Supplemental File

Appendix A: Table of Notations

Table 2 Notations

$\mathcal{M}$:	set of suppliers; $\mathcal{M} = \{\, i : i = 1, \cdots, m\,\}$;
$\mathcal{N}$:	set of demand classes; $\mathcal{N} = \{\, j : j = 1, \cdots, n\,\}$;
θ_t^i :	advance supply signal for supplier i in period t, $\theta_t = (\theta_t^1, \theta_t^2, \cdots, \theta_t^m)$;
$\mathcal{T}$:	planning horizon; $\mathcal{T} = \{\, t : t = 1, \ldots, T\,\}$;
K_t^i :	capacity of supplier i in period t, $K_t = (K_t^1, K_t^2, \cdots, K_t^m)$;
D_t^j :	demand from demand class j in period t, $D_t = (D_t^1, D_t^2, \cdots, D_t^n)$;
$[D_t]_1^j$:	$= \sum_{k=1}^j D_t^k$;
$[K_t]_1^i$:	$= \sum_{k=1}^i K_t^k$;
I_t :	starting inventory level in period t;
x_t^i :	order quantity from supplier i in period t, $x_t = (x_t^1, x_t^2, \cdots, x_t^m)$;
$\|x_t\|$:	total procurement quantity in period t, i.e., $\|x_t\| = \sum_{i=1}^m x_t^i$;
y_t^j :	selling quantity to demand class j in period t, $y_t = (y_t^1, \ldots, y_t^n)$;
$\|y_t\|$:	total selling quantity in period t, i.e., $\|y_t\| = \sum_{j=1}^n y_t^j$;
J_t :	post-delivery inventory level in period t, $J_t = I_t + \|x_t\|$;
c_t^i :	unit purchasing cost of supplier i in period t, $c_t = (c_t^1, c_t^2, \cdots, c_t^m)$;
$\tilde{r}_t^j$:	unit marginal revenue of demand class j in period t, $\tilde{r}_t = (\tilde{r}_t^1, \tilde{r}_t^2, \cdots, \tilde{r}_t^n)$;
b_t^j :	unit rejection cost of demand class j in period t, $b_t = (b_t^1, b_t^2, \cdots, b_t^n)$;
r_t^j :	unit effective marginal revenue of demand class j in period t, $r_t^j = \tilde{r}_t^j + b_t^j$, $r_t = (r_t^1, r_t^2, \cdots, r_t^n)$;
$h_t(\cdot)$:	inventory (holding and shortage) cost in period t;
$V_t(I_t, K_t, \theta_t)$:	maximal total profits in periods $\{t, t-1, \cdots, 1\}$, given state (I_t, K_t, θ_t) in period t;
$H_t(I_t, x_t, \theta_t)$:	maximal total profits in periods $\{t, t-1, \cdots, 1\}$, given state (I_t, θ_t) and procurement decision x_t;
$W_t(J_t, D_t, \theta_t)$:	maximal total profits in periods $\{t, t-1, \cdots, 1\}$, given state θ_t, post-delivery inventory level J_t, and demand D_t;
$G_t(J_t, y_t, \theta_t)$:	maximal total profits in periods $\{t, t-1, \cdots, 1\}$, given state θ_t, post-delivery inventory level J_t, and selling decision y_t;
$x_t^{i*}(I_t, K_t, \theta_t)$:	optimal order quantity from supplier i in period t, given (I_t, K_t, θ_t);
$y_t^{j*}(J_t, D_t, \theta_t)$:	optimal selling quantity to demand class j in period t, given (J_t, D_t, θ_t);
$\alpha_t^i(\theta_t)$:	optimal base-stock level for supplier i in period t, given state θ_t;
$\beta_t^j(\theta_t)$:	optimal demand rationing level for demand class j in period t, given state θ_t;
$\leq_{s.d.}$ $(\geq_{s.d.})$:	first-order stochastic dominance;
$\leq_{cx}$ $(\geq_{cx})$:	convex order;
$\mathbf{1}_{\mathcal{A}}$:	indicator function of event $\mathcal{A}$;
a^+ :	$= \max\{a, 0\}$;
a^- :	$= \max\{-a, 0\}$.

Appendix B: Concavity and Supermodularity

The following lemma summarizes the properties of concave functions and supermodular functions necessary for establishing our structural results. Its proof can be found in Boyd and Vandenberghe (2004), Topkis (1998), and Simchi-Levi et al. (2005).

LEMMA 2. *(i) Define $h \circ g(x) = h(g_1(x), \ldots, g_m(x))$, with $h\colon \mathbb{R}^m \to \mathbb{R}$, $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$. Then $h \circ g(x)$ is concave if h is concave and nondecreasing in each argument, and g_i is concave for each i.*

(ii) If $h\colon \mathbb{R}^m \to \mathbb{R}$ is a concave function, then $h(Ax + b)$ is also a concave function of x, where $A \in \mathbb{R}^m \times \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

(iii) Assume that for any $x \in \mathbb{R}^n$, there is an associated convex set $C(x) \subset \mathbb{R}^m$ and $\{\,(x, y) : y \in C(x), x \in \mathbb{R}^n\,\}$ is a convex set. If $h(x, y)$ is concave and the function $g(x) = \sup_{y \in C(x)} h(x, y)$ is well defined, then $g(x)$ is concave over $\mathbb{R}^n$.

(iv) If $f(x)$ and $g(x)$ are concave [supermodular] on X and $\alpha,\beta>0$, then $\alpha f(x)+\beta g(x)$ is concave [supermodular] on X.

(v) Assume that $f(x,y)$ is concave [supermodular] in x on a convex set [lattice] X for each $y\in Y$. Let Z be a random variable on Y and, for each $x\in X$, $f(x,Z)$ is integrable. Then $g(x)=\mathbb{E}_Z[f(x,Z)]$ is concave [supermodular] in x on X.

(vi) If X and Y are lattices, S is a sublattice of $X\times Y$, S_y is the section of S at y in Y, and $f(x,y)$ is supermodular in (x,y) on S, then $\arg\max_{x\in S_y} f(x,y)$ is increasing in y on $\{\, y\in Y : \arg\max_{x\in S_y} f(x,y)\neq\emptyset \,\}$.

