$$

Therefore, $\partial_{I_t}v_t(x_t(N_t)-,N_t)=\partial_{I_t}v_t(x_t(N_t)+,N_t)=c$. This completes the induction and, thus, the proof of Lemma 3. *Q.E.D.*

Proof of Theorem 1: Parts (a)-(b) follow immediately from the joint concavity of $J_t(\cdot,\cdot,N_t)$ in (x_t,p_t) for any $N_t\ge 0$.

We now show **part (c)** by backward induction. More specifically, we prove that if $x_{t-1}(N_{t-1})>0$ for all $N_{t-1}\ge 0$, $x_t(N_t)>0$ for all $N_t\ge 0$. Since $v_0(I_0,N_0)=cI_0$, $\Psi_1(x,y)\equiv 0$. Since $D_1\ge 0$ with probability 1, $\partial_x\Lambda(-\bar{V}_1+p_1-\gamma(N_1))=0$ for all $p_1\in[\underline{p},\bar{p}]$ and $N_1\ge 0$. Hence, for any $p_1\in[\underline{p},\bar{p}]$ and $N_1\ge 0$,

$$\partial_{x_1}J_1(0,p_1,N_1)=\beta-\partial_x\Lambda(-\bar{V}_1+p_1-\gamma(N_1))=\beta>0.$$

Hence, $x_1(N_1)>0$ for any $N_1\ge 0$. Thus, the initial condition is satisfied.

Now we assume that $x_{t-1}(N_{t-1})>0$ for all $N_{t-1}\ge 0$ and $x_t(\tilde{N}_t)\le 0$ for some $\tilde{N}_t\ge 0$. Thus,

$$I_{t-1}=x_t(\tilde{N}_t)-D_t(p_t(\tilde{N}_t),\tilde{N}_t)\le 0<x_{t-1}(\tilde{N}_{t-1})$$

almost surely, where

$$\tilde{N}_{t-1}=\eta\tilde{N}_t+\theta D_t(p_t(\tilde{N}_t),\tilde{N}_t)-\sigma(D_t(p_t(\tilde{N}_t),N_t)-x_t(\tilde{N}_t))^++\epsilon_t.$$

Thus, by **part (a)**, $\partial_{I_{t-1}} v_{t-1}(I_{t-1}, \tilde{N}_{t-1}) = c$ almost surely, when conditioned on $N_t = \tilde{N}_t$. Hence, conditioned on $N_t = \tilde{N}_t$, $\partial_x \Psi_t(x,y) = \alpha \mathbb{E}\{\partial_{I_{t-1}} v_{t-1}(I_{t-1}, \tilde{N}_{t-1}) - c | N_t = \tilde{N}_t\} = c - c = 0$, when (x_t, p_t) lies in the neighborhood of $(x_t(\tilde{N}_t), p_t(\tilde{N}_t))$. Since $x_t(\tilde{N}_t) \leq 0$, $\partial_x \Lambda(x_t(\tilde{N}_t) - \bar{V}_t + p_t - \gamma(\tilde{N}_t)) = 0$ for all $p_t \in [\underline{p}, \bar{p}]$. Hence, for any $p_t \in [\underline{p}, \bar{p}]$,

$$\partial_{x_t} J_t(x_t(\tilde{N}_t), p_t, \tilde{N}_t) = \beta - \partial_x \Lambda(x_t(\tilde{N}_t) - \bar{V}_t + p_t - \gamma(\tilde{N}_t)) = \beta > 0.$$

Hence, $x_t(\tilde{N}_t) > 0$, which contradicts the assumption that $x_t(\tilde{N}_t) \leq 0$ is the optimizer of (6) when $N_t = \tilde{N}_t$. Therefore, $x_t(N_t) > 0$ for all $N_t \geq 0$, whenever $x_t(\tilde{N}_t) > 0$ for all $\tilde{N}_t$. This completes the induction and, thus, the proof of **part (c)**. *Q.E.D.*

Proof of Lemma 1: We first show that (8) holds for the case $\sigma = 0$. Specifically, with backward induction, we show that, if $\sigma = 0$, (i) (8) holds for each period t and (ii) $\Delta_t(N_t) = \Delta_*$ for all t and N_t, where $\Delta_* = \arg\max_\Delta \{\beta\Delta + \Lambda(\Delta)\}$. When $t = 1$, $\Psi_t(\cdot, \cdot) \equiv 0$, optimizing the objective function in period 1, (12), indicates that $\Delta_1(N_1) = \Delta_*$ for all N_1 and (8) automatically holds.

We now show that if (i) and (ii) hold for period $t-1$, they also hold for period t. First, we prove that $\Delta_t(N_t) \leq \Delta_*$. If, to the contrary, $\Delta_t(N_t) > \Delta_*$, Lemma 2 yields that

$$\begin{aligned} &\partial_{\Delta_t}[Q_t(p_t(N_t), N_t) + \beta\Delta_t(N_t) + \Lambda(\Delta_t(N_t)) + \mathbb{E}[\Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] \\ &\geq \partial_\Delta[\beta\Delta + \Lambda(\Delta)], \end{aligned}$$

i.e.,

$$\beta + \Lambda'(\Delta_t(N_t)) + \mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] \geq \beta + \Lambda'(\Delta_*).$$

The concavity of $\Lambda(\cdot)$ implies that $\Lambda'(\Delta_t(N_t)) \leq \Lambda'(\Delta_*)$. Moreover, since $\Psi_t(x,y)$ is decreasing in x, $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] \leq 0$. Therefore, $\Lambda'(\Delta_t(N_t)) = \Lambda'(\Delta_*)$ and $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] = 0$. Thus, by the first-order condition with respect to Δ_t, $(p_t(N_t), \Delta_*)$ is also the optimal price and safety-stock level, which is strictly lexicographically smaller than $(p_t(N_t), \Delta_t(N_t))$. This contradicts the assumption that we select the lexicographically smallest optimizer in each period. Hence, $\Delta_t(N_t) \leq \Delta_*$ for all $N_t \geq 0$.

We now show that (8) holds. Note that, conditioned on N_t,

$$\begin{aligned} x_t(N_t) - D_t(p_t(N_t), N_t) - x_{t-1}(N_{t-1}) &= \Delta_t(N_t) - \xi_t - x_{t-1}(N_{t-1}) \\ &\leq \Delta_* - \xi_t - x_{t-1}(N_{t-1}) \\ &= -y_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) - \xi_t \\ &= y_{t-1}(\bar{p}, 0) - \xi_t \\ &\stackrel{d}{=} y_{t-1}(\bar{p}, 0) - \xi_{t-1} \\ &= -D_{t-1}(\bar{p}, 0) \end{aligned}$$

where the first inequality follows from $\Delta_t(N_t) \leq \Delta_*$, the second equality from the inductive hypothesis that $x_{t-1}(N_{t-1}) = y_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) + \Delta_*$ for all $N_{t-1} \geq 0$, the second inequality from $y_{t-1}(p_t, N_t) \geq$

$y_{t-1}(\bar{p},0)$, and the third equality from from $\xi_{t-1} \stackrel{d}{=} \xi_t$. Because $D_{t-1}(\bar{p},0) \geq 0$ with probability 1, conditioned on N_t, we have

$$\mathbb{P}[x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})|N_t] = \mathbb{P}[-D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \leq 0|N_t] = 1,$$

i.e., conditioned on N_t, $x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})$ with probability 1, i.e., (8) holds for period t.

To complete the induction, we show that $\Delta_t(N_t) = \Delta_*$. If, to the contrary $\Delta_t(N_t) < \Delta_*$, Lemma 2 implies that

$$\begin{aligned}&\partial_{\Delta_t}[Q_t(p_t(N_t), N_t) + \beta\Delta_t(N_t) + \Lambda(\Delta_t(N_t)) + \mathbb{E}[\Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] \\ \leq\ &\partial_{\Delta}[\beta\Delta + \Lambda(\Delta)].\end{aligned}$$

On the other hand, (8) implies that $\mathbb{E}[\partial_x \Psi_t(\Delta_t(N_t) - \xi_t, \theta y_t(p_t(N_t), N_t) + \eta N_t + \theta\xi_t)] = 0$. Thus,

$$\beta + \Lambda'(\Delta_t(N_t)) \leq \beta + \Lambda'(\Delta_*).$$

The concavity of $\Lambda(\cdot)$ indicates that $\Lambda'(\Delta_t(N_t)) = \Lambda'(\Delta_*)$. Since Δ_* is the smallest minimizer of $[\beta\Delta + \Lambda(\Delta)]$, we have $\Delta_t(N_t) = \Delta_*$. This completes the induction and, thus, the proof of the sample path property (8) for the case with $\sigma = 0$.

To complete the proof, it suffices to show (8) holds for the case $0 < \sigma \leq \theta$. Observe that the above argument continues to hold if $\xi_t < \Delta_t(N_t)$, since, in this case, the inventory stocking level does not affect future network size evolutions. By Theorem 1(c), if $\xi_t \geq \Delta_t(N_t)$, when conditioned on N_t,

$$x_t(N_t) - D_t(p_t(N_t), N_t) = \Delta_t(N_t) - \xi_t \leq 0 < x_{t-1}(N_{t-1}) \text{ with probability 1,}$$

i.e., (8) holds for this case. Therefore, for any $\sigma \in [0, \theta]$, the sample-path property (8) holds. This completes the proof of Lemma 1. *Q.E.D.*

Proof of Lemma 4:. By parts (a) and (b) of Theorem 1, if $I_t \leq x_t(N_t)$,

$$v_t(I_t, N_t) = cI_t + \pi_t(N_t),$$

where

$$\pi_t(N_t) := \max\{J_t(x_t, p_t, N_t) : x_t \geq 0, p_t \in [\underline{p}, \bar{p}]\}.$$

By Lemma 3, $\pi_t(\cdot)$ is concavely increasing and continuously differentiable in N_t.

By Lemma 1, for each $N_t \geq 0$, $x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})$ with probability 1. Since $v_{t-1}(I_{t-1}, N_{t-1}) = cI_{t-1} + \pi_{t-1}(N_{t-1})$ for all $I_{t-1} \leq x_{t-1}(N_{t-1})$,

$$\begin{aligned}&v_{t-1}(x_t(N_t) - D_t(p_t(N_t), N_t), \theta D_t(p_t(N_t), N_t) + \eta N_t - \sigma(D_t(p_t(N_t), N_t) - x_t(N_t))^+ + \epsilon_t) \\ =&c[x_t(N_t) - D_t(p_t(N_t), N_t)] + \pi_{t-1}(\theta D_t(p_t(N_t), N_t) + \eta N_t - \sigma(D_t(p_t(N_t), N_t) - x_t(N_t))^+)\end{aligned}$$

with probability 1. Taking expectation with respect to ϵ_t, we have, for all $N_t \geq 0$ and $x_t \leq x_t(N_t)$,

$$\begin{aligned}&\Psi_t(\Delta_t(N_t) - \xi_t, \theta(y_t(p_t(N_t), N_t) + \xi_t) + \eta N_t - \sigma(\xi_t - \Delta_t(N_t))^+) \\ =&\alpha\mathbb{E}[\pi_{t-1}(\theta(y_t(p_t(N_t), N_t) + \xi_t) + \eta N_t - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Therefore, for all $N_t \geq 0$, if $\Delta_t \leq \Delta_t(N_t)$,

$$O_t(\Delta_t, p_t, N_t) = Q_t(p_t, N_t) + \beta\Delta_t + \Lambda(\Delta_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

where $G_t(y) := \alpha\mathbb{E}[\pi_{t-1}(y + \epsilon_t)]$.

Finally, it remains to show that $(\Delta_t(N_t), p_t(N_t))$ maximizes the right-hand side of (13). Note that Theorem 1(c) and Lemma 1 imply that, if $I_t \leq x_t(N_t)$, with probability 1, $I_\tau \leq x_\tau(N_\tau)$ for all $\tau = t, t-1, \cdots, 1$ and, hence, $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}_{\tau=t,t-1,\cdots,1}$ is the optimal policy in periods $t, t-1, \cdots, 1$. In particular, $(x_t(N_t), p_t(N_t))$ maximizes the total expected discounted profit given that the firm adopts $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$ for $\tau = t-1, \cdots, 1$. It's straightforward to check that if the firm adopts the policy $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$ for $\tau = t-1, \cdots, 1$; and sets the safety-stock level Δ_t and charges p_t in period t, the (normalized) total expected discounted profit of the firm in period t is given by the right-hand side of (13). Since $(\Delta_t(N_t), p_t(N_t))$ maximizes the (normalized) total expected discounted profit in period t, it also maximizes the right-hand side of (13) for each t. This proves Lemma 4. *Q.E.D.*

Proof of Theorem 2: By Theorem 1(c) and Lemma 1, if $I_T \leq x_T(N_t)$, $I_t \leq x_t(N_t)$ for all $t = T, T-1, \cdots, 1$ with probability 1. Therefore, by Theorem 1(a), $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$ with probability 1 if $I_T \leq x_T(N_T)$. The characterization of $(\Delta_t(N_t), p_t(N_t))$ follows immediately from Lemma 4 and its discussions. *Q.E.D.*

Before giving the proof of Theorem 3, we first show Theorem 4.

Proof of Theorem 4: Part (a). If $\sigma = 0$, $O_t(\Delta_t, p_t, N_t) = f_1(\Delta_t) + f_2(p_t, N_t)$, where

$$f_1(\Delta_t) := \beta\Delta_t + \Lambda(\Delta_t) \text{ and}$$

$$f_2(p_t, N_t) := Q_t(p_t, N_t) + \mathbb{E}[G_t(\eta N_t + \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t))].$$

Since $G_t(\cdot)$ is concave, $f_2(\cdot,\cdot)$ is supermodular in (p_t, N_t). Thus, $p_t(\hat{N}_t) \geq p_t(N_t)$ follows immediately (see Topkis 1998). Now we only consider the case $\sigma > 0$.

Assume, to the contrary, $p_t(\hat{N}_t) < p_t(N_t)$, Lemma 2 implies that $\partial_{p_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}
&\partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) - \theta\mathbb{E}[G_t'(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] \\
\leq &\partial_{p_t} Q_t(p_t(N_t), N_t) - \theta\mathbb{E}[G_t'(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].
\end{aligned}$$

Since $Q_t(\cdot,\cdot)$ is supermodular in (p_t, N_t) and strictly concave in p_t, $\partial_{p_t} Q_t(p_t(\hat{N}_t), \hat{N}_t) > \partial_{p_t} Q_t(p_t(N_t), N_t)$. Hence,

$$\mathbb{E}[G_t'(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] > \mathbb{E}[G_t'(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{18}$$

Since $\hat{N}_t > N_t$ and $p_t(\hat{N}_t) < p_t(N_t)$, $y_t(p_t(\hat{N}_t), \hat{N}_t) > y_t(p_t(N_t), N_t)$. The concavity of $G_t(\cdot)$ and (18) imply that $\Delta_t(\hat{N}_t) < \Delta_t(N_t)$. Thus, invoking Lemma 2, we have $\partial_{\Delta_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}
&\beta + \Lambda'(\Delta_t(\hat{N}_t)) + \sigma\mathbb{E}[G_t'(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t), \hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)1_{\{\xi_t \geq \Delta_t(\hat{N}_t)\}}] \\
\leq &\beta + \Lambda'(\Delta_t(N_t)) + \sigma\mathbb{E}[G_t'(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \geq \Delta_t(N_t)\}}].
\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t(\hat{N}_t)) \geq \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}&\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t\geq\Delta_t(\hat{N}_t)\}}]\\ \leq&\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t\geq\Delta_t(N_t)\}}].\end{aligned} \tag{19}$$

Since $\Delta_t(\hat{N}_t) < \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t<\Delta_t(\hat{N}_t\}}\\ \leq&G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t<\Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t<\Delta_t(\hat{N}_t)\}}]\\ \leq&\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t<\Delta_t(N_t)\}}].\end{aligned} \tag{20}$$

Sum up (19) and (20) and we have:

$$\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)]\leq\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)],$$

which contradicts (18). Therefore, $p_t(\hat{N}_t) \geq p_t(N_t)$ for all $\hat{N}_t > N_t$. This proves part (a).

Part (b). If $\sigma = 0$, $O_t(\Delta_t, p_t, N_t) = f_1(\Delta_t) + f_2(p_t, N_t)$, so $\Delta_t(\hat{N}_t) = \Delta_t(N_t)$. Now we restrict ourselves to the case $\sigma > 0$.

Assume, to the contrary, that $\Delta_t(\hat{N}_t) > \Delta_t(N_t)$. Lemma 2 implies that $\partial_{\Delta_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\beta+\Lambda'(\Delta_t(\hat{N}_t))+\sigma\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t\geq\Delta_t(\hat{N}_t)\}}]\\ \geq&\beta+\Lambda'(\Delta_t(N_t))+\sigma\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t\geq\Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t(\hat{N}_t)) \leq \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}&\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t\geq\Delta_t(\hat{N}_t)\}}]\\ \geq&\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t\geq\Delta_t(N_t)\}}].\end{aligned} \tag{21}$$

The concavity of $G_t(\cdot)$ implies that $\eta\hat{N}_t + \theta y_t(p_t(\hat{N}_t), \hat{N}_t) < \eta N_t + \theta y_t(p_t(N_t), N_t)$ and, thus, $p_t(\hat{N}_t) > p_t(N_t)$. Since $\Delta_t(\hat{N}_t) > \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t<\Delta_t(\hat{N}_t)\}}\\ \geq&G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t<\Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)1_{\{\xi_t<\Delta_t(\hat{N}_t)\}}]\\ \geq&\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)1_{\{\xi_t<\Delta_t(N_t)\}}].\end{aligned} \tag{22}$$

Sum up (21) and (22) and we have:

$$\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)]\geq\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)]. \tag{23}$$

Since $p_t(\hat{N}_t) > p_t(N_t)$, Lemma 2 implies that $\partial_{p_t} O_t(\Delta_t(\hat{N}_t), p_t(\hat{N}_t), \hat{N}_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t}Q_t(p_t(\hat{N}_t),\hat{N}_t)-\theta\mathbb{E}[G'_t(\eta\hat{N}_t+\theta(y_t(p_t(\hat{N}_t),\hat{N}_t)+\xi_t)-\sigma(\xi_t-\Delta_t(\hat{N}_t))^+)]\\ \geq&\partial_{p_t}Q_t(p_t(N_t),N_t)-\theta\mathbb{E}[G'_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)].\end{aligned}$$

Since $y_t(p_t(\hat{N}_t),\hat{N}_t) < y_t(p_t(N_t),N_t)$,

$$\partial_{p_t} Q_t(p_t(\hat{N}_t),\hat{N}_t) = y_t(p_t(\hat{N}_t),\hat{N}_t) - p_t(\hat{N}_t) + c < y_t(p_t(N_t),N_t) - p_t(N_t) + c = \partial_{p_t} Q_t(p_t(N_t),N_t).$$

Thus,

$$\mathbb{E}[G'_t(\eta\hat{N}_t + \theta(y_t(p_t(\hat{N}_t),\hat{N}_t) + \xi_t) - \sigma(\xi_t - \Delta_t(\hat{N}_t))^+)] < \mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t(N_t),N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (23). Therefore, $\Delta_t(\hat{N}_t) \le \Delta_t(N_t)$ for any $\hat{N}_t > N_t$. This proves part (b).

Part (c). Since $\gamma(\hat{N}_t) = \gamma(N_t)$, $p_t(\hat{N}_t) \ge p_t(N_t)$ implies that $y_t(p_t(\hat{N}_t),\hat{N}_t) \le y_t(p_t(N_t),N_t)$. Moreover, by part (b), $\Delta_t(\hat{N}_t) \le \Delta_t(N_t)$. Therefore,

$$x_t(\hat{N}_t) = \Delta_t(\hat{N}_t) + y_t(p_t(\hat{N}_t),\hat{N}_t) \le \Delta_t(N_t) + y_t(p_t(N_t),N_t) = x_t(N_t).$$

This proves part (c). *Q.E.D.*

Proof of Theorem 3: Part (a). Since $\gamma(\cdot) \equiv \gamma_0$, $\partial_{N_t} v_t(\cdot,\cdot) \equiv 0$, $\pi'_t(\cdot) \equiv 0$, and thus $G'_t(\cdot) \equiv 0$. Therefore, optimizing (9) yields that $\Delta_t(N_t) \equiv \Delta_*$ for any t and N_t. To show $\hat{\Delta}_t(N_t) \ge \Delta_*$, we assume, to the contrary, that $\hat{\Delta}_t(N_t) < \Delta_*$. Lemma 2 yields that $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \le \beta + \Lambda'(\Delta_*)$, i.e.,

$$\beta + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma\mathbb{E}[G'_t(\eta N_t + \theta(y_t(\hat{p}_t(N_t),N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}] \le \beta + \Lambda'(\Delta_*).$$

Since $\sigma\mathbb{E}[G'_t(\eta N_t + \theta(y_t(\hat{p}_t(N_t),N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}] \ge 0$, we have $\Lambda'(\hat{\Delta}_t(N_t)) \le \Lambda'(\Delta_*)$. The concavity of $\Lambda(\cdot)$ indicates that $\Lambda'(\hat{\Delta}_t(N_t)) = \Lambda'(\Delta_*)$ and thus, by our assumption that Δ_* is the lexicographically smallest optimizer, $\hat{\Delta}_t(N_t) \ge \Delta_*$. This contradicts with $\hat{\Delta}_t(N_t) < \Delta_*$. Thus, $\hat{\Delta}_t(N_t) \ge \Delta_*$.

If $\hat{\gamma}'(\cdot) > 0$, $\hat{G}'_t(\cdot) > 0$. Thus, for any p_t,

$$\begin{aligned}\partial_{\Delta_t} \hat{O}_t(\Delta_*, p_t, N_t) =& \beta + \Lambda'(\Delta_*) + \sigma\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(y_t(p_t,N_t) + \xi_t) - \sigma(\xi_t - \Delta_*)^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}] \\ =& \sigma\mathbb{E}[G'_t(\eta N_t + \theta(y_t(p_t,N_t) + \xi_t) - \sigma(\xi_t - \Delta_*)^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}] \\ >& 0,\end{aligned}$$

where the second equality follows from the first-order condition $\beta + \Lambda'(\Delta_*) = 0$ and the inequality from $\sigma > 0$ and $\hat{G}'(\cdot) > 0$. Hence, $\hat{\Delta}_t(N_t) > \Delta_*$. This proves part (a).

Part (b). We rewrite the objective function (9) in (Δ_t, y_t, N_t), where $y_t = y_t(p_t, N_t) = \bar{V}_t - p_t + \gamma(N_t)$ is the expected demand in period t. It is clear that, given the net rating N_t, price p_t and expected demand y_t have a one-to-one correspondence. Hence, optimizing over (Δ_t, p_t, N_t) is equivalent to optimizing over (Δ_t, y_t, N_t). We transform the objective function $O_t(\cdot,\cdot,\cdot)$ into

$$K_t(\Delta_t, y_t, N_t) = (\bar{V}_t + \gamma(N_t) - y_t - c)y_t + \beta\Delta_t + \Lambda(\Delta_t) + \mathbb{E}[G_t(\eta N_t + \theta(y_t + \xi_t) - \sigma(\xi_t - \Delta_t)^+)].$$

Let $y_t(N_t)$ be the optimal expected demand in period t with net rating N_t. We have $y_t(N_t) = y_t(p_t(N_t), N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t)$.

We now show that $\hat{y}_t(N_t) \ge y_t(N_t)$ for all N_t. Assume, to the contrary, that $\hat{y}_t(N_t) < y_t(N_t)$. Lemma 2 yields that $\partial_{y_t} \hat{K}_t(\hat{\Delta}_t(N_t), \hat{y}_t(N_t), N_t) \le \partial_{y_t} K_t(\Delta_t(N_t), y_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\bar{V}_t - c - 2\hat{y}_t(N_t) + \hat{\gamma}(N_t) + \theta\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ \le& \bar{V}_t - c - 2y_t(N_t) + \gamma(N_t) + \theta\mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Because $\hat{G}'_t(\cdot) \geq G'_t(\cdot) \equiv 0$ and $\hat{y}_t(N_t) < y_t(N_t)$, the concavity of $\hat{G}_t(\cdot)$ implies that

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \geq \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].$$

Since $\hat{\gamma}(N_t) \geq \gamma(N_t) \equiv 0$, we have $-2\hat{y}_t(N_t) \leq -2y_t(N_t)$, which contradicts the assumption that $\hat{y}_t(N_t) < y_t(N_t)$. Hence, $\hat{y}_t(N_t) \geq y_t(N_t)$.

We now show $y_t(N_t) \equiv y_*(t)$. Observe that, since $\gamma(\cdot) \equiv 0$ and $G'_t(\cdot) \equiv 0$, optimizing (9) yields that

$$p_t(N_t) = \arg\max_{p_t \in [\underline{p}, \bar{p}]} Q_t(p_t, N_t) = \min\{\max\{\frac{\bar{V}_t + c}{2}, \underline{p}\}, \bar{p}\} =: p_*(t).$$

Clearly, $p_*(t)$ is independent of the net rating N_t. Hence, $y_t(N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t) = \bar{V}_t - p_*(t) =: y_*(t)$, which is independent of the net rating N_t as well.

Putting everything together, we have

$$\hat{x}_t(N_t) = \hat{y}_t(N_t) + \hat{\Delta}_t(N_t) \geq y_t(N_t) + \Delta_t(N_t) = y_*(t) + \Delta_* =: x_*(t).$$

Here, $x_*(t)$ is independent of the net rating N_t. By part (a), the inequality is strict if $\sigma > 0$ and $\gamma'(\cdot) > 0$. This proves part (b).

Part (c). The equality $p_t(N_t) \equiv p_*(t)$ has been shown in part (b). To show the existence of the threshold $\mathfrak{N}_t$, we first prove that $\hat{p}_t(0) \leq p_t(0)$. Observe that $\hat{p}_t(0) = \bar{V}_t + \hat{\gamma}(0) - \hat{y}_t(0)$ and $p_t(0) = \bar{V}_t + \gamma(0) - y_t(0)$. By the proof of part (b), $\hat{y}_t(0) \geq y_t(0)$. Moreover, since $\hat{\gamma}(0) = \gamma(0) = 0$, $\hat{p}_t(0) \leq p_t(0)$. By part (b), $p_t(N_t) \equiv p_t(0)$ for all N_t. Recall from Theorem 4(a) that $\hat{p}_t(N_t)$ is increasing in N_t. The joint concavity of $\hat{O}_t(\cdot,\cdot,\cdot)$ implies that $\hat{p}_t(N_t)$ is continuously increasing in N_t. Thus, let $\mathfrak{N}_t$ be the smallest N_t such that $\hat{p}_t(N_t) \geq p_t(N_t) = p_*(t)$. The monotonicity of $\hat{p}_t(\cdot)$ then suggests that $\hat{p}_t(N_t) \leq p_t(N_t) \equiv p_*(t)$ if $N_t \leq \mathfrak{N}_t$, and $\hat{p}_t(N_t) \geq p_t(N_t) \equiv p_*(t)$ if $N_t \geq \mathfrak{N}_t$. This proves part (c). *Q.E.D.*

Proof of Theorem 5: We show Theorem 5 by backward induction. More specifically, we show that if $\bar{V}_\tau \equiv \bar{V}$ for all τ and $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$ for all $N \geq 0$, (i) $p_t(N) \leq p_{t-1}(N)$ for all $N \geq 0$, (ii) $\Delta_t(N) \geq \Delta_{t-1}(N)$ for all $N \geq 0$, (iii) $x_t(N) \geq x_{t-1}(N)$ for all $N \geq 0$, and (iv) $\pi'_t(N) \geq \pi'_{t-1}(N)$ for all $N \geq 0$. Since $\pi'_1(N) \geq \pi'_0(N) \equiv 0$ for all N, the initial condition is satisfied.