(vii) Suppose that Y is a convex subset of $\mathbb{R}^1$, X is a sublattice of $\mathbb{R}^n$, $a_i>0$ for $i=1,\ldots,n$, $\sum_{i=1}^n a_i x_i\in Y$ for $x\in X$. If $g(y)$ is concave in y on Y, then $f(x):=g(\sum_{i=1}^n a_i x_i)$ is submodular in x on X.

(viii) Suppose that Y is a convex subset of $\mathbb{R}^1$, X is a sublattice of $\mathbb{R}^2$, $a_1>0$ and $a_2<0$, $\sum_{i=1}^2 a_i x_i\in Y$ for $x\in X$. If $g(y)$ is concave in y on Y, then $f(x):=g(\sum_{i=1}^2 a_i x_i)$ is supermodular in x on X.

The following lemma on the preservation of supermodularity/submodularity is central to the proof of our analytical results.

LEMMA 3. *(i) If $V(I,K):\mathbb{R}^1\times\mathbb{R}^m\to\mathbb{R}$ is supermodular [submodular] in (I,K^i) for $i=1,2,\cdots,m$, and $K^i(\theta)$ is increasing in $\theta\in\mathbb{R}$ for $i=1,2,\cdots,m$, then $V(I,K(\theta))$ is supermodular [submodular] in (I,θ), where $K(\theta):=(K^1(\theta),K^2(\theta),\cdots,K^m(\theta))$.*

(ii) If, for $i=1,2,\cdots,m$, $V(I,K):\mathbb{R}^1\times\mathbb{R}^m\to\mathbb{R}$ is supermodular [submodular] in (I,K^i) and $K^i(\theta_1)\geq_{s.d.} K^i(\theta_2)$ for all $\theta_1\geq\theta_2$ ($\theta_1,\theta_2\in\mathbb{R}$), then $\mathbb{E}[V(I,K(\theta))]$ is supermodular [submodular] in (I,θ), where $K(\theta):=(K^1(\theta),K^2(\theta),\cdots,K^m(\theta))$.

(iii) (Corollary 1 in Chen et al. 2013) Assume that $g(y,\theta)$ is a supermodular function in (y,θ) on a sublattice $\mathbf{D}\subset\mathbb{R}^{n+1}$ and jointly concave in y for any θ. For every θ, assume that the section $\mathbf{D}_\theta$ is convex. Let $f(I,\theta):=\max_y\{g(y,\theta):\sum_{i=1}^n a_i y^i+b\theta=I,(y^1,y^2,\cdots,y^n,\theta)\in\mathbf{D}\}$ and $\mathbf{S}:=\{(\sum_{i=1}^n a_i y^i+b\theta,\theta):(y,\theta)\in\mathbf{D}\}$, where $a_1,a_2,\cdots,a_n,b\geq 0$. We have: $f(I,\theta)$ is supermodular on $\mathbf{S}$ and concave in I for any θ.

(iv) If $g(I,\theta)$ is supermodular [submodular] in (I,θ) and concave in I, then

$$f(I,\theta):=\max_{a^i\leq y^i\leq b^i,1\leq i\leq n} g(I-\textstyle\sum_{i=1}^n y^i,\theta)+c\cdot y$$

is also supermodular [submodular] in (I,θ) and, for any θ, concave in I, where $c=(c^1,c^2,\cdots,c^n)$ is a constant vector.

(v) Suppose $g_1(I)$ and $g_2(I)$ are continuously differentiable and concave in I, with $g_2'(I)\geq g_1'(I)$ for any I. Let

$$f_j(I):=\max_{a^i\leq y^i\leq b^i,1\leq i\leq n} g_j(I-\textstyle\sum_{i=1}^n y^i)+c_j\cdot y,\ \text{for } j=1,2,$$

where $c_j=(c_j^1,c_j^2,\cdots,c_j^n)$ is a constant vector, with $c_2^i\geq c_1^i$ for any $1\leq i\leq n$. We have $f_2'(I)\geq f_1'(I)$ for any I.

Appendix C: Proofs

Proof of Lemma 3

Part (i). We only show the supermodularity part, whereas the submodularity part follows from the same argument. Assume that $V(I,K)$ is supermodular in (I,K^i) for $i=1,2,\cdots,m$. For $I_1<I_2$ and $\theta_1<\theta_2$, we have

$$\begin{aligned}V(I_2,K(\theta_1))-V(I_1,K(\theta_1)) &= V(I_2,K^1(\theta_1),K^2(\theta_1),\ldots,K^m(\theta_1))-V(I_1,K^1(\theta_1),K^2(\theta_1),\cdots,K^m(\theta_1))\\ &\leq V(I_2,K^1(\theta_2),K^2(\theta_1),\cdots,K^m(\theta_1))-V(I_1,K^1(\theta_2),K^2(\theta_1),\cdots,K^m(\theta_1))\end{aligned}$$

$$\begin{aligned}
&\cdots \\
&\le V(I_2, K^1(\theta_2), K^2(\theta_2), \ldots, K^m(\theta_2)) - V(I_1, K^1(\theta_2), K^2(\theta_2), \ldots, K^m(\theta_2)) \\
&= V(I_2, K(\theta_2)) - V(I_1, K(\theta_2)),
\end{aligned}$$

where, for any i, the ith inequality holds since V is supermodular in (I, K^i) for given $K^{-i}(\theta_1) := (K^1(\theta_1), K^2(\theta_1), \cdots, K^{i-1}(\theta_1), K^{i+1}(\theta_1), \cdots, K^m(\theta_1))$, and $K^i(\theta)$ is increasing in θ. This completes the proof of **part (i)**.

Part (ii). We only show the supermodularity part, whereas the submodularity part follows from the same argument. Assume that $V(I, K)$ is supermodular in (I, K^i) for $i = 1, 2, \cdots, m$. For $I_1 < I_2$ and $\theta_1 < \theta_2$, since, for each $i = 1, 2, \cdots, m$, $K^i(\theta)$ is stochastically increasing in θ, there exist two random vectors $\hat{K}(\theta_1)$ and $\hat{K}(\theta_2)$ defined on the same probability space, such that $K(\theta_1) =_d \hat{K}(\theta_1)$ and $K(\theta_2) =_d \hat{K}(\theta_2)$, and $\hat{K}(\theta_1) \le \hat{K}(\theta_2)$ with probability 1. Therefore, we have:

$$\begin{aligned}
\mathbb{E}V(I_2, K(\theta_1)) - \mathbb{E}V(I_1, K(\theta_1)) &= \mathbb{E}V(I_2, \hat{K}(\theta_1)) - \mathbb{E}V(I_1, \hat{K}(\theta_1)) \\
&\le \mathbb{E}V(I_2, \hat{K}(\theta_2)) - \mathbb{E}V(I_1, \hat{K}(\theta_2)) \\
&= \mathbb{E}V(I_2, K(\theta_2)) - \mathbb{E}V(I_1, K(\theta_2)),
\end{aligned}$$

where the equalities follow from the construction and the inequality follows from the supermodularity of $V(I, K(\theta))$ in (I, θ) and $\hat{K}(\theta_1) \le \hat{K}(\theta_2)$ with probability 1.