Note that $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$ for all $N \geq 0$ implies that

$$G'_t(y) = \alpha\mathbb{E}[\pi'_{t-1}(y + \epsilon_t)] \geq \alpha\mathbb{E}[\pi'_{t-2}(y + \epsilon_t)] = G'_{t-1}(y),$$

for all y. Since $\bar{V}_t \equiv \bar{V}$, $Q_t(p_t, N_t) = (p_t - c)(\bar{V} - p_t + \gamma(N_t)) =: Q(p_t, N_t)$ for all t. We use $y_t(N) := y_t(p_t(N), N)$ to denote the expected demand in period t with net rating N under the optimal policy.

We first prove that $p_t(N) \leq p_{t-1}(N)$ for all N. Assume, to the contrary, that $p_t(N) > p_{t-1}(N)$ for some N. Lemma 2 implies that $\partial_{p_t} O_t(\Delta_t(N), p_t(N), N) \geq \partial_{p_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$, i.e.,

$$\begin{aligned}&\partial_p Q(p_t(N), N) - \theta\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)]\\ \geq &\partial_p Q(p_{t-1}(N), N) - \theta\mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)].\end{aligned}$$

Since $Q(\cdot, N)$ is strictly concave in p and $p_t(N) > p_{t-1}(N)$, $\partial_p Q(p_t(N), N) < \partial_p Q(p_{t-1}(N), N)$. Thus,

$$\mathbb{E}[G'_t(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] < \mathbb{E}[G'_{t-1}(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)]. \tag{24}$$

Note that $G_t'(\cdot) \geq G_{t-1}'(\cdot)$ for all y, $p_t(N) > p_{t-1}(N)$, and $G_t(\cdot)$ and $G_{t-1}(\cdot)$ are concave. We have (24) implies that $\sigma > 0$ and $\Delta_t(N) > \Delta_{t-1}(N)$. Thus, Lemma 2 implies that $\partial_{\Delta_t} O_t(\Delta_t(N), p_t(N), N) \geq \partial_{\Delta_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$, i.e.,

$$\begin{aligned}&\beta + \Lambda'(\Delta_t(N)) + \sigma\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t \geq \Delta_t(N)\}}] \\ \geq &\beta + \Lambda'(\Delta_{t-1}(N)) + \sigma\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t(N_t)) \leq \Lambda'(\Delta_{t-1}(N))$ and, thus,

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t \geq \Delta_t(N)\}}] \\ \geq &\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}].\end{aligned} \tag{25}$$

Since $\Delta_t(N) > \Delta_{t-1}(N)$ and $G_t'(\cdot) \geq G_{t-1}'(\cdot) \geq 0$, it follows immediately that, for any realization of $\xi_t = \xi_{t-1}$,

$$\begin{aligned}&G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N)\}} \\ \geq &G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}.\end{aligned}$$

Integrate over ξ_t and ξ_{t-1} and we have

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N)\}}] \\ \geq &\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}].\end{aligned} \tag{26}$$

Sum up (25) and (26) and we have:

$$\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)] \geq \mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)],$$

which contradicts (24). Therefore, $p_t(N) \leq p_{t-1}(N)$ for all N.

Next, we show that $\Delta_t(N) \geq \Delta_{t-1}(N)$. If $\sigma = 0$, it is straightforward to show that $\Delta_t(N) = \Delta_{t-1}(N) = \Delta_*$. Hence, we confine ourselves to the interesting case of $\sigma > 0$.

Assume, to the contrary, that $\Delta_t(N) < \Delta_{t-1}(N)$. Lemma 2 implies that $\partial_{\Delta_t} O_t(\Delta_t(N), p_t(N), N) \leq \partial_{\Delta_{t-1}} O_{t-1}(\Delta_{t-1}(N), p_{t-1}(N), N)$, i.e.,

$$\begin{aligned}&\beta + \Lambda'(\Delta_t(N)) + \sigma\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t \geq \Delta_t(N)\}}] \\ \leq &\beta + \Lambda'(\Delta_{t-1}(N)) + \sigma\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t(N)) \geq \Lambda'(\Delta_{t-1}(N))$ and, thus,

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t \geq \Delta_t(N)\}}] \\ \leq &\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} \geq \Delta_{t-1}(N)\}}].\end{aligned} \tag{27}$$

The concavity of $G_t(\cdot)$ and $G_{t-1}(\cdot)$ and that $G_t'(\cdot) \geq G_{t-1}'(\cdot)$ imply that $y_t(N) > y_{t-1}(N_t)$ and, thus, $p_t(N) < p_{t-1}(N)$. Since $\Delta_t(N) < \Delta_{t-1}(N)$, it follows immediately that, for any realization of $\xi_t = \xi_{t-1}$,

$$\begin{aligned}&G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N)\}} \\ \leq &G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}.\end{aligned}$$

Integrate over ξ_t and ξ_{t-1} and we have

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N + \theta(y_t(N) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N)\}}] \\ \leq &\mathbb{E}[G_{t-1}'(\eta N + \theta(y_{t-1}(N) + \xi_{t-1}) - \sigma(\xi_{t-1} - \Delta_{t-1}(N))^+)1_{\{\xi_{t-1} < \Delta_{t-1}(N)\}}].\end{aligned} \tag{28}$$

Sum up (27) and (28) and we have:

$$\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\le\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)]. \tag{29}$$

By Lemma 2, $p_t(N)<p_{t-1}(N)$ yields that $\partial_{p_t}O_t(\Delta_t(N),p_t(N),N)\le\partial_{p_{t-1}}O_{t-1}(\Delta_{t-1}(N),p_{t-1}(N),N)$, i.e.,

$$\begin{aligned}&\partial_pQ(p_t(N),N)-\theta\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ \le&\partial_pQ(p_{t-1}(N),N)-\theta\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)].\end{aligned}$$

Since $Q(\cdot,N)$ is strictly concave in p, $\partial_pQ(p_t(N),N)>\partial_pQ(p_{t-1}(N),N)$. Thus,

$$\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]>\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)],$$

which contradicts inequality (29). Therefore, $\Delta_t(N)\ge\Delta_{t-1}(N)$ for any N.

Next, we show $x_t(N)\ge x_{t-1}(N)$. Note that $p_t(N)\le p_{t-1}(N)$ implies that $y_t(N)\ge y_{t-1}(N)$. Thus,

$$x_t(N)=y_t(N)+\Delta_t(N)\ge y_{t-1}(N)+\Delta_{t-1}(N)=x_{t-1}(N).$$

Finally, to complete the induction, we show that $\pi_t'(N)\ge\pi_{t-1}'(N)$ for all N. By the envelope theorem,

$$\pi_t'(N)=(p_t(N)-c)\gamma'(N)+(\eta+\theta\gamma'(N))\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)],$$

and

$$\pi_{t-1}'(N)=(p_{t-1}(N)-c)\gamma'(N)+(\eta+\theta\gamma'(N))\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)],$$

If $p_t(N)=p_{t-1}(N)$ and $\Delta_t(N)=\Delta_{t-1}(N)$, $\pi_t'(N)\ge\pi_{t-1}'(N)$ follows immediately from $\gamma'(N)\ge 0$ and $G_t'(\cdot)\ge G_{t-1}'(\cdot)$.

If $p_t(N)=p_{t-1}(N)$ and $\Delta_t(N)>\Delta_{t-1}(N)$, Lemma 2 yields that $\partial_{\Delta_t}O_t(\Delta_t(N),p_t(N),N)\ge\partial_{\Delta_{t-1}}O_{t-1}(\Delta_{t-1}(N),p_{t-1}(N),N)$, i.e.,

$$\begin{aligned}&\beta+\Lambda'(\Delta_t(N))+\sigma\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)1_{\{\xi_t\ge\Delta_t(N)\}}]\\ \ge&\beta+\Lambda'(\Delta_{t-1}(N))+\sigma\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)1_{\{\xi_{t-1}\ge\Delta_{t-1}(N)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t(N))\le\Lambda'(\Delta_{t-1}(N))$ and, thus,

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)1_{\{\xi_t\ge\Delta_t(N)\}}]\\ \ge&\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)1_{\{\xi_{t-1}\ge\Delta_{t-1}(N)\}}].\end{aligned} \tag{30}$$

Since $\Delta_t(N)>\Delta_{t-1}(N)$, it follows immediately that, for any realization of $\xi_t=\xi_{t-1}$,

$$\begin{aligned}&G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)1_{\{\xi_t<\Delta_t(N)\}}\\ \ge&G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)1_{\{\xi_{t-1}<\Delta_{t-1}(N)\}}.\end{aligned}$$

Integrate over ξ_t and ξ_{t-1} and we have

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)1_{\{\xi_t<\Delta_t(N)\}}]\\ \ge&\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)1_{\{\xi_{t-1}<\Delta_{t-1}(N)\}}].\end{aligned} \tag{31}$$

Sum up (30) and (31) and we have:

$$\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\ge\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)]. \tag{32}$$

Plugging (32) into the formulas of $\pi_t'(\cdot)$ and $\pi_{t-1}'(\cdot)$, we have that the inequality $\pi_t'(N)\geq\pi_{t-1}'(N)$ follows immediately from $p_t(N)=p_{t-1}(N)$.

If $p_t(N)<p_{t-1}(N)$, Lemma 2 yields that $\partial_{p_t}O_t(\Delta_t(N),p_t(N),N)\leq\partial_{p_{t-1}}O_{t-1}(\Delta_{t-1}(N),p_{t-1}(N),N)$, i.e.,

$$\begin{aligned}&\partial_pQ(p_t(N),N)-\theta\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ \leq&\partial_pQ(p_{t-1}(N),N)-\theta\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)],\end{aligned}$$

i.e.,

$$\begin{aligned}&\bar V+c-2p_t(N)+\gamma(N)-\theta\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ \leq&\bar V+c-2p_{t-1}(N)+\gamma(N)-\theta\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)].\end{aligned}$$

Thus,

$$\begin{aligned}&(p_t(N)-p_{t-1}(N))+\theta(\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ &-\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)])\\ \geq&p_{t-1}(N)-p_t(N)\\ >&0.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ &-\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)]\\ \geq&\frac{2}{\theta}(p_{t-1}(N)-p_t(N))\\ >&0.\end{aligned}$$

Therefore,

$$\begin{aligned}\pi_t'(N)-\pi_{t-1}'(N)=&((p_t(N)-p_{t-1}(N))+\theta(\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ &-\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)])\gamma'(N)\\ &+\eta(\mathbb{E}[G_t'(\eta N+\theta(y_t(N)+\xi_t)-\sigma(\xi_t-\Delta_t(N))^+)]\\ &-\mathbb{E}[G_{t-1}'(\eta N+\theta(y_{t-1}(N)+\xi_{t-1})-\sigma(\xi_{t-1}-\Delta_{t-1}(N))^+)])\\ \geq&0.\end{aligned}$$

Hence, $\pi_t'(N)\geq\pi_{t-1}'(N)$ for all N. This completes the induction and, thus, the proof of Theorem 5. *Q.E.D.*

Proof of Theorem 6: We show Theorem 6 by backward induction. More specifically, we show that if $\hat\alpha>\alpha$ and $\hat\pi_{t-1}'(N_{t-1})\geq\pi_{t-1}'(N_{t-1})$ for all $N_{t-1}\geq0$, (i) $\hat p_t(N_t)\leq p_t(N_t)$ for all $N_t\geq0$, (ii) $\hat\Delta_t(N_t)\geq\Delta_t(N_t)$ for all $N_t\geq0$, (iii) $\hat x_t(N_t)\geq x_t(N_t)$ for all $N_t\geq0$, and (iv) $\hat\pi_t'(N_t)\geq\pi_t'(N_t)$ for all $N_t\geq0$. Since $\hat\pi_0'(N_0)=\pi_0'(N_0)\equiv0$ for all N, the initial condition is satisfied.

Note that $\hat\pi_{t-1}'(N_{t-1})\geq\pi_{t-1}'(N_{t-1})$ for all $N_{t-1}\geq0$ implies that

$$\hat G_t'(y)=\hat\alpha\mathbb{E}[\hat\pi_{t-1}'(y+\epsilon_t)]\geq\alpha\mathbb{E}[\pi_{t-1}'(y+\epsilon_t)]=G_t'(y),$$

for all y. We use $\hat y_t(N_t):=y_t(\hat p_t(N_t),N_t)$ and $y_t(N_t):=y_t(p_t(N_t),N_t)$ to denote the expected demand in period t under the optimal policy with discount factor $\hat\alpha$ and α, respectively. Since $\hat\alpha>\alpha$, $\hat\beta=b-(1-\hat\alpha)c>b-(1-\alpha)c=\beta$.