Part (iii). See Chen et al. (2013).

Part (iv). To show the concavity and supermodularity/submodularity of $f(\cdot, \cdot)$, we invoke **part (iii)**. If $g(I, \theta)$ is supermodular in (I, θ) and, for any θ, concave in I, by **part (iii)**, $f(I, \theta) = \max_y \{g(y^0, \theta) + c \cdot y : \sum_{i=0}^n y^i = I, y^i \in [a^i, b^i]$ for $1 \le i \le n\}$ is also supermodular in (I, θ) and, for any θ, concave in I.

If $g(I, \theta)$ is submodular in (I, θ) and, for any θ, concave in I, $g_0(I, \theta) := g(-I, \theta)$ is supermodular in (I, θ) and, for any θ, concave in I. Let $f_0(I, \theta) := f(-I, \theta)$, so

$$\begin{aligned}
f_0(I, \theta) &= \max_{a^i \le y^i \le b^i, 1 \le i \le n} \{g(-I - \textstyle\sum_{i=1}^n y^i, \theta) + c \cdot y\} \\
&= \max_{a^i \le y^i \le b^i, 1 \le i \le n} \{g_0(I + \textstyle\sum_{i=1}^n y^i, \theta) + c \cdot y\} \\
&= \max_{-b^i \le y^i \le -a^i, 1 \le i \le n} \{g_0(I - \textstyle\sum_{i=1}^n y^i, \theta) - c \cdot y\}
\end{aligned}$$

is supermodular in (I, θ) and, for any θ, concave in I. Therefore, $f(I, \theta) = f_0(-I, \theta)$ is submodular in (I, θ) and, for any θ, concave in I.

Part (v). Let $g(I, \theta) := g_\theta(I)$ and $f(I, \theta) := f_\theta(I)$. $g_2'(I) \ge g_1'(I)$ implies that $g(I, \theta)$ is supermodular in (I, θ) and, for any θ, concave in I. By the envelope theorem, $f(I, \theta)$ is continuously differentiable in I for any θ. Therefore, by **part (iii)**,

$$f(I, \theta) = \max_y \{g(y^0, \theta) + c_\theta \cdot y : \textstyle\sum_{i=0}^n y^i = I, y^i \in [a^i, b^i] \text{ for } 1 \le i \le n\} \text{ is supermodular in } (I, \theta).$$

Hence, $f_2'(I) = \partial_I f(I, 2) \ge \partial_I f(I, 1) = f_1'(I)$ for any I. ■

Proof of Theorem 1

We show the concavity and differentiability of the functions $H_t(\cdot,\cdot,\theta_t)$, $G_t(\cdot,\cdot,\theta_t)$, $V_t(\cdot,\cdot,\theta_t)$ and $W_t(\cdot,\cdot,\theta_t)$ for any given θ_t together by backward induction. For $t=0$, $V_t(\cdot,\cdot,\cdot)=0$, so the initial condition holds.

Suppose that the joint concavity holds for $t-1$. We will show that it also holds for t. Fix an advance supply signal θ_t. First consider $G_t(J_t,y_t,\theta_t)$: the first term $r_t\cdot y_t$ is a linear function of y_t and hence jointly concave in (J_t,y_t) by part (iv) of Lemma 2; the second term $-h_t(J_t-|y_t|)$ is the composition of a concave function $-h_t(\cdot)$ and an affine function $(J_t-|y_t|)$ of (J_t,y_t), thus jointly concave in (J_t,y_t) by part (ii) of Lemma 2; the concavity of the third term, for given θ_t, $\gamma\mathbb{E}_{K_{t-1},\theta_{t-1}}[V_{t-1}(J_t-|y_t|,K_{t-1}(\theta_t),\theta_{t-1})|\theta_t]$, follows from the induction hypothesis, part (ii) and part (v) of Lemma 2. Since summation preserves concavity (Lemma 2(iv)), we conclude that $G_t(J_t,y_t,\theta_t)$ is jointly concave in (J_t,y_t) for any θ_t. Since concavity is preserved under maximization (Lemma 2(iii)), $\max_{0\le y_t\le D_t}G_t(J_t,y_t,\theta_t)$ is jointly concave in (J_t,D_t) for each θ_t, and so is $W_t(J_t,D_t,\theta_t)$. By a similar argument, the concavity of $H_t(\cdot,\cdot,\theta_t)$ and $V_t(\cdot,\cdot,\theta_t)$ follows analogously. This completes the induction step for the proof of concavity.

Next, we show the differentiability. Suppose that the differentiability holds for period $t-1$. We will show that it also holds for period t. For fixed θ_t, the differentiability of $G_t(\cdot,\cdot,\theta_t)$ follows from the induction hypothesis and the differentiability of $h_t(\cdot)$, while that of $W(\cdot,\cdot,\theta_t)$ follows from the envelope theorem. Analogously, the differentiability of $H_t(\cdot,\cdot,\theta_t)$ follows from that of $W_t(\cdot,\cdot,\theta_t)$, whereas that of $V_t(\cdot,\cdot,\theta_t)$ follows from the envelope theorem. This completes the induction step for proof of differentiability.

Now we show that $V_t(I_t,K_t,\theta_t)$ is increasing in K_t. This is readily verified because, for any $K_t\le K_t'$, any feasible procurement decision x_t under realized capacity vector K_t must also be feasible under K_t'. Hence,

$$V_t(I_t,K_t,\theta_t)=\max_{0\le x_t\le K_t}H_t(I_t,x_t,\theta_t)\le\max_{0\le x_t\le K_t'}H_t(I_t,x_t,\theta_t)\le V_t(I_t,K_t',\theta_t).$$

■

Proof of Theorem 2

For **part (i)**, the inequalities follow directly from the concavity and differentiability of $W_t(\cdot,\theta_t)$ and $c_t^1<c_t^2<\cdots<c_t^m$.

For **parts (ii) and (iii)**, we first show that i_t is well-defined. If $I_t+|K_t|\le\alpha_t^m(\theta_t)$, $i_t=m$. Otherwise, $\{I_t+[K_t]_1^i\}_{i\in\mathcal{M}}$ is increasing in i, and $\{\alpha_t^i(\theta_t)\}_{i\in\mathcal{M}}$ is decreasing in i. Thus, $i_t=\min\{i\in\mathcal{M}:I_t+[K_t]_1^i\ge\alpha_t^i(\theta_t)\}$ exists and is unique. Let $J_t^*=I_t+|x_t^*(I_t,K_t,\theta_t)|$ denote the optimal post-delivery inventory level. We now show that, if $x_t^{i*}(I_t,K_t,\theta_t)>0$, $x_t^{j*}(I_t,K_t,\theta_t)=K_t^j$ for all $j<i$. Since $x_t^{i*}(I_t,K_t,\theta_t)>0$,

$$\partial_{J_t}W_t(J_t^*,\theta_t)-c_t^j>\partial_{J_t}W_t(J_t^*,\theta_t)-c_t^i\ge0,\ \text{for all } j<i.$$

Hence, $x_t^{j*}(I_t,K_t,\theta_t)=K_t^j$ for all $j<i$. In particular, if $x_t^{i_t*}(I_t,K_t,\theta_t)>0$, $x_t^{i*}(I_t,K_t,\theta_t)=K_t^i$ for all $i<i_t$. Otherwise, $x_t^{i_t*}(I_t,K_t,\theta_t)=0$, $J_t^*\le I_t+[K_t]_1^{i_t-1}$. In this case, we have $\partial_{J_t}W_t(J_t^*,\theta_t)-c_t^i\ge0$, for all $i\le i_t-1$. Hence, $x_t^{i*}(I_t,K_t,\theta_t)=K_t^i$ for all $i\le i_t-1$. If $i>i_t$, $\partial_{J_t}W_t(J_t^*,\theta_t)-c_t^i\le\partial_{J_t}W_t(I_t+[K_t]_1^{i_t},\theta_t)-c_t^{i_t+1}<0$, by the definition of i_t. Hence, $x_t^{i*}(I_t,K_t,\theta_t)=0$ for $i>i_t$. If $i=i_t$, it is optimal to order, from supplier i, up to $\alpha_t^i(\theta_t)$, but constrained by its capacity K_t^i. Since $x_t^{i*}(I_t,K_t,\theta_t)=K_t^i$ for all $i\le i_t-1$, $x_t^{i_t*}(I_t,K_t,\theta_t)=\min\{\alpha_t^i(\theta_t)-I_t-[K_t]_1^{i_t-1},K_t^i\}$. ■

Proof of Theorem 3

The proof is analogous to that of Theorem 2 and hence omitted for brevity. ■

Proof of Theorem 4

For **part (i)**, since (11) implies (10), we only show (11). We first observe that if $i < i_t$, $\alpha_t^i(\theta_t) - I_t - [K_t]_t^{i-1} > K_t^i$; if $i > i_t$, $\alpha_t^i(\theta_t) - I_t - [K_t]_t^{i-1} < 0$. Therefore, $x_t^{i*}(I_t, K_t, \theta_t) = \min\{(\alpha_t^i(\theta_t) - I_t - [K_t]_t^{i-1})^+, K_t^i\}$ for all $i \in \mathcal{M}$. It's clear that $x_t^{i*}(I_t, K_t, \theta_t)$ is decreasing in I_t for all $i \in \mathcal{M}$. Hence, $|x_t^*(I_t + \delta, K_t, \theta_t)| \le |x_t^*(I_t, K_t, \theta_t)|$.

To prove the other inequality in (11), we first show that $x_t^{i*}(I_t + \delta, K_t, \theta_t) \ge x_t^{i*}(I_t, K_t, \theta_t) - \delta$ for all $i \in \mathcal{M}$ and any $\delta > 0$. For any $\delta > 0$,

$$\begin{aligned} x_t^{i*}(I_t+\delta, K_t, \theta_t) &= \min\{(\alpha_t^i(\theta_t) - I_t - \delta - [K_t]_t^{i-1})^+, K_t^i\} \\ &\ge \min\{(\alpha_t^i(\theta_t) - I_t - [K_t]_t^{i-1})^+ - \delta, K_t^i\} \\ &\ge \min\{(\alpha_t^i(\theta_t) - I_t - [K_t]_t^{i-1})^+, K_t^i\} - \delta \\ &= x_t^{i*}(I_t, K_t, \theta_t) - \delta, \end{aligned} \tag{25}$$

where the first inequality follows from $(a+\delta)^+ - a^+ \le \delta$ for any $\delta > 0$, and the second from $\min\{a+\delta, k\} - \min\{a, k\} \le \delta$ for any $\delta > 0$. Note that, for $\delta > 0$ small enough, there is at most one i, such that $x_t^{i*}(I_t + \delta, K_t, \theta_t) \ne x_t^{i*}(I_t, K_t, \theta_t)$. Therefore, for any $\delta > 0$, there exists a partition $I_t = I_t + \delta_0 < I_t + \delta_1 < I_t + \delta_2 < \cdots < I_t + \delta_k = I_t + \delta$ of $[I_t, I_t + \delta]$, such that for any $\delta', \delta'' \in [\delta_l, \delta_{l+1}]$, $x_t^{i_l*}(I_t + \delta'', K_t, \theta_t) \ne x_t^{i_l*}(I_t + \delta', K_t, \theta_t)$ and $x_t^{i*}(I_t + \delta'', K_t, \theta_t) = x_t^{i*}(I_t + \delta', K_t, \theta_t)$ for $i \ne i_l$. Therefore,

$$\begin{aligned} |x_t^*(I_t+\delta, K_t, \theta_t)| &= |x_t^*(I_t, K_t, \theta_t)| - \textstyle\sum_{l=0}^{k-1}(|x_t^*(I_t+\delta_l, K_t, \theta_t)| - |x_t^*(I_t+\delta_{l+1}, K_t, \theta_t)|) \\ &\ge |x_t^*(I_t, K_t, \theta_t)| - \textstyle\sum_{l=0}^{k-1}(\delta_{l+1} - \delta_l) \\ &= |x_t^*(I_t, K_t, \theta_t)| - \delta, \end{aligned}$$

where the inequality follows from (25), i.e., (11) follows.