We first prove that $\hat{p}_t(N_t) \le p_t(N_t)$ for all N_t. Assume, to the contrary, that $\hat{p}_t(N_t) > p_t(N_t)$ for some N_t. Lemma 2 implies that $\partial_{p_t} O_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \ge \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t} Q_t(\hat{p}_t(N_t), N_t) - \theta \mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ \ge &\partial_{p_t} Q_t(p_t(N_t), N_t) - \theta \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Since $Q_t(\cdot, N_t)$ is strictly concave in p_t and $\hat{p}_t(N_t) > p_t(N_t)$, $\partial_{p_t} \hat{Q}_t(p_t(N_t), N_t) < \partial_{p_t} Q(p_t(N_t), N_t)$. Thus,

$$\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] < \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{33}$$

Note that $\hat{G}_t'(\cdot) \ge G_t'(\cdot)$ for all y, $\hat{p}_t(N_t) > p_t(N_t)$, and $\hat{G}_t(\cdot)$ and $G_t(\cdot)$ are concave. We have (33) implies that $\sigma > 0$ and $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$. Thus, Lemma 2 implies that $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \ge \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\hat{\beta} + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}]\\ \ge &\beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\hat{\Delta}_t(N_t)) \le \Lambda'(\Delta_t(N_t))$. In addition, $\hat{\beta} > \beta$, thus,

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}]\\ \ge &\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned} \tag{34}$$

Since $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$ and $\hat{G}_t'(\cdot) \ge G_t'(\cdot) \ge 0$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}\\ \ge &G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}]\\ \ge &\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}].\end{aligned} \tag{35}$$

Sum up (34) and (35) and we have:

$$\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \ge \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts (33). Therefore, $\hat{p}_t(N_t) \le p_t(N_t)$ for all N.

Next, we show that $\hat{\Delta}_t(N_t) \ge \Delta_t(N_t)$. If $\sigma = 0$, $\hat{\Delta}_t(N_t) = \arg\max_{\Delta_t}[\hat{\beta}\Delta_t + L(\Delta_t)]$, whereas $\Delta_t(N_t) = \arg\max_{\Delta_t}[\beta\Delta_t + L(\Delta_t)]$. Since $\hat{\beta} > \beta$, it follows immediately that $\hat{\Delta}_t(N_t) \ge \Delta_t(N_t)$. Hence, we confine ourselves to the interesting case of $\sigma > 0$.

Assume, to the contrary, that $\hat{\Delta}_t(N_t) < \Delta_t(N_t)$. Lemma 2 implies that $\partial_{\Delta_t} \hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \le \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\hat{\beta} + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma \mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}]\\ \le &\beta + \Lambda'(\Delta_t(N_t)) + \sigma \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\hat{\Delta}_t(N_t)) \ge \Lambda'(\Delta_t(N_t))$. Since $\hat{\beta} > \beta$, we have

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+) 1_{\{\xi_t \ge \hat{\Delta}_t(N_t)\}}]\\ \le &\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned} \tag{36}$$

The concavity of $\hat{G}_t(\cdot)$ and $G_t(\cdot)$ and that $\hat{G}'_t(\cdot) \geq G'_t(\cdot)$ imply that $\hat{y}_t(N_t) > y_t(N_t)$ and, thus, $\hat{p}_t(N) < p_t(N_t)$. Since $\hat{\Delta}_t(N_t) < \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t < \hat{\Delta}_t(N_t)\}} \\ \leq & G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}] \\ \leq & \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}].\end{aligned} \tag{37}$$

Sum up (36) and (37) and we have:

$$\begin{aligned}&\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ \leq & \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned} \tag{38}$$

By Lemma 2, $\hat{p}_t(N_t) < p_t(N_t)$ yields that $\partial_{p_t}\hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t}\hat{Q}(\hat{p}_t(N_t), N_t) - \theta\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] \\ \leq & \partial_{p_t} Q(p_t(N_t), N_t) - \theta\mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Since $Q_t(\cdot, N_t)$ is strictly concave in p_t, $\partial_{p_t} Q_t(\hat{p}_t(N_t), N_t) > \partial_{p_t} Q_t(p_t(N_t), N_t)$. Thus,

$$\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)] > \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (38). Therefore, $\hat{\Delta}_t(N_t) \geq \Delta_t(N_t)$ for any N_t.

Next, we show $\hat{x}_t(N_t) \geq x_t(N_t)$. Note that $\hat{p}_t(N_t) \leq p_t(N_t)$ implies that $\hat{y}_t(N_t) \geq y_t(N_t)$. Thus,

$$\hat{x}_t(N_t) = \hat{y}_t(N_t) + \hat{\Delta}_t(N_t) \geq y_t(N_t) + \Delta_t(N_t) = x_t(N_t).$$

Finally, to complete the induction, we show that $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$ for all N_t. By the envelope theorem,

$$\hat{\pi}'_t(N_t) = (\hat{p}_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)],$$

and

$$\pi'_t(N_t) = (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

If $\hat{p}_t(N_t) = p_t(N_t)$ and $\hat{\Delta}_t(N_t) = \Delta_t(N_t)$, $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$ follows immediately from $\gamma'(N_t) \geq 0$ and $\hat{G}'_t(\cdot) \geq G'_t(\cdot)$.

If $\hat{p}_t(N_t) = p_t(N_t)$ and $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$, Lemma 2 yields that $\partial_{\Delta_t}\hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\beta + \Lambda'(\hat{\Delta}_t(N_t)) + \sigma\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ \geq & \beta + \Lambda'(\Delta_t(N_t)) + \sigma\mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \geq \Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\hat{\Delta}_t(N_t)) \leq \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}&\mathbb{E}[\hat{G}'_t(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t \geq \hat{\Delta}_t(N_t)\}}] \\ \geq & \mathbb{E}[G'_t(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \geq \Delta_t(N_t)\}}].\end{aligned} \tag{39}$$

Since $\hat{\Delta}_t(N_t) > \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}\\ \geq &G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)1_{\{\xi_t < \hat{\Delta}_t(N_t)\}}]\\ \geq &\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)1_{\{\xi_t < \Delta_t(N_t)\}}].\end{aligned} \tag{40}$$

Sum up (39) and (40) and we have:

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ \geq &\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N))^+)].\end{aligned} \tag{41}$$

Plugging (41) into the formulas of $\hat{\pi}_t'(\cdot)$ and $\pi_t'(\cdot)$, we have that the inequality $\hat{\pi}_t'(N_t) \geq \pi_t'(N_t)$ follows immediately from $\hat{p}_t(N_t) = p_t(N_t)$.

If $\hat{p}_t(N_t) < p_t(N_t)$, Lemma 2 yields that $\partial_{p_t}\hat{O}_t(\hat{\Delta}_t(N_t), \hat{p}_t(N_t), N_t) \leq \partial_{p_t}O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t}Q_t(\hat{p}_t(N_t), N_t) - \theta\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ \leq &\partial_{p_t}Q_t(p_t(N_t), N_t) - \theta\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],\end{aligned}$$

i.e.,

$$\begin{aligned}&\bar{V}_t + c - 2\hat{p}_t(N_t) + \gamma(N_t) - \theta\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ \leq &\bar{V}_t + c - 2p_t(N_t) + \gamma(N_t) - \theta\mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Thus,

$$\begin{aligned}&(\hat{p}_t(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ &- \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)])\\ \geq &\hat{p}_t(N_t) - p_t(N_t)\\ > &0.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ &- \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]\\ \geq &\frac{2}{\theta}(p_t(N_t) - \hat{p}_t(N_t))\\ > &0.\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{\pi}_t'(N_t) - \pi_t'(N_t) = &((\hat{p}_t(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ &- \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)])\gamma'(N_t)\\ &+ \eta(\mathbb{E}[\hat{G}_t'(\eta N_t + \theta(\hat{y}_t(N_t) + \xi_t) - \sigma(\xi_t - \hat{\Delta}_t(N_t))^+)]\\ &- \mathbb{E}[G_t'(\eta N_t + \theta(y_t(N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)])\\ \geq &0.\end{aligned}$$

Hence, $\hat{\pi}'_t(N_t) \geq \pi'_t(N_t)$ for all N_t. This completes the induction and, thus, the proof of Theorem 6. *Q.E.D.*

Proof of Theorem 7: Part (a). For any $\hat{N}_t > N_t$, since $p_t(\hat{N}_t) \geq p_t(N_t)$ (Theorem 4(a)),

$$y_t(\hat{N}_t) - y_t(N_t) = \gamma(\hat{N}_t) - \gamma(N_t) - (p_t(\hat{N}_t) - p_t(N_t)) \leq \gamma(\hat{N}_t) - \gamma(N_t) \leq (\hat{N}_t - N_t)\gamma'(N_t), \tag{42}$$

where the last inequality follows from the concavity of $\gamma(\cdot)$.

Let $\mathcal{N} := \min\{N_t \geq 0 : \theta\gamma'(N_t) \leq 1 - \eta\}$. Since $\lim_{N_t \to +\infty} \gamma'(N_t) = 0$, $\mathcal{N} < +\infty$. Since $I_T \leq x_T(N_T)$, $I_t \leq x_t(N_t)$ for all t with probability 1. Thus,

$$\mathbb{E}[N_{t-1}|N_t] = \eta N_t + \theta y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t) + \xi_t)^+$$

We now show that, if $N_t > \mathcal{N}$, $\mathbb{E}[N_{t-1}|N_t] - N_t$ is decreasing in N_t. For any $\hat{N}_t > N_t > \mathcal{N}$, we have

$$\begin{aligned}
\mathbb{E}[N_{t-1}|\hat{N}_t] - \hat{N}_t - (\mathbb{E}[N_{t-1}|N_t] - N_t) =& \eta\hat{N}_t + \theta y_t(\hat{N}_t) - \sigma\mathbb{E}(\xi_t - \Delta_t(\hat{N}_t))^+ - \hat{N}_t - (\eta N_t + \theta y_t(N_t) - \sigma\mathbb{E}(\xi_t - \Delta_t(N_t))^+ - N_t) \\
=& -(1-\eta)(\hat{N}_t - N_t) + \theta(y_t(\hat{N}_t) - y_t(N_t)) - \sigma(\mathbb{E}(\xi_t - \Delta_t(\hat{N}_t))^+ - \mathbb{E}(\xi_t - \Delta_t(N_t))^+) \\
\leq& -(1-\eta)(\hat{N}_t - N_t) + \theta\gamma'(N_t)((\hat{N}_t - N_t)) \\
<& [(1-\theta) - (1-\theta)]((\hat{N}_t - N_t)) \\
=& 0,
\end{aligned} \tag{43}$$

where the first inequality follows from (42) and $\Delta_t(\hat{N}_t) \leq \Delta_t(N_t)$ (Theorem 4(b)), and the second inequality from the definition of $\mathcal{N}$.

Let $g_t(N_t) := \mathbb{E}[N_{t-1}|N_t] - N_t$. Clearly, $g_t(\cdot)$ is continuous in N_t. On the other hand, $g_t(0) = \mathbb{E}[N_{t-1}|0] = \theta\mathbb{E}(x_t(0) \wedge D_t(p_t(0), 0)) + (\theta - \sigma)\mathbb{E}(D_t(p_t(0), n) - x_t(0))^+$. By Theorem 1(a), $x_t(0) > 0$ and thus $g_t(0) > 0$. Since $\lim_{N_t \to +\infty} \gamma'(N_t) = 0$, when N_t is sufficiently large, $g_t(N_t) \leq -(1-\eta)N_t + C$ for some constant C. Hence, $\lim_{N_t \to +\infty} g_t(N_t) = -\infty$. By the concavity of $\gamma(\cdot)$ and Lemma 7 in Appendix D, it is straightforward to check that, if $g_t(\cdot)$ is decreasing at the point N_t, it is strictly decreasing at any point $\hat{N}_t > N_t$. By (43), there exists a threshold $\bar{\mathcal{N}}_t \leq \mathcal{N}$, such that $g_t(N_t) > 0$ on the region $[0, \bar{\mathcal{N}}_t]$ and it is strictly decreasing in N_t on the region $[\bar{\mathcal{N}}_t, +\infty)$, with $\lim_{N_t \to +\infty} g_t(N_t) = -\infty$. Therefore, there exists a unique $\bar{N}_t > \bar{\mathcal{N}}_t$ such that $g_t(N_t) > 0$ for $N_t < \bar{N}_t$ and $g_t(N_t) < 0$ for $N_t > \bar{N}_t$, i.e., $\mathbb{E}[N_{t-1}|N_t] > N_t$ if $N_t < \bar{N}_t$ and $\mathbb{E}[N_{t-1}|N_t] < N_t$ if $N_t > \bar{N}_t$. Since $g_t(0) > 0$ and $\lim_{N_t \to +\infty} g_t(N_t) = -\infty$, $\bar{N}_t \in (0, +\infty)$.

Part (b). By Theorem 5, $\Delta_t(N)$ is increasing, whereas $p_t(N)$ is decreasing, in t for any $N \geq 0$. Therefore, $\Delta(\cdot) := \lim_{t \to +\infty} \Delta_t(\cdot)$ and $p(\cdot) := \lim_{t \to +\infty} p_t(\cdot)$ is the optimal safety-stock and price in the infinite-horizon discounted reward model ($T = +\infty$). Hence, in this case,

$$N_{t-1} = \eta N_t + \theta(y(N_t) + \xi_t) - \sigma(-\Delta(N_t) + \xi_t)^+,$$

where $y(N_t) = \mathbb{E}[D_t(p(N_t), N_t)] = \bar{V} - p(N_t) + \gamma(N_t)$. By Theorem 11.10 in Stokey et al. (1989), the Markov process $\{N_t : t \in \mathbb{Z}\}$ is ergodic and, thus, has a stationary distribution $\nu(\cdot)$.

Using the same technique as the proof of part (a), we can show that the threshold $\bar{N}$ exists. The convergence result $\bar{N} = \lim_{t \to +\infty} \bar{N}_t$ follows immediately from the monotonicity that $\Delta_t(\cdot)$ is increasing and $p_t(\cdot)$ is decreasing in t.

Part (c). It suffices to show that, as $\eta \uparrow 1$, $\bar{N}_t \uparrow +\infty$ for each t. Since $\mathbb{E}[N_{t-1}|N_t] = \eta N_t + \theta y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t)+\xi_t)^+$ is increasing in η for any N_t. Note that $\bar{N}_t = \inf\{N_t : \mathbb{E}[N_{t-1}|N_t] < N_t\}$, so it follows immediately from the monotonicity of $\mathbb{E}[N_{t-1}|N_t]$ in η that $\bar{N}_t$ is increasing in η. Hence, $\bar{N} = \lim_{t\to+\infty}\bar{N}_t$ is increasing in η as well.