Part (ii) follows from the same argument as **part (i)** except that $y_t^{j*}(J_t, D_t, \theta_t) = \min\{(J_t - [D_t]_1^{j-1} - \beta_t^j(\theta_t))^+, D_t^j\}$, so we omit its proof. ■

Before giving the proofs of Lemma 1 and Theorem 5, we present the proofs of Theorem 7 and Theorem 9 first.

Proof of Theorem 7

Part (i). We prove this part by backward induction. Since $\tilde{V}_0(\cdot,\cdot,\cdot) = V_0(\cdot,\cdot,\cdot) = 0$, the initial condition is satisfied. It suffices to show that if $\tilde{V}_s(I_s, K_s, \theta_s) \le V_s(I_s, K_s, \theta_s)$ for $s = t-1$ and $K_{t-1}^i(\theta_t) \le_{cx} \tilde{K}_{t-1}^i(\theta_t)$ for all $i \in \mathcal{M}$, $\tilde{V}_t(I_t, K_t, \theta_t) \le V_t(I_t, K_t, \theta_t)$. Since $\tilde{V}_s(I_s, K_s, \theta_s) \le V_s(I_s, K_s, \theta_s)$ for $s = t-1$ and $\tilde{V}_s(I_s, K_s, \theta_s)$ and $V_s(I_s, K_s, \theta_s)$ are concave in K_s,

$$\tilde{V}_s(I_s|\theta_t) = \mathbb{E}_{\tilde{K}_s, \theta_s}[\tilde{V}_s(I_s, K_s, \theta_s)|\theta_t] \le \mathbb{E}_{K_s, \theta_s}[\tilde{V}_s(I_s, K_s, \theta_s)|\theta_t] \le \mathbb{E}_{K_s, \theta_s}[V_s(I_s, K_s, \theta_s)|\theta_t] = V_s(I_s|\theta_t), \text{ for } s = t-1.$$

Since monotonicity is preserved under maximization and expectation, $\tilde{G}_t(J_t, y_t, \theta_t) \le G_t(J_t, y_t, \theta_t)$, $\tilde{W}_t(J_t, D_t, \theta_t) \le W_t(J_t, D_t, \theta_t)$, $\tilde{H}_t(I_t, x_t, \theta_t) \le H_t(I_t, x_t, \theta_t)$ and $\tilde{V}_t(I_t, K_t, \theta_t) \le V_t(I_t, K_t, \theta_t)$. This completes the proof of **part (i)**.

Part (ii). If $I_t + |K_t| \le \alpha_t^m(\theta_t)$, the total optimal post-delivery inventory level $J_t^*(I_t, K_t, \theta_t) := I_t + |x_t^*(I_t, K_t, \theta_t)| = I_t + |K_t|$ is increasing in K_t^i for any $i \in \mathcal{M}$. If $I_t + |K_t| > \alpha_t^m(\theta_t)$, $J_t^*(I_t, K_t, \theta_t)$ equals to $\alpha_t^{i_t}(\theta_t)$. Moreover, since $\{I_t +$

$[K_t]_1^i\}$ is increasing in K_t^j for any j, i_t is decreasing in K_t^i for any i by the definition of i_t. Because $\alpha_t^i(\theta_t)$ is decreasing in i, $\alpha_t^{i_t}(\theta_t)$ is increasing in K_t^i for any i. Therefore, $J_t^*(I_t,K_t,\theta_t)$ is increasing in K_t^i for any $i\in\mathcal{M}$. By the envelope theorem, $\partial_{I_t}V_t(I_t,K_t,\theta_t)=\partial_{J_t}W_t(J_t^*(I_t,K_t,\theta_t),\theta_t)$, which is decreasing in $J_t^*(I_t,K_t,\theta_t)$ by the concavity of $W_t(\cdot,\theta_t)$. Thus, $\partial_{I_t}V_t(I_t,K_t,\theta_t)$ is decreasing in K_t^i for any $i\in\mathcal{M}$, i.e., $V_t(I_t,K_t,\theta_t)$ is submodular in (I_t,K_t^i) for any $i\in\mathcal{M}$.

We now show that $\partial_{J_t}\tilde{W}_t(J_t,\theta_t)\le\partial_{J_t}W_t(J_t,\theta_t)$, $\partial_{I_{t-1}}\tilde{V}_{t-1}(I_{t-1}|\theta_t)\le\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_t)$, and $\partial_{I_t}\tilde{V}_t(I_t,K_t,\theta_t)\le\partial_{I_t}V_t(I_t,K_t,\theta_t)$ by backward induction. We will show that if $\partial_{I_s}\tilde{V}_s(I_s,K_s,\theta_s)\le\partial_{I_s}V_s(I_s,K_s,\theta_s)$ for $s=t-1$ and $K_{t-1}^i(\theta_t)\le_{s.d.}\tilde{K}_{t-1}^i(\theta_t)$ for each $i\in\mathcal{M}$, $\partial_{I_s}\tilde{V}_s(I_s|\theta_t)\le\partial_{I_s}V_s(I_s|\theta_t)$, $\partial_{J_t}\tilde{W}_t(J_t,\theta_t)\le\partial_{J_t}W_t(J_t,\theta_t)$ and $\partial_{I_t}\tilde{V}_t(I_t,K_t,\theta_t)\le\partial_{I_t}V_t(I_t,K_t,\theta_t)$. Since $\tilde{V}_0(\cdot,\cdot,\cdot)=V_0(\cdot,\cdot,\cdot)=0$, the initial condition is satisfied. Since $\partial_{I_s}\tilde{V}_s(I_s,K_s,\theta_s)\le\partial_{I_s}V_s(I_s,K_s,\theta_s)$, $\tilde{V}_s(I_s,K_s,\theta_s)$ and $V_s(I_s,K_s,\theta_s)$ are submodular in (I_s,K_s^i) for each $i\in\mathcal{M}$, the proof of Lemma 3(ii) yields that:

$$\partial_{I_s}\tilde{V}_s(I_s|\theta_t)=\mathbb{E}_{\tilde{K}_s,\theta_s}[\partial_{I_s}\tilde{V}_s(I_s,K_s,\theta_s)|\theta_t]\le\mathbb{E}_{K_s,\theta_s}[\partial_{I_s}\tilde{V}_s(I_s,K_s,\theta_s)|\theta_t]\le\mathbb{E}_{K_s,\theta_s}[\partial_{I_s}V_s(I_s,K_s,\theta_s)|\theta_t]=\partial_{I_s}V_s(I_s|\theta_t).$$

Moreover, by the concavity of $\tilde{V}_s(\cdot|\theta_t)$ and $V_s(\cdot|\theta_t)$ and Lemma 3(v), $\partial_{I_s}\tilde{V}_s(I_s|\theta_t)\le\partial_{I_s}V_s(I_s|\theta_t)$ implies that $\partial_{J_t}\tilde{W}_t(J_t,\theta_t)\le\partial_{J_t}W_t(J_t,\theta_t)$ and, by the concavity of $\tilde{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$ and Lemma 3(v), $\partial_{I_t}\tilde{V}_t(I_t,K_t,\theta_t)\le\partial_{I_t}V_t(I_t,K_t,\theta_t)$. This establishes **part (ii)**.

Part (iii). By (6), the inequality $\tilde{\alpha}_t^i(\theta_t)\le\alpha_t^i(\theta_t)$ follows from $\partial_{J_t}\tilde{W}_t(J_t,\theta_t)\le\partial_{J_t}W_t(J_t,\theta_t)$ and the concavity of $\tilde{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$. By the proof of Theorem 4,

$$\tilde{x}_t^{i*}(I_t,K_t,\theta_t)=\min\{(\tilde{\alpha}_t^i(\theta_t)-[K]_t^{i-1}-I_t)^+,K_t^i\}\le\min\{(\alpha_t^i(\theta_t)-[K]_t^{i-1}-I_t)^+,K_t^i\}=x_t^{i*}(I_t,K_t,\theta_t),\ \text{for any } i\in\mathcal{M}.$$

Analogously, by (8), the inequality $\tilde{\beta}_t^i(\theta_t)\le\beta_t^i(\theta_t)$ follows from $\partial_{I_{t-1}}\tilde{V}_{t-1}(I_{t-1}|\theta_t)\le\partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_t)$ and the concavity of $-h_t(\cdot)$, $\tilde{V}_{t-1}(\cdot|\theta_t)$, and $V_{t-1}(\cdot|\theta_t)$. By the proof of Theorem 4,

$$\tilde{y}_t^{j*}(J_t,D_t,\theta_t)=\min\{(J_t-[D]_t^{j-1}-\tilde{\beta}_t^j(\theta_t))^+,D_t^j\}\ge\min\{(J_t-[D]_t^{j-1}-\beta_t^j(\theta_t))^+,D_t^j\}=y_t^{j*}(J_t,D_t,\theta_t),\ \text{for any } j\in\mathcal{N}.$$

This establishes **part (iii)**. ■

Theorem 9 and its Proof

We now characterize the optimal policy in the model with stationary forecast (see Section 5.1 for details). Let $\hat{V}_{t-1}(I_{t-1}):=\mathbb{E}_{\mathcal{K}_{t-1}}[\hat{V}_{t-1}(I_{t-1},\mathcal{K}_{t-1})]$ and

$$\hat{W}_t(J_t):=\mathbb{E}_{D_t}\{\max_{0\le y_t\le D_t}[\sum_{j=1}^n r_t^j y_t^i-h_t(J_t-|y_t|)+\gamma\mathbb{E}_{\mathcal{K}_{t-1}}(\hat{V}_{t-1}(J_t-|y_t|,\mathcal{K}_{t-1}))]-\sum_{j=1}^n b_t^j D_t^j\}.$$

THEOREM 9. *(i) $\hat{V}_t(I_t,K_t)$ is concave and continuously differentiable in I_t for any K_t, and is submodular in (I_t,K_t^i), $\forall i\in\mathcal{M}$. (ii) Let*

$$\hat{\alpha}_t^i:=\min\{J_t\in\mathbb{R}:c_t^i\ge\partial_{J_t}\hat{W}_t(J_t)\},\ i\in\mathcal{M},$$

where $\hat{\alpha}_t^i:=-\infty$ if $\{J_t\in\mathbb{R}:c_t^i<\partial_{J_t}\hat{W}_t(J_t)\}=\emptyset$. $\{\hat{\alpha}_t^i\}_{i\in\mathcal{M}}$ are independent of the starting inventory level I_t and capacity K_t, and decreasing in $i\in\mathcal{M}$. The optimal procurement decision is

$$\hat{x}_t^{i*}(I_t,K_t)=K_t^i\cdot\mathbf{1}_{\{i<\hat{i}_t\}}+\min\{\hat{\alpha}_t^i-I_t-[K]_1^{i-1},K_t^i\}\cdot\mathbf{1}_{\{i=\hat{i}_t\}},\quad i\in\mathcal{M},\tag{26}$$

where $\hat{i}_t := m \cdot \mathbf{1}_{\{I_t+|K_t|\le \hat{\alpha}_t^m\}} + \min\{i : I_t + [K_t]_1^i \ge \hat{\alpha}_t^i\} \cdot \mathbf{1}_{\{I_t+|K_t|>\hat{\alpha}_t^m\}}$.

(iii) Let

$$\hat{\beta}_t^j := \min\{I_{t-1} \in \mathbb{R} : r_t^j \ge -\partial_{I_{t-1}} h_t(I_{t-1}) + \gamma \partial_{I_{t-1}} \hat{V}_{t-1}(I_{t-1})\},\ j \in \mathcal{N},$$

where $\hat{\beta}_t^j := -\infty$ *if* $\{I_{t-1} \in \mathbb{R} : r_t^j < -\partial_{I_{t-1}} h_t(I_{t-1}) + \gamma \partial_{I_{t-1}} \hat{V}_{t-1}(I_{t-1})\} = \emptyset$. $\{\hat{\beta}_t^j\}_{j\in\mathcal{N}}$ *are independent of the post-delivery inventory level* J_t *and realized demand* D_t*, and increasing in* $j \in \mathcal{N}$*. The optimal selling decision is*

$$\hat{y}_t^{j*}(J_t, D_t) = D_t^j \cdot \mathbf{1}_{\{j<\hat{j}_t\}} + \min\{J_t - [D_t]_1^{j-1} - \hat{\beta}_t^j, D_t^j\} \cdot \mathbf{1}_{\{j=\hat{j}_t\}}, \quad j \in \mathcal{N}, \tag{27}$$

where $\hat{j}_t := n \cdot \mathbf{1}_{\{J_t-|D_t|\ge \hat{\beta}_t^n\}} + \min\{j : J_t - [D_t]_1^j \le \hat{\beta}_t^j\} \cdot \mathbf{1}_{\{J_t-|D_t|<\hat{\beta}_t^n\}}$.