Finally, it remains to show that as $\eta \uparrow 1$, $\bar{N}_t \uparrow +\infty$. Assume, to the contrary, that there exists a uniform upper bound $\Gamma_t \in (0,+\infty)$ such that $\bar{N}_t < \Gamma_t$ for all $\eta < 1$. Hence,

$$g_t(N_t) = -(1-\eta)N_t + y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t)+\xi_t)^+ < 0, \text{ for all } N_t \geq \Gamma_t \text{ and } \eta < 1.$$

Note that $y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t)+\xi_t)^+$ is the average number of customers who join the network in period t. Since $D_t(p_t, N_t) \geq 0$ with probability 1, we have $y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t)+\xi_t)^+ \geq \underline{y}$ for some $\underline{y} > 0$. Therefore, if $\eta > \max\{1-\underline{y}/(2\Gamma_t), 0\}$,

$$g_t(N_t) = -(1-\eta)N_t + y_t(N_t) - \sigma\mathbb{E}(-\Delta_t(N_t)+\xi_t)^+ > -\left(1-\left(1-\frac{\underline{y}}{2\Gamma_t}\right)\right)N_t + \underline{y} = \underline{y}\left(1-\frac{N_t}{2\Gamma_t}\right).$$

Thus, if $N_t = 1.5\Gamma_t > \Gamma_t$, $g_t(N_t) > 0.25\underline{y} > 0$, which contradicts that $g_t(N_t) < 0$ for all $N_t \geq \Gamma_t$. Therefore, as $\eta \uparrow 1$, $\bar{N}_t \uparrow +\infty$ and thus $\bar{N} = \lim_{t\to+\infty}\bar{N}_t$ also increases to infinity as η increases to 1. *Q.E.D.*

Proof of Theorem 8: Part (a). First $v_t^w(\cdot,\cdot) \leq v_t(\cdot,\cdot)$ follows immediately from that $v_t(\cdot,\cdot)$ is the optimal expected profit in periods $t, t-1, \cdots, 1$.

If $w \geq t$, the length of the moving time-window exceeds the total planning horizon length. Therefore, if $w \geq t$, the w-heuristic is the optimal policy and, hence, $v_t^w(\cdot,\cdot) = v_t(\cdot,\cdot)$.

It remains to show that if $w \leq t-2$, $v_t^w(\cdot,\cdot) \leq v_t^{w+1}(\cdot,\cdot)$. Note that

$$\begin{aligned}
v_t(I_t, N_t) =& \mathbb{E}[\sum_{1\leq\tau\leq t}\alpha^{t-\tau}\{Q_\tau(p_\tau, N_\tau) + \beta\Delta_\tau + \Lambda(\Delta_\tau) + cI_\tau\} + \alpha^t I_0 | I_t, N_t],\\
s.t.&\text{ for each } 1\leq\tau\leq t:\\
p_\tau =& p_\tau(N_\tau);\\
\Delta_\tau =& \Delta_\tau(N_\tau);\\
N_{\tau-1} =& \eta N_\tau + \theta D_\tau(p_\tau, N_\tau) - \sigma(\xi_\tau - \Delta_\tau(p_\tau, N_\tau))^+ + \epsilon_\tau;\\
I_{\tau-1} =& \bar{V}_\tau + \gamma(N_\tau) - p_\tau + \Delta_\tau(N_\tau) - D_\tau(p_\tau, N_\tau).
\end{aligned}$$

Analogously, we have

$$
\begin{aligned}
v_t^w(I_t,N_t) =& \mathbb{E}[\sum_{1\le\tau\le t}\alpha^{t-\tau}\{Q_\tau(p_\tau,N_\tau)+\beta\Delta_\tau+\Lambda(\Delta_\tau)+cI_\tau\}+\alpha^t I_0|I_t,N_t],\\
s.t.&\text{ for each } 1\le\tau\le w:\\
&p_\tau = p_\tau(N_\tau);\\
&\Delta_\tau = \Delta_\tau(N_\tau);\\
&\text{for each } w+1\le\tau\le t:\\
&p_\tau = p_w(N_\tau);\\
&\Delta_\tau = \Delta_w(N_\tau);\\
&\text{for each } 1\le\tau\le t:\\
&N_{\tau-1} = \eta N_\tau+\theta D_\tau(p_\tau,N_\tau)-\sigma(\xi_\tau-\Delta_\tau(p_\tau,N_\tau))^+ +\epsilon_\tau;\\
&I_{\tau-1} = \bar{V}_\tau+\gamma(N_\tau)-p_\tau+\Delta_\tau(N_\tau)-D_\tau(p_\tau,N_\tau);
\end{aligned}
$$

and

$$
\begin{aligned}
v_t^{w+1}(I_t,N_t) =& \mathbb{E}[\sum_{1\le\tau\le t}\alpha^{t-\tau}\{Q_\tau(p_\tau,N_\tau)+\beta\Delta_\tau+\Lambda(\Delta_\tau)+cI_\tau\}+\alpha^t I_0|I_t,N_t],\\
s.t.&\text{ for each } 1\le\tau\le w+1:\\
&p_\tau = p_\tau(N_\tau);\\
&\Delta_\tau = \Delta_\tau(N_\tau);\\
&\text{for each } w+2\le\tau\le t:\\
&p_\tau = p_w(N_\tau);\\
&\Delta_\tau = \Delta_w(N_\tau);\\
&\text{for each } 1\le\tau\le t:\\
&N_{\tau-1} = \eta N_\tau+\theta D_\tau(p_\tau,N_\tau)-\sigma(\xi_\tau-\Delta_\tau(p_\tau,N_\tau))^+ +\epsilon_\tau;\\
&I_{\tau-1} = \bar{V}_\tau+\gamma(N_\tau)-p_\tau+\Delta_\tau(N_\tau)-D_\tau(p_\tau,N_\tau);
\end{aligned}
$$

Since $I_T\le x_T(N_t)$, Theorem 5 implies that $\Delta_\tau(\cdot)\ge\Delta_{w+1}(\cdot)\ge\Delta_w(\cdot)$ and $p_\tau(\cdot)\le p_{w+1}(\cdot)\le p_w(\cdot)$ for all $\tau\ge w+1$. Moreover, $(\Delta_\tau(\cdot),p_\tau(\cdot))$ is the optimal policy for the dynamic program. Putting everything together, it follows immediately that $v_t^w(\cdot,\cdot)\le v_t^{w+1}(\cdot,\cdot)\le v_t(\cdot,\cdot)$ for all $t\ge w+2$. This finishes the proof of part (a).

Part (b). The inequality $v^w(\cdot,\cdot)\le v^{w+1}(\cdot,\cdot)\le v(\cdot,\cdot)$ follows from $v_t^w(\cdot,\cdot)\le v_t^{w+1}(\cdot,\cdot)\le v_t(\cdot,\cdot)$ by letting T approach infinity. Note that $v_w(\cdot,\cdot)\le v^w(\cdot,\cdot)\le v(\cdot,\cdot)$. Hence, $\sup|v^w(\cdot,\cdot)-v(\cdot,\cdot)|\le\sup|v_w(\cdot,\cdot)-v(\cdot,\cdot)|$. Let $\mathcal{T}$ be the operator acted on a concave and continuously differentiable function $f(\cdot,\cdot)$ that satisfies

$$
\begin{aligned}
\mathcal{T}[f(I,N)] &= \max_{(x,p)\in\hat{\mathcal{F}}(I)}\{R(p,N)+\beta x+\Lambda(x-y(p,N))\\
&\quad+\mathbb{E}[\Psi(x_t-y(p,N)-\xi,\eta N+\theta(y(p,N)+\xi)-\sigma(y(p,N)+\xi-x)^+)]\},\\
\text{with } \Psi(x,y) &:= \alpha\mathbb{E}\{[f(x,y+\epsilon_t)-cx]\},\\
R(p,N) &:= (p-\alpha c-b)(\bar{V}+\gamma(N)-p),\\
y(p,N) &:= \bar{V}+\gamma(N)-p.
\end{aligned}
$$

By Theorem 9.6 in Stokey et al. (1989), $\mathcal{T}$ is a contraction mapping with contraction factor α under the sup norm. Since $v(\cdot,\cdot)$ is the fixed point of $\mathcal{T}$ and $v_w = \mathcal{T}^w[v_0(\cdot,\cdot)]$, we have

$$\sup|v_w(\cdot,\cdot) - v(\cdot,\cdot)| \leq \alpha^w \sup|v_0(\cdot,\cdot) - v(\cdot,\cdot)|.$$

Let $C := \sup|v_0(\cdot,\cdot) - v(\cdot,\cdot)| > 0$ for a given initial state (I,N), and $\delta := -\log(\alpha) > 0$. Thus, we have

$$\sup|v^w(\cdot,\cdot) - v(\cdot,\cdot)| \leq \sup|v_w(\cdot,\cdot) - v(\cdot,\cdot)| \leq C\mathrm{e}^{-\delta w}. \tag{44}$$

By (44), $\lim_{w\to+\infty} v^w(\cdot,\cdot) = v(\cdot,\cdot)$ follows immediately for any initial state (I,N). This proves part (b). *Q.E.D.*

Proof of Lemma 5: Part (a). Part (a) follows from the same argument as the proof of Lemma 3, so we omit its proof for brevity.

Part (b). The optimal value function $v_t^e(I_t,N_t)$ satisfies the following recursive scheme:

$$v_t^e(I_t,N_t) = cI_t + \max_{(x_t,p_t,n_t)\in\hat{\mathcal{F}}_e(I_t)} J_t^e(x_t,p_t,n_t,N_t), \tag{45}$$

where $\hat{\mathcal{F}}_e(I_t) := [I_t,+\infty)\times[\underline{p},\bar{p}]\times[0,+\infty)$ denotes the set of feasible decisions and

$$\begin{aligned} J_t^e(x_t,p_t,n_t,N_t) =& R_t(p_t,N_t) + \beta x_t + \Lambda(x_t - y_t(p_t,N_t)) - c_n(n_t) \\ &+ \mathbb{E}[\Psi_t^e(x_t - y_t(p_t,N_t) - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(y_t(p_t,N_t)+\xi_t - x_t)^+ + n_t)], \end{aligned} \tag{46}$$

with $\Psi_t^e(x,y) := \alpha\mathbb{E}\{v_{t-1}^e(x, y+\epsilon_t) - cx\}$.

The derivation of (46) is given as follows:

$$\begin{aligned} J_t^e(x_t,p_t,n_t,N_t) :=& -cI_t + \mathbb{E}\{p_tD_t(p_t,N_t) - c(x_t - I_t) - h(x_t - D_t(p_t,N_t))^+ - b(x_t - D_t(p_t,N_t))^- - c_n(n_t) \\ &+\alpha v_{t-1}^e(x_t - D_t(p_t,N_t), \theta D_t(p_t,N_t) + \eta N_t - \sigma(D_t(p_t,N_t) - x_t)^+ + n_t + \epsilon_t)|N_t\}, \\ =& (p_t - \alpha c - b)y_t(p_t,N_t) + (b - (1-\alpha)c)x_t - c_n(n_t) + \mathbb{E}\{-(h+b)(x_t - y_t(p_t,N_t) - \xi_t)^+ \\ &+\alpha[v_{t-1}^e(x_t - y_t(p_t,N_t) - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(y_t(p_t,N_t)+\xi_t - x_t)^+ + n_t + \epsilon_t) \\ &-c(x_t - y_t(p_t,N_t) - \xi_t)]|N_t\} \\ =& R_t(p_t,N_t) + \beta x_t + \Lambda(x_t - y_t(p_t,N_t)) - c_n(n_t) \\ &+\mathbb{E}[\Psi_t^e(x_t - y_t(p_t,N_t) - \xi_t, \eta N_t + \theta(y_t(p_t,N_t)+\xi_t) - \sigma(y_t(p_t,N_t)+\xi_t - x_t)^+ + n_t)]. \end{aligned}$$

We use $(\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t))$ as the unconstrained optimizer of (46). The same argument as the proof of Lemma 3 yields that $J_t^e(\cdot,\cdot,\cdot,\cdot)$ is jointly concave in (x_t,p_t,n_t,N_t). Hence, if $I_t \leq \hat{x}_t^e(N_t)$, $(x_t^{e*}(I_t,N_t), p_t^{e*}(I_t,N_t), n_t^*(I_t,N_t)) = (\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t))$; otherwise $(I_t > \hat{x}_t^e(N_t))$ $x_t^{e*}(I_t,N_t) = I_t$.

The same argument as the proof of Lemma 1 implies that $\mathbb{P}[\hat{x}_t^e(N_t) - D_t(\hat{p}_t^e(N_t),N_t) \leq \hat{x}_{t-1}^e(N_{t-1})|N_t] = 1$. Hence, the same argument as the proof of Lemma 4 and its discussions enables us to transform the objective function from $J_t^e(x_t,p_t,n_t,N_t)$ to $O_t^e(\Delta_t,p_t,n_t,N_t)$ by letting $\Delta_t = x_t - \mathbb{E}[D_t(p_t,N_t)] = x_t - y_t(p_t,N_t)$, and we have that $(\hat{x}_t^e(N_t), \hat{p}_t^e(N_t), \hat{n}_t(N_t)) = (x_t^e(N_t), p_t^e(N_t), n_t(N_t))$ for all $N_t \geq 0$, where $x_t^e(N_t) = \Delta_t^e(N_t) + y_t(p_t^e(N_t),N_t)$. Hence, as in Theorem 2, if $I_T \leq x_T^e(N_T)$, $I_t \leq x_t^e(N_t)$ for all t with probability 1. Hence, part (b) follows. *Q.E.D.*

Proof of Theorem 9: Part (a). We first show that if (10) holds, $n_t^*(I_t, N) > 0$ for all I_t. Observe that, since $\partial_y \Psi_{t-1}^e(x,y) \geq 0$,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \geq (\underline{p} - b - \alpha c)\gamma'(N_{t-1}) - \gamma'(N_{t-1})\Lambda'(\Delta_{t-1}^*),$$

where $\Delta_{t-1}^* := x_{t-1}^{e*}(I_{t-1}, N_{t-1}) - y_{t-1}(I_{t-1}, N_{t-1})$. The first-order condition with respect to x_{t-1} yields that $\Lambda'(\Delta_{t-1}^*) \leq -\beta$ for any realization of ξ_t and ϵ_t. Thus, for any realization of ξ_t and ϵ_t,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \geq (\underline{p} - c)\gamma'(N_{t-1}). \tag{47}$$

Therefore, for any Δ_t and $p_t \in [\underline{p}, \bar{p}]$,

$$\begin{aligned}
\partial_{n_t} O_t^e(\Delta_t, p_t, 0, N) \geq & \alpha \mathbb{E}\{\partial_{N_{t-1}} v_{t-1}^e(x_t - D_t(p_t, N), N_{t-1}) | N_t = N\} - c_n'(0) \\
\geq & \alpha \mathbb{E}\{(\underline{p} - c)\gamma'(N_{t-1}) | N_t = N\} - c_n'(0) \\
\geq & \alpha(1-\iota)(\underline{p} - c)\gamma'(\bar{S}(N)) - c_n'(0) \\
> & 0,
\end{aligned} \tag{48}$$

where the second inequality follows from (47), and the fourth from the assumption (10). The third inequality of (48) follows from the following inequality:

$$\begin{aligned}
\alpha \mathbb{E}[(\underline{p} - c)\gamma'(N_{t-1}) | N_t = N] \;=\; & \alpha \mathbb{E}_{N_{t-1} \geq \bar{S}(N)}[(\underline{p} - c)\gamma'(N_{t-1}) | N_t = N] \\
& + \alpha \mathbb{E}_{N_{t-1} < \bar{S}(N)}[(\underline{p} - c)\gamma'(N_{t-1}) | N_t = N] \\
\geq\; & 0 + \alpha \mathbb{E}_{N_{t-1} < \bar{S}(N)}[(\underline{p} - c)\gamma'(\bar{S}(N))] \\
\geq\; & \alpha(1-\iota)(\underline{p} - c)\gamma'(\bar{S}(N)),
\end{aligned}$$

where the first inequality follows from the concavity of $\gamma(\cdot)$, and the second from the definition of $\bar{S}(N)$. The inequality (48) yields that $n_t^*(I_t, N) > 0$ for all I_t.

Since $\gamma(\cdot)$ is continuously increasing in N_t, $\bar{S}(N)$ is continuously increasing in N. The concavity of $\gamma(\cdot)$ implies that $\gamma'(\bar{S}(N))$ is continuously decreasing in N. Therefore, let

$$N^*(\iota) := \max\{N \geq 0 : \alpha(1-\iota)[(\underline{p} - c)\gamma'(\bar{S}(N))] > c_n'(0)\}.$$

We have (10) holds for all $N < N^*(\iota)$. This completes the proof of part (a).

Part (b). Since $\gamma(\cdot)$ is concavely increasing in N_t,

$$\partial_{N_{t-1}} v_{t-1}^e(I_{t-1}, N_{t-1}) \leq \partial_{N_{t-1}} v_t^e(I_{t-1}, 0) \leq (\sum_{\tau=1}^{t-1} (\alpha\eta)^{\tau-1})(\bar{p} - c)\gamma'(0).$$

Thus, if $\alpha(\sum_{\tau=1}^{t-1}(\alpha\eta)^{\tau-1})(\bar{p} - c)\gamma'(0) \leq c_n'(0)$, then for any (x_t, p_t, n_t, N_t),

$$\begin{aligned}
\partial_{n_t} J_t^e(x_t, p_t, n_t, N_t) \leq & \alpha \mathbb{E}\{\partial_{N_{t-1}} v_{t-1}^e(x_t - D_t(p_t, N_t), N_{t-1} + n_t) | N_t = N\} - c_n'(0) \\
\leq & \alpha(\sum_{\tau=1}^{t-1} (\alpha\eta)^{\tau-1})(\bar{p} - c)\gamma'(0) - c_n'(0) \\
\leq & 0.
\end{aligned}$$

Hence, $n_t^*(I_t, N_t) = 0$ for all (I_t, N_t). This completes the proof of part (b). *Q.E.D.*

Proof of Theorem 10: Parts (a)-(c). We prove parts (a)-(c) together by backward induction. More specifically, we show that if $\partial_{N_{t-1}}\pi^e_{t-1}(\cdot) \le \partial_{N_{t-1}}\pi_{t-1}(\cdot)$ for all $N_{t-1} \ge 0$, (i) $p^e_t(N_t) \ge p_t(N_t)$, (ii) $\Delta^e_t(N_t) \le \Delta_t(N_t)$, (iii) $x^e_t(N_t) \le x_t(N_t)$, and (iv) $\partial_{N_t}\pi^e_t(\cdot) \le \partial_{N_t}\pi_t(\cdot)$ for all $N_t \ge 0$. Since $\partial_{N_0}\pi^e_0(\cdot) = \partial_{N_0}\pi_0(\cdot) \equiv 0$, the initial condition is satisfied. Note that $\partial_{N_{t-1}}\pi^e_{t-1}(N_{t-1}) \le \partial_{N_{t-1}}\pi_{t-1}(N_{t-1})$ for all $N_{t-1} \ge 0$ implies that

$$\partial_y G^e_t(y) = \alpha\mathbb{E}\{\partial_{N_{t-1}}\pi^e_{t-1}(y+\epsilon_t)\} \le \alpha\mathbb{E}\{\partial_{N_{t-1}}\pi_{t-1}(y+\epsilon_t)\} = \partial_y G_t(y),$$

for all y.

We first show that $p^e_t(N_t) \ge p_t(N_t)$ for all N_t. Assume, to the contrary, that $p^e_t(N_t) < p_t(N_t)$ for some N_t. Lemma 2 implies that $\partial_{p_t}O^e_t(\Delta^e_t(N_t), p^e_t(N_t), n_t(N_t), N_t) \le \partial_{p_t}O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t}Q(p^e_t(N_t), N_t) - \theta\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)]\\ \le&\partial_{p_t}Q_t(p_t(N_t), N_t) - \theta\mathbb{E}[\partial y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Since $Q_t(\cdot, N_t)$ is strictly concave in p_t and $p^e_t(N_t) < p_t(N_t)$, $\partial_{p_t}Q(p^e_t(N_t), N_t) > \partial_{p_t}Q_t(p_t(N_t), N_t)$. Thus,

$$\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)] > \mathbb{E}[\partial y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{49}$$

Note that $\partial_y G^e_t(\cdot) \le \partial_y G_t(\cdot)$ for all y, $p^e_t(N_t) < p_t(N_t)$, and $n_t(N_t) \ge 0$. Thus, the concavity of $G^e_t(\cdot)$ and $G_t(\cdot)$, together with the inequality (49), implies that $\sigma > 0$ and $\Delta^e_t(N_t) < \Delta_t(N_t)$. Thus, Lemma 2 implies that $\partial_{\Delta_t}O^e_t(\Delta^e_t(N_t), p^e_t(N_t), n_t(N_t), N_t) \le \partial_{\Delta_t}O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\beta + \Lambda'(\Delta^e_t(N_t)) + \sigma\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)1_{\{\xi_t \ge \Delta^e_t(N_t)\}}]\\ \le&\beta + \Lambda'(\Delta_t(N_t)) + \sigma\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta^e_t(N_t)) \ge \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}&\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)1_{\{\xi_t \ge \Delta^e_t(N_t)\}}]\\ \le&\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \ge \Delta_t(N_t)\}}].\end{aligned} \tag{50}$$

Since $\Delta^e_t(N_t) < \Delta_t(N_t)$ and $0 \le \partial y G^e_t(\cdot) \le \partial_y G_t(\cdot)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)1_{\{\xi_t < \Delta^e_t(N_t)\}}\\ \le&\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)1_{\{\xi_t < \Delta^e_t(N_t)\}}]\\ \le&\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}].\end{aligned} \tag{51}$$

Sum up (50) and (51) and we have:

$$\mathbb{E}[\partial_y G^e_t(\eta N_t + \theta(y_t(p^e_t(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta^e_t(N_t))^+)] \le \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts (49). Therefore, $p^e_t(N_t) \ge p_t(N_t)$ for all N_t.

Next, we show that $\Delta^e_t(N_t) \ge \Delta_t(N_t)$. If $\sigma = 0$, it is straightforward to show that $\Delta^e_t(N_t) = \Delta_t(N_t) = \Delta_*$. Hence, we restrict ourselves to the interesting case of $\sigma > 0$.

Assume, to the contrary, that $\Delta_t^e(N_t) > \Delta_t(N_t)$. Lemma 2 implies that $\partial_{\Delta_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\beta + \Lambda'(\Delta_t^e(N_t)) + \sigma\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ \geq&\beta + \Lambda'(\Delta_t(N_t)) + \sigma\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \geq \Delta_t(N_t)\}}].\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t^e(N_t)) \leq \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}&\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\ \geq&\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t \geq \Delta_t(N_t)\}}].\end{aligned} \tag{52}$$

The concavity of $G_t^e(\cdot)$ and $G_t(\cdot)$ and that $n_t(N_t) \geq 0$ and $\partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$ imply that $y_t(p_t^e(N_t), N_t) < y_t(p_t(N_t), N_t)$ and, thus, $p_t^e(N_t) > p_t(N_t)$. Since $\Delta_t^e(N_t) < \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}&\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)1_{\{\xi_t < \Delta_t^e(N_t)\}} \\ \geq&\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}.\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}&\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)1_{\{\xi_t < \Delta_t^e(N_t)\}}] \\ \geq&\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)1_{\{\xi_t < \Delta_t(N_t)\}}].\end{aligned} \tag{53}$$

Sum up (52) and (53) and we have:

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \geq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{54}$$

By Lemma 2, $p_t^e(N_t) > p_t(N_t)$ yields that $\partial_{p_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&\partial_{p_t} Q(p_t^e(N_t), N_t) - \theta\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\ \geq&\partial_{p_t} Q_t(p_t(N_t), N_t) - \theta\mathbb{E}[\partial y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].\end{aligned}$$

Since $Q_t(\cdot, N_t)$ is strictly concave in p_t, $\partial_{p_t} Q(p_t^e(N_t), N_t) < \partial_{p_t} Q_t(p_t(N_t), N_t)$. Thus,

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] < \mathbb{E}[\partial y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],$$

which contradicts inequality (54). Therefore, $\Delta_t^e(N_t) \leq \Delta_t(N_t)$ for any N_t.

Next, we show $x_t^e(N_t) \leq x_t(N_t)$. Note that $p_t^e(N_t) \geq p_t(N_t)$ implies that $y_t(p_t^e(N_t), N_t) \leq y_t(p_t(N_t), N_t)$. Thus,

$$x_t^e(N_t) = y_t(p_t^e(N_t), N_t) + \Delta_t^e(N_t) \leq y_t(p_t(N_t), N_t) + \Delta_t(N_t) = x_t(N_t).$$

Finally, to complete the induction, we show that $\partial_{N_t}\pi_t^e(N_t) \geq \partial_{N_t}\pi_t(N_t)$ for all $N_t \geq 0$. By the envelope theorem,

$$\partial_{N_t}\pi_t^e(N_t) = (p_t^e(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)],$$

and

$$\partial_{N_t}\pi_t(N_t) = (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\mathbb{E}[\partial y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].$$

If $p_t^e(N_t) = p_t(N_t)$ and $\Delta_t^e(N_t) = \Delta_t(N_t)$, $\partial_{N_t}\pi_t^e(N_t) \leq \partial_{N_t}\pi_t(N_t)$ follows immediately from $\gamma'(N) \geq 0$ and $\partial_y G_t^e(\cdot) \leq \partial_y G_t(\cdot)$ for all y.

If $p_t^e(N_t) = p_t(N_t)$ and $\Delta_t^e(N_t) < \Delta_t(N_t)$, Lemma 2 yields that $\partial_{\Delta_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \leq \partial_{\Delta_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}
&\beta + \Lambda'(\Delta_t^e(N_t)) + \sigma\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\
\leq &\beta + \Lambda'(\Delta_t(N_t)) + \sigma\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}].
\end{aligned}$$

The concavity of $\Lambda(\cdot)$ suggests that $\Lambda'(\Delta_t^e(N_t)) \geq \Lambda'(\Delta_t(N_t))$ and, thus,

$$\begin{aligned}
&\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t \geq \Delta_t^e(N_t)\}}] \\
\leq &\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t \geq \Delta_t(N_t)\}}].
\end{aligned} \tag{55}$$

Since $\Delta_t^e(N_t) < \Delta_t(N_t)$, it follows immediately that, for any realization of ξ_t,

$$\begin{aligned}
&\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}} \\
\leq &\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}.
\end{aligned}$$

Integrate over ξ_t and we have

$$\begin{aligned}
&\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+) 1_{\{\xi_t < \Delta_t^e(N_t)\}}] \\
\leq &\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+) 1_{\{\xi_t < \Delta_t(N_t)\}}].
\end{aligned} \tag{56}$$

Sum up (55) and (56) and we have:

$$\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \leq \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]. \tag{57}$$

Plugging (57) into the formulas of $\partial_{N_t}\pi_t^e(\cdot)$ and $\partial_{N_t}\pi_t(\cdot)$, we have the inequality $\partial_{N_t}\pi_t^e(N_t) \leq \partial_{N_t}\pi_t(N_t)$ follows immediately from $p_t^e(N) = p_t(N_t)$.