Part (i) follows from the same argument as the proof of Theorem 1 and Theorem 7(ii). **Part (ii)** [**part (iii)**] follows from the same argument as the proof of Theorem 2 [Theorem 3]. Hence, we omit the proof of **Theorem 9**. ■

Proof of Lemma 1

Let Ξ^i be the set of all advance supply signals for supplier i. Without loss of generality, we assume that Ξ^i is compact for each i. We take $\bar{\theta}_t^i = \max\{\Xi^i\}$ and $\underline{\theta}_t^i = \min\{\Xi^i\}$. Let $\{\hat{\theta}_t\}_{t\in\mathcal{T}}$ be $i.i.d.$ random vectors with the stationary distribution of $\{\theta_t\}_{t\in\mathcal{T}}$. Since $\Theta_t^i(\theta_t^i)$ are stochastically increasing in θ_t^i for any $i \in \mathcal{M}$, conditioned on $\theta_{t^*} = \bar{\theta}_{t^*}$, $\theta_t^i \ge_{s.d.} \hat{\theta}_t^i$ for $t \le t^*$ and $i \in \mathcal{M}$. Analogously, conditioned on $\theta_{t^*} = \underline{\theta}_{t^*}$, $\hat{\theta}_t^i \ge_{s.d.} \theta_t^i$ for $t \le t^*$ and $i \in \mathcal{M}$. Since $K_t^i(\theta_t^i)$ is stochastically increasing in θ_t^i for any $i \in \mathcal{M}$, conditioned on $\theta_{t^*} = \bar{\theta}_{t^*}$, $K_t^i \ge_{s.d.} \mathcal{K}_t^i$, for $t \le t^*$ and any $i \in \mathcal{M}$; and, conditioned on $\theta_{t^*} = \underline{\theta}_{t^*}$, $\mathcal{K}_t^i \ge_{s.d.} K_t^i$, for $t \le t^*$ and any $i \in \mathcal{M}$. By Theorem 7(ii) and Theorem 9(i), both $V_t(I_t, K_t, \theta_t)$ and $\hat{V}_t(I_t, K_t)$ are submodular in (I_t, K_t^i) for any $i \in \mathcal{M}$. Therefore, we show inequality (15) by backward induction, with the same argument as the proof of Theorem 7(ii). More specifically, we show that, for any $t \le t^*$, if $\partial_{I_s}\mathbb{E}_{\mathcal{K}_s}\hat{V}_s(I_s, \mathcal{K}_s) \ge \partial_{I_s}\mathbb{E}_{K_s,\theta_s}[V_s(I_s, K_s, \theta_s)|\theta_{t^*} = \bar{\theta}_{t^*}]$ for $s = t-1$ and $K_t^i \ge_{s.d.} \mathcal{K}_t^i$ for all $i \in \mathcal{M}$, then we have $\partial_{I_t}\mathbb{E}_{\mathcal{K}_t}\hat{V}_t(I_t, \mathcal{K}_t) \ge \partial_{I_t}\mathbb{E}_{K_t,\theta_t}[V_t(I_t, K_t, \theta_t)|\theta_{t^*} = \bar{\theta}_{t^*}]$; and if $\partial_{I_s}\mathbb{E}_{\mathcal{K}_s}\hat{V}_s(I_s, \mathcal{K}_s) \le \partial_{I_s}\mathbb{E}_{K_s,\theta_s}[V_s(I_s, K_s, \theta_s)|\theta_{t^*} = \underline{\theta}_{t^*}]$ for $s = t-1$ and $\mathcal{K}_t^i \ge_{s.d.} K_t^i$ for all $i \in \mathcal{M}$, then we have $\partial_{I_t}\mathbb{E}_{\mathcal{K}_t}\hat{V}_t(I_t, \mathcal{K}_t) \le \partial_{I_t}\mathbb{E}_{K_t,\theta_t}[V_t(I_t, K_t, \theta_t)|\theta_{t^*} = \underline{\theta}_{t^*}]$. The above two backward inductions follow from the same argument as the proof of Theorem 7(ii). We omit their proofs for brevity. This establishes Lemma 1. ■

Proof of Theorem 5

By Lemma 1, Theorem 5 follows from the same argument as the proof of Theorem 7(ii,iii). ■

Proof of Theorem 6

Part (i). We prove this part by backward induction. Because, $\Theta_{t-1}^i(\theta_t^i)$ is stochastically increasing in θ_t^i for any $i \in \mathcal{M}$, it suffices to show that, for any $t \in \mathcal{T}$ and $i \in \mathcal{M}$, if $V_s(I_s, K_s, \theta_s)$ is increasing in θ_s^i for $s = t-1$, $V_t(I_t, K_t, \theta_t)$ is increasing in θ_t^i. Since $V_0(\cdot,\cdot,\cdot) = 0$, the initial condition is satisfied. Assume that $V_s(I_s, K_s, \theta_s)$ is increasing in θ_s^i for $s = t-1$ and any $i \in \mathcal{M}$. By Theorem 1(ii), $V_s(I_s, K_s, \theta_s)$ is increasing in K_s^i for any $i \in \mathcal{M}$ and $s = t-1$. Since $K_t^i(\theta_t)$ is stochastically increasing in θ_t for all i, $V_{t-1}(I_{t-1}|\theta_t)$ and, thus, $G_t(J_t, y_t, \theta_t)$ are increasing in θ_t^i for any $i \in \mathcal{M}$. Because monotonicity is preserved under maximization and expectation, $W_t(J_t, D_t, \theta_t)$, $H_t(I_t, x_t, \theta_t)$ and $V_t(I_t, K_t, \theta_t)$ are all increasing in θ_t^i for any $i \in \mathcal{M}$. This establishes **part (i)**.