If $p_t^e(N_t) > p_t(N_t)$, Lemma 2 yields that $\partial_{p_t} O_t^e(\Delta_t^e(N_t), p_t^e(N_t), n_t(N_t), N_t) \geq \partial_{p_t} O_t(\Delta_t(N_t), p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}
&\partial_{p_t} Q_t(p_t^e(N_t), N_t) - \theta\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\
\geq &\partial_{p_t} Q_t(p_t(N_t), N_t) - \theta\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\bar{V}_t + c - 2p_t^e(N_t) + \gamma(N_t) - \theta\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\
\leq &\bar{V}_t + c - 2p_t(N_t) + \gamma(N_t) - \theta\mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)].
\end{aligned}$$

Thus,

$$\begin{aligned}
&(p_t^e(N_t) - p_t(N_t)) + \theta(\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\
&- \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)]) \\
\geq &p_t^e(N_t) - p_t(N_t) \\
> &0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}[\partial_y G_t^e(\eta N_t + \theta(y_t(p_t^e(N_t), N_t) + \xi_t) + n_t(N_t) - \sigma(\xi_t - \Delta_t^e(N_t))^+)] \\
&- \mathbb{E}[\partial_y G_t(\eta N_t + \theta(y_t(p_t(N_t), N_t) + \xi_t) - \sigma(\xi_t - \Delta_t(N_t))^+)] \\
\geq &\frac{2}{\theta}(p_t^e(N_t) - p_t(N_t)) \\
> &0.
\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_{N_t}\pi_t^e(N_t)-\partial_{N_t}\pi_t(N_t)=&((p_t^e(N_t)-p_t(N_t))+\theta(\mathbb{E}[\partial_y G_t^e(\eta N_t+\theta(y_t(p_t^e(N_t),N_t)+\xi_t)+n_t(N_t)-\sigma(\xi_t-\Delta_t^e(N_t))^+)]\\&-\mathbb{E}[\partial_y G_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)])\gamma'(N)\\&+\eta(\mathbb{E}[\partial_y G_t^e(\eta N_t+\theta(y_t(p_t^e(N_t),N_t)+\xi_t)+n_t(N_t)-\sigma(\xi_t-\Delta_t^e(N_t))^+)]\\&-\mathbb{E}[\partial_y G_t(\eta N_t+\theta(y_t(p_t(N_t),N_t)+\xi_t)-\sigma(\xi_t-\Delta_t(N_t))^+)])\\\geq&0.\end{aligned}$$

Hence, $\partial_{N_t}\pi_t^e(N_t)\geq\partial_{N_t}\pi_t(N_t)$ for all N_t. This completes the induction and, thus, the proof of parts (a)-(c).

Part (d). Note that $\pi_t(\cdot)$ is the normalized optimal profit with the Bellman equation (9) and feasible decision set $\{(x_t,p_t,n_t):\Delta_t\in\mathbb{R},p_t\in[\underline{p},\bar{p}],n_t=0\}\subset\mathcal{F}_e$, which is the feasible decision set associated with the profit $\pi_t^e(\cdot)$. Thus, $\pi_t^e(N_t)\geq\pi_t(N_t)$ for all t and any $N_t\geq 0$. If $n_t(N_t)>0$, we must have $\pi_t^e(N_t)>\pi_t(N_t)$. Otherwise there are two lexicographically different policies (one with $n_t(N_t)=0$ and the other with $n_t(N_t)>0$) that generate the same optimal normalized profit $\pi_t(N_t)$. This contradicts the assumption that the lexicographically smallest policy is selected. Thus, $\pi_t^e(N_t)>\pi_t(N_t)$, which establishes part (d). *Q.E.D.*

Proof of Theorem 11: Part (a). We show part (a) by backward induction. More specifically, we show that if $\sigma=\eta=0$ and $v_{t-1}(\cdot,\cdot)$ is supermodular in (I_{t-1},N_{t-1}), $v_t(\cdot,\cdot)$ is supermodular in (I_t,N_t). Since $v_0(I_0,N_0)=cI_0$, the initial condition is satisfied.

Since supermodularity is preserved under expectation, $\bar{\Psi}_t(x,y):=\alpha\mathbb{E}\{[v_{t-1}(x-\xi_t,y+\theta\xi_t+\epsilon_t)-cx]\}$ is supermodular in (x,y). Let $y_t=\bar{V}_t-p_t+\gamma(N_t)$. Observe that

$$\begin{aligned}J_t(x_t,p_t,N_t)=&R_t(p_t,N_t)+\beta x_t+\Lambda(x_t-\bar{V}_t+p_t-\gamma(N_t))+\bar{\Psi}_t(x_t-\bar{V}_t+p_t-\gamma(N_t),\theta(\bar{V}_t-p_t+\gamma(N_t)))\\=&(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta x_t+\Lambda(x_t-y_t)+\bar{\Psi}_t(x_t-y_t,\theta y_t).\end{aligned}$$

Hence,

$$v_t(I_t,N_t)=cI_t+\max_{(x_t,y_t)\in\mathcal{F}_t'(I_t,N_t)}\{(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta x_t+\Lambda(x_t-y_t)+\bar{\Psi}_t(x_t-y_t,\theta y_t)\},$$

where $\mathcal{F}_t'(I_t,N_t):=\{(x_t,y_t):x_t\geq I_t,y_t\in[\bar{V}_t+\gamma(N_t)-\bar{p},\bar{V}_t+\gamma(N_t)-\underline{p}]\}$. Because $\gamma(\cdot)$ is increasing in N_t, $\Lambda(\cdot)$ is concave, and $\bar{\Psi}_t(\cdot,\cdot)$ is concave and supermodular, $(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta x_t+\Lambda(x_t-y_t)+\bar{\Psi}_t(x_t-y_t,\theta y_t)$ is supermodular in (x_t,y_t,N_t). Moreover, it's straightforward to verify that the feasible set $\{(x_t,y_t,I_t,N_t):N_t\geq 0,(x_t,y_t)\in\mathcal{F}_t'(I_t)\}$ is a lattice in $\mathbb{R}^4$. Therefore, $v_t(I_t,N_t)$ is supermodular in (I_t,N_t). This completes the induction and, thus, the proof of part (a).

Part (b). The continuity results in parts (b)-(e) all follow from the joint concavity and continuous differentiability of $J_t(\cdot,\cdot,\cdot)$ in (x_t,p_t,N_t). Since $x_t^*(I_t,N_t)=\max\{I_t,x_t(N_t)\}$, $x_t^*(I_t,N_t)$ is increasing in I_t. Moreover, because the objective function $(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta x_t+\Lambda(x_t-y_t)+\bar{\Psi}_t(x_t-y_t,\theta y_t)$ is supermodular in (x_t,y_t,N_t), $x_t^*(I_t,N_t)$ is increasing in N_t as well. This proves part (b).

Part (c). If $I_t\leq x_t(N_t)$, $p_t^*(I_t,N_t)=p_t(N_t)$, which is independent of I_t. If $I_t>x_t(N_t)$, $x_t^*(I_t,N_t)=I_t$ and, thus,

$$J_t(x_t^*(I_t,N_t),p_t,N_t)=R_t(p_t,N_t)+\beta I_t+\Lambda(I_t-\bar{V}_t+p_t-\gamma(N_t))+\bar{\Psi}_t(I_t-\bar{V}_t+p_t-\gamma(N_t),\theta(\bar{V}_t-p_t+\gamma(N_t))).\tag{58}$$

Since $\Lambda(\cdot)$ is concave and $\bar{\Psi}_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is submodular in (I_t,p_t). Hence, $p_t^*(I_t,N_t)$ is decreasing in I_t for all (I_t,N_t). By Theorem 4(d), if $I_t \le x_t(N_t)$, $p_t^*(I_t,N_t)=p_t(N_t)$ is increasing in N_t. If $I_t > x_t(N_t)$, we observe from (58) that $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is supermodular in (p_t,N_t). Hence, $p_t^*(I_t,N_t)$ is increasing in N_t for all (I_t,N_t). This proves part (c).

Part (d). If $I_t \le x_t(N_t)$, $y_t^*(I_t,N_t)=y_t(N_t)$, which is independent of I_t. If $I_t > x_t(N_t)$, $x_t^*(I_t,N_t)=I_t$ and, thus,

$$J_t(x_t^*(I_t,N_t),p_t,N_t)=(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta I_t+\Lambda(I_t-y_t)+\bar{\Psi}_t(I_t-y_t,\theta y_t).$$

Since $\Lambda(\cdot)$ is concave and $\Psi_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is supermodular in (I_t,y_t) and its domain is a sublattice of $\mathbb{R}^2$. Hence, $y_t^*(I_t,N_t)$ is increasing in I_t for all (I_t,N_t). By Theorem 4(d), if $I_t \le x_t(N_t)$, $y_t^*(I_t,N_t)=y_t(N_t)$ is increasing in N_t. If $I_t > x_t(N_t)$, $x_t^*(I_t,N_t)=I_t$ and, thus, $J_t(x_t^*(I_t,N_t),p_t,N_t)=(\bar{V}_t+\gamma(N_t)-y_t-\alpha c-b)y_t+\beta I_t+\Lambda(I_t-y_t)+\bar{\Psi}_t(I_t-y_t,\theta y_t)$. The supermodularity of $J_t(x_t^*(I_t,N_t),p_t,N_t)$ in (y_t,N_t) follows directly from that $\gamma(\cdot)$ is increasing in N_t. Moreover, the feasible set $\{(y_t,N_t): y_t\in[\bar{V}_t+\gamma(N_t)-\bar{p},\bar{V}_t+\gamma(N_t)-\underline{p}]\}$ is clearly a sublattice of $\mathbb{R}^2$. Therefore, $y_t^*(I_t,N_t)$ is increasing in N_t for all (I_t,N_t). This proves part (d).

Part (e). If $I_t \le x_t(N_t)$, optimizing (9) yields that $\Delta_t^*(I_t,N_t)=\Delta_*$ is independent of I_t and N_t. If $I_t > x_t(N_t)$, since $I_t-\Delta_t=y_t$,

$$J_t(x_t^*(I_t,N_t),p_t,N_t)=(\bar{V}_t+\gamma(N_t)+\Delta_t-I_t-\alpha c-b)(I_t-\Delta_t)+\beta I_t+\Lambda(\Delta_t)+\bar{\Psi}_t(\Delta_t,\theta(I_t-\Delta_t)).$$

Since $\bar{\Psi}_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is supermodular in (I_t,Δ_t). Moreover, the feasible set $\{(I_t,\Delta_t):\Delta_t\in[I_t-\bar{V}_t-\gamma(N_t)+\underline{p},I_t-\bar{V}_t-\gamma(N_t)+\bar{p}]\}$ is clearly a sublattice of $\mathbb{R}^2$. Hence, $\Delta_t^*(I_t,N_t)$ is increasing in I_t for all (I_t,N_t). Moreover, since $\Delta_t^*(I_t,N_t)=I_t-y_t^*(I_t,N_t)$, by part (d), $\Delta_t^*(I_t,N_t)$ is decreasing in N_t. This proves part (e). *Q.E.D.*

Appendix D: Additional Discussions on Assumption 1

In this section, we present some additional discussions on the key technical assumption of our model, Assumption 1, which assumes that $R_t(\cdot,\cdot)$ is jointly concave on its domain. Specifically, we present the necessary and sufficient conditions for $R_t(\cdot,\cdot)$ to be jointly concave in Section D.1. Then, in Section D.2, we give concrete examples of concave $R_t(\cdot,\cdot)$ functions for specific forms of the function $\gamma(\cdot)$.

D.1. Necessary and Sufficient Conditions for Assumption 1

First, we give the necessary and sufficient condition for the joint concavity of $R_t(\cdot,\cdot)$.

Lemma 6. *Assumption 1 holds for period t, if and only if, for all $N_t\ge 0$,*

$$-2(\underline{p}-\alpha c-b)\gamma''(N_t)\ge(\gamma'(N_t))^2. \tag{59}$$

Proof: Since $\gamma(\cdot)$ is twice continuously differentiable, $R_t(\cdot,\cdot)$ is twice continuously differentiable, and jointly concave in (p_t,N_t) if and only if the Hessian of $R_t(\cdot,\cdot)$ is negative semi-definite, i.e., $\partial^2_{p_t}R_t(p_t,N_t)\le 0$, and $\partial^2_{p_t}R_t(p_t,N_t)\partial^2_{N_t}R_t(p_t,N_t)\ge(\partial_{p_t}\partial_{N_t}R_t(p_t,N_t))^2$, where $\partial^2_{p_t}R_t(p_t,N_t)=-2$, $\partial^2_{N_t}R_t(p_t,N_t)=(p_t-b-\alpha c)\gamma''(N_t)$, and $\partial_{p_t}\partial_{N_t}R_t(p_t,N_t)=\gamma'(N_t)$. Hence, $R_t(\cdot,\cdot)$ is jointly concave on $[\underline{p},\bar{p}]\times[0,+\infty)$ if and only if $-2(p_t-b-\alpha c)\gamma''(N_t)\ge(\gamma'(N_t))^2$ for all (p_t,N_t). Since $-2(p_t-b-\alpha c)\gamma''(N_t)\ge -2(\underline{p}-b-\alpha c)\gamma''(N_t)$,

$-2(p_t - b - \alpha c)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all (p_t, N_t) if and only if $-2(\underline{p} - b - \alpha c)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all $N_t \geq 0$. *Q.E.D.*

The necessary and sufficient condition for Assumption 1 characterized by inequality (59) offers little insight on what Assumption 1 means in practice. Thus, we give a simpler condition for Assumption 1.

LEMMA 7. *Let* $M := \sup\{-(\gamma'(N_t))^2/\gamma''(N_t) : N_t \geq 0\}$. $R_t(\cdot,\cdot)$ *is jointly concave if and only if* $\underline{p} \geq \alpha c + b + \frac{M}{2}$.

Proof: By Lemma 6, if Assumption 1 holds, we have $-2(\underline{p} - \alpha c - b)\gamma''(N_t) \geq (\gamma'(N_t))^2$. Since $\gamma''(N_t) \leq 0$, (59) implies that $\underline{p} - \alpha c - b \geq -\frac{1}{2}((\gamma'(N_t))^2/\gamma''(N_t))$ for any $N_t \geq 0$. Taking supreme, we have that $\underline{p} \geq \alpha c + b + \frac{M}{2}$.

If $\underline{p} \geq \alpha c + b + \frac{M}{2}$, we have $-2(\underline{p} - \alpha c - b)\gamma''(N_t) \geq -M\gamma''(N_t)$ for any N_t. Since $M = \sup\{-(\gamma'(N_t))^2/\gamma''(N_t) : N_t \geq 0\}$, $M \geq -(\gamma'(N_t))^2/\gamma''(N_t)$ for any $N_t \geq 0$. Therefore, $-M\gamma''(N_t) \geq (\gamma'(N_t))^2$ for any N_t. Putting everything together, we have $-2(\underline{p} - \alpha c - b)\gamma''(N_t) \geq -M\gamma''(N_t) \geq (\gamma'(N_t))^2$. By Lemma 6, $R_t(\cdot,\cdot)$ is jointly concave. *Q.E.D.*

D.2. Examples of Concave $R_t(\cdot,\cdot)$ Functions

We continue our discussion by giving some concrete examples of jointly concave $R_t(\cdot,\cdot)$ functions (see Assumption 1). We characterize the necessary and sufficient conditions under which $R_t(\cdot,\cdot)$ is jointly concave for some specific forms of the function $\gamma(\cdot)$. We discuss three families of $\gamma(\cdot)$: (a) exponential functions; (b) power functions; and (c) logarithm functions. We demonstrate that the necessary and sufficient conditions characterized in Lemmas 7 and 6 can be satisfied by these simple $\gamma(\cdot)$ functions under certain conditions, which are presented in model primitives and easy to verify.

First, we specify the functional form of $\gamma(\cdot)$ as $\gamma(N_t) = \gamma_0 - \gamma_0 \exp(-kN_t)$ for $N_t \geq 0$ $(\gamma_0, k > 0)$. First, we compute the first and second order derivatives of $\gamma(\cdot)$:

$$\begin{cases} \gamma'(N_t) = k\gamma_0 \exp(-kN_t), \\ \gamma''(N_t) = -k^2\gamma_0 \exp(-kN_t). \end{cases} \tag{60}$$

Note that $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \gamma_0 \exp(-kN_t) \leq \gamma_0$. Hence, the necessary condition characterized in Lemma 7 for $R_t(\cdot,\cdot)$ to be jointly concave is satisfied for this family of $\gamma(\cdot)$'s. Next we characterize the necessary and sufficient condition for $R_t(\cdot,\cdot)$ to be jointly concave for an exponential $\gamma(\cdot)$ function.

LEMMA 8. *If* $\gamma(N_t) = \gamma_0 - \gamma_0 \exp(-kN_t)$ $(\gamma_0, k > 0)$, *we have* $R_t(\cdot,\cdot)$ *is jointly concave in* (p_t, N_t) *if and only if*

$$2(\underline{p} - \alpha c - b) \geq \gamma_0. \tag{61}$$

Proof: Plug (60) into (59), and we have that $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b)k^2\gamma_0 \exp(-kN_t) \geq k^2\gamma_0^2 \exp(-2kN_t), \text{ for any } N_t \geq 0. \tag{62}$$

Direct algebraic manipulation yields that (62) is equivalent to that $2(\underline{p} - \alpha c - b)\exp(kN_t) \geq \gamma_0$ for any $N_t \geq 0$. Therefore, $R_t(\cdot,\cdot)$ is jointly concave if and only if (61) holds. *Q.E.D.*

Lemma 8 specifies the necessary and sufficient conditions characterized in Lemmas 7 and 6 in the case with an exponential $\gamma(\cdot)$ function. In short, $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if $\underline{p}$ is sufficiently large relative to γ_0, which is equivalent to that the price elasticity of demand is sufficiently high compared with the rating elasticity of demand.

Next, we specify the functional form of $\gamma(\cdot)$ as $\gamma(N_t) = \gamma_0 - \gamma_0(N_t+1)^{-k}$ for $N_t \geq 0$ $(\gamma_0, k > 0)$. First, we compute the first and second order derivatives of $\gamma(\cdot)$:

$$\begin{cases} \gamma'(N_t) = k\gamma_0(N_t+1)^{-k-1}, \\ \gamma''(N_t) = -k(k+1)\gamma_0(N_t+1)^{-k-2}. \end{cases} \tag{63}$$

Note that for $N_t \geq 0$, $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \frac{k\gamma_0}{k+1}(N_t+1)^{-k} \leq \frac{k\gamma_0}{k+1}$. Hence, the necessary condition characterized in Lemma 7 for $R_t(\cdot,\cdot)$ to be jointly concave is satisfied. Next we characterize the necessary and sufficient condition for $R_t(\cdot,\cdot)$ to be jointly concave for a power $\gamma(\cdot)$ function.

Lemma 9. *If $\gamma(N_t) = \gamma_0 - \gamma_0(N_t+1)^{-k}$ for $N_t \geq 0$ $(\gamma_0, k > 0)$, we have $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if*

$$2(\underline{p} - \alpha c - b)(k+1) \geq \gamma_0 k. \tag{64}$$

Proof: Plug (63) into (59), and we have that $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b)k(k+1)\gamma_0(N_t+1)^{-k-2} \geq k^2\gamma_0^2(N_t+1)^{-2k-2}, \text{ for any } N_t \geq 0. \tag{65}$$

Direct algebraic manipulation yields that (65) is equivalent to $2(\underline{p} - \alpha c - b)(k+1)(N_t+1)^k \geq k\gamma_0$ for all $N_t \geq 0$. Therefore, $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if (64) holds. *Q.E.D.*

Lemma 9 specifies the necessary and sufficient conditions characterized in Lemmas 7 and 6 in the case with a power $\gamma(\cdot)$ function. As in the case with exponential network externalities functions, $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if $\underline{p}$ is sufficiently large relative to γ_0, which is equivalent to that the price elasticity of demand is sufficiently high compared with the rating elasticity of demand.

Finally, we specify the functional form of $\gamma(\cdot)$ as $\gamma(N_t) = \gamma_0 \log(N_t+1)$ $(\gamma_0 > 0)$. First, we compute the first and second order derivatives of $\gamma(\cdot)$:

$$\begin{cases} \gamma'(N_t) = \frac{\gamma_0}{N_t+1}, \\ \gamma''(N_t) = -\frac{\gamma_0}{(N_t+1)^2}. \end{cases} \tag{66}$$

Note that $-\frac{(\gamma'(N_t))^2}{\gamma''(N_t)} = \gamma_0$ for all N_t. Hence, the necessary condition characterized in Lemma 7 for $R_t(\cdot,\cdot)$ to be jointly concave is satisfied for this family of $\gamma(\cdot)$'s. Next we characterize the necessary and sufficient condition for $R_t(\cdot,\cdot)$ to be jointly concave for a logarithm $\gamma(\cdot)$ function.

Lemma 10. *If $\gamma(N_t) = \gamma_0 \log(N_t+1)$ $(\gamma_0 > 0)$, we have $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if:*

$$2(\underline{p} - \alpha c - b) \geq \gamma_0. \tag{67}$$

Proof: Plug (66) into (59), and we have that $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$2(\underline{p} - \alpha c - b)\frac{\gamma_0}{(N_t+1)^2} \geq \frac{\gamma_0^2}{(N_t+1)^2}. \tag{68}$$

Direct algebraic manipulation yields that (68) is equivalent to $2(\underline{p} - \alpha c - b) \geq \gamma_0$ for all $N_t \geq 0$. Therefore, $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if (67) holds. *Q.E.D.*

Lemma 10 specifies the necessary and sufficient conditions characterized in Lemmas 7 and 6 in the case of power $\gamma(\cdot)$ functions. As in the cases with exponential and power $\gamma(\cdot)$ functions, $R_t(\cdot,\cdot)$ is jointly concave on its domain if and only if $\underline{p}$ is sufficiently large relative to γ_0, which is equivalent to that the price elasticity of demand is sufficiently high compared with the rating elasticity of demand.

Lemmas 8-10 confirm our previous insight delivered by Lemma 7 that when the price elasticity of demand (i.e., $|\frac{\mathrm{d}\mathbb{E}[D_t(p_t,N_t)]/\mathbb{E}[D_t(p_t,N_t)]}{\mathrm{d}p_t/p_t}|$) is sufficiently high relative to the *rating elasticity of demand* (i.e., $|\frac{\mathrm{d}\mathbb{E}[D_t(p_t,N_t)]/\mathbb{E}[D_t(p_t,N_t)]}{\mathrm{d}N_t/N_t}|$), $R_t(\cdot,\cdot)$ is jointly concave in (p_t,N_t) on its domain. Therefore, Assumption 1 can be satisfied for a wide variety of $\gamma(\cdot)$ functions. To conclude this section, we remark that the above method can be easily adapted to characterize the conditions under which $R_t(\cdot,\cdot)$ is jointly concave with other families of $\gamma(\cdot)$ functions.

Appendix E: Parameters for the Numerical Illustrations in Figures 1-2

This section offers the parameter specifications for the numerical illustrations in Figures 1-2, which plot the behaviors of the optimal policy throughout the planning horizon for different values of the aggregate net rating. The numerical example belongs to the family of numerical experiments examined in Section 5.3. The parameters are given in Table 2.

Table 2 Parameter Specifications for Figures 1-2

$\bar{V}_t = 30$:	Stationary market throughout the planning horizon
$T = 10$:	planning horizon length
$\gamma(N_t) = 0.5N_t$:	impact of aggregate net rating on demand
$D_t(p_t,N_t) = 30 + 0.5N_t - p_t + \xi_t$:	demand function
$\xi_t \sim N(0,2)$:	demand perturbation follows a truncated normal distribution to ensure $D_t(p_t,N_t) \geq 0$
$\alpha = 0.99$:	discount factor
$c = 8$:	inventory purchasing cost
$b = 10$:	backlogging cost
$h = 1$:	holding cost
$[\underline{p},\bar{p}] = [0,25]$:	price range
$\eta = 0.5$:	discount factor for reviews
$\theta = 0.5$:	net rating contribution ratio of demand
$\sigma = 0$:	impact of inventory availability on net rating

Appendix F: Micro-Foundation for the Paid-Review Strategy

In this section, we provide the micro-foundation for the model with the paid-review strategy. Assume that the firm sends conditional cash rewards or coupons to Z customers who recently purchased the product without leaving any review. If a customer opts to leave a review, she will receive an equivalent monetary reward of w. A customer is characterized by his willingness-to-review $\omega \geq 0$, which follows a continuous distribution with CDF $\Omega(\cdot)$ satisfying the log-concave property (i.e., $\log(\Omega(\cdot))$ is concave, see Bagnoli and Bergstrom, 2006). Hence, a customer would choose to leave a review if and only if $\omega \leq w$. We further assume that, for different customers, ω are *i.i.d.*. To ensure the effectiveness of paid-review strategy, the firm only sends cash rewards/coupons to customers who receive the product immediately. Conditioned on that a customer chooses to leave a review, i.e., $\omega \leq w$, a customer would leave a positive review with probability ς^+ and a negative review with probability ς^-, where $\varsigma := \varsigma^+ - \varsigma^- > 0$.

We are now ready to compute the function $c_n(\cdot)$. Given the monetary reward w, the total number of customers who opt to leave a review is $n_t = \varsigma Z \mathbb{P}(\omega \le w) = \varsigma Z \Omega(w)$. Hence, $w = \Omega^{-1}\left(\frac{n_t}{\varsigma Z}\right)$ and the total cost of the paid-review strategy increase n_t aggregate net rating is given by:

$$c_n(n_t) = Z\mathbb{P}(\omega \le w)w = \frac{n_t}{\varsigma}\Omega^{-1}\left(\frac{n_t}{\varsigma Z}\right) \tag{69}$$

Since the distribution of ω, $\Omega(\cdot)$, satisfies the log-concave property, we can show that $c_n(\cdot)$ is continuously differentiable and convexly increasing in n_t.

LEMMA 11. *Assume that $\Omega(\cdot)$ satisfies the log-concave property. Then $c_n(n_t)$, defined by* (69)*, is convexly increasing in n_t.*

Proof. The continuous differentiability of $c_n(\cdot)$ follows immediately from that ω follows a continuous distribution. Let $h(x) := \log \Omega(x)$. Since $h(x)$ is concave, we have $h'(x) = \frac{\Omega'(x)}{\Omega(x)}$ is decreasing in x (equivalently, $\frac{\Omega(x)}{\Omega'(x)}$ is increasing in x). Therefore,

$$c_n'(n_t) = \frac{1}{\varsigma}\Omega^{-1}\left(\frac{n_t}{\varsigma Z}\right) + \frac{1}{\varsigma} \cdot \frac{\Omega\left(\Omega^{-1}\left(\frac{n_t}{\varsigma Z}\right)\right)}{\Omega'\left(\frac{n_t}{\varsigma Z}\right)} > 0,$$

which is increasing in n_t, given that both $\Omega^{-1}(x)$ and $\frac{\Omega(x)}{\Omega'(x)}$ are increasing in x. This proves that $c_n(\cdot)$ is convexly increasing in n_t. *Q.E.D.*

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