Parts (ii) and (iii). We prove **parts (ii) and (iii)** together by backward induction. We will show that: if $\partial_{I_s} V_s(I_s, K_s, \hat{\theta}_s) \leq \partial_{I_s} V_s(I_s, K_s, \theta_s)$ for any $\hat{\theta}_s \geq \theta_s$, for $s = t-1$, then we have: (a) $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_t) \leq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_t)$ and $\partial_{J_t} W_t(J_t, \hat{\theta}_t) \leq \partial_{J_t} W_t(J_t, \theta_t)$; (b) $\alpha_t^i(\hat{\theta}_t) \leq \alpha_t^i(\theta_t)$ and $x_t^{i*}(I_t, K_t, \hat{\theta}_t) \leq x_t^{i*}(I_t, K_t, \theta_t)$ for any $i \in \mathcal{M}$; (c) $\beta_t^j(\hat{\theta}_t) \leq \beta_t^j(\theta_t)$ and $y_t^{j*}(J_t, D_t, \hat{\theta}_t) \geq y_t^{j*}(J_t, D_t, \theta_t)$ for any $j \in \mathcal{N}$; and (d) $\partial_{I_t} V_t(I_t, K_t, \hat{\theta}_t) \leq \partial_{I_t} V_t(I_t, K_t, \theta_t)$. Since $V_0(\cdot,\cdot,\cdot) = 0$, the initial condition is satisfied.

Since $\Theta_{t-1}^i(\theta_t^i)$ and $K_{t-1}^i(\theta_t^i)$ are stochastically increasing in θ_t^i and independent of θ_t^j for $j \neq i$, and $\partial_{I_s} V_s(I_s, K_s, \theta_s)$ is decreasing in θ_s^i and K_s^i and any $i \in \mathcal{M}$, Lemma 3(ii) implies that $\partial_{I_s} V_s(I_s|\hat{\theta}_t) \leq \partial_{I_s} V_s(I_s|\theta_t)$. Thus, by (8) and the concavity of $V_s(\cdot|\hat{\theta}_t)$ and $V_s(\cdot|\theta_t)$, $\beta_t^j(\hat{\theta}_t) \leq \beta_t^j(\theta_t)$ for any $j \in \mathcal{N}$. Hence, by the same argument as the proof of Theorem 7(iii), $y_t^{j*}(J_t, D_t, \hat{\theta}_t) \geq y_t^{j*}(J_t, D_t, \theta_t)$ $j \in \mathcal{N}$. Since $V_s(I_s|\hat{\theta}_t)$ and $V_s(I_s|\theta_t)$ are concave in I_s and $\partial_{I_s} V_s(I_s|\hat{\theta}_t) \leq \partial_{I_s} V_s(I_s|\theta_t)$, Lemma 3(v) implies that $\partial_{J_t} W_t(J_t, D_t, \hat{\theta}_t) \leq \partial_{J_t} W_t(J_t, D_t, \theta_t)$. Hence, $\partial_{J_t} W_t(J_t, \hat{\theta}_t) \leq \partial_{J_t} W_t(J_t, \theta_t)$.

By (6), the concavity of $W_t(\cdot, \hat{\theta}_t)$ and $W_t(\cdot, \theta_t)$, and that $\partial_{J_t} W_t(J_t, D_t, \hat{\theta}_t) \leq \partial_{J_t} W_t(J_t, D_t, \theta_t)$, $\alpha_t^i(\hat{\theta}_t) \leq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$. Hence, by the same argument as the proof of Theorem 7(iii), $x_t^{i*}(I_t, K_t, \hat{\theta}_t) \leq x_t^{i*}(I_t, K_t, \theta_t)$ for any $i \in \mathcal{M}$. Since $W_t(J_t, \hat{\theta}_t)$ $W_t(J_t, \theta_t)$ are concave in J_t and $\partial_{J_t} W_t(J_t, \hat{\theta}_t) \leq \partial_{J_t} W_t(J_t, \theta_t)$, Lemma 3(v) implies that $\partial_{I_t} V_t(I_t, K_t, \hat{\theta}_t) \leq \partial_{I_t} V_t(I_t, K_t, \theta_t)$. This completes the induction and, thus, the proof of **parts (ii) and (iii)**. ■

Before giving the proof of Theorem 8, we present Theorem 10 and its proof first.

Theorem 10 and its Proof

We now characterize the optimal policy in the model without discretionary selling (see Section 6 for details). We define

$$\check{W}_t(J_t, \theta_t) := \mathbb{E}_{D_t}\{\tilde{r}_t D_t - h_t(J_t - |D_t|) + \gamma \mathbb{E}_{(K_{t-1}, \theta_{t-1})}[\check{V}_{t-1}(J_t - |D_t|, K_{t-1}, \theta_{t-1})|\theta_t]|\theta_t\},$$

where $J_t = I_t + |x_t|$. Hence,

$$\check{V}_t(I_t, K_t, \theta_t) = \max_{0 \leq x_t \leq K_t} \{\check{W}_t(J_t, \theta_t) - c_t x_t\}. \tag{28}$$

THEOREM 10. *(i)* $\check{V}_t(I_t, K_t, \theta_t)$ *is concave and continuously differentiable in* I_t, $\forall t$ *and* (K_t, θ_t).

(ii) Let

$$\check{\alpha}_t^i(\theta_t) := \min\{J_t \in \mathbb{R} : c_t^i \geq \partial_{J_t} \check{W}_t(J_t, \theta_t)\},\ i \in \mathcal{M}, \tag{29}$$

where $\check{\alpha}_t^i(\theta_t) := -\infty$ *if* $\{J_t \in \mathbb{R} : c_t^i < \partial_{J_t} \check{W}_t(J_t, \theta_t)\} = \emptyset$. *We have* $\{\check{\alpha}_t^i(\theta_t)\}_{i \in \mathcal{M}}$ *are independent of the starting inventory level* I_t *and capacity* K_t, *and decreasing in* $i \in \mathcal{M}$, $\forall t$ *and* θ_t. *The optimal decision is*

$$\check{x}_t^{i*}(I_t, K_t, \theta_t) = K_t^i \cdot \mathbf{1}_{\{i < \check{i}_t\}} + \min\{\check{\alpha}_t^i(\theta_t) - I_t - [K_t]_1^{i-1}, K_t^i\} \cdot \mathbf{1}_{\{i = \check{i}_t\}}, \quad i \in \mathcal{M}, \tag{30}$$

where $\check{i}_t := m \cdot \mathbf{1}_{\{I_t + |K_t| \leq \check{\alpha}_t^m(\theta_t)\}} + \min\{i : I_t + [K_t]_1^i \geq \check{\alpha}_t^i(\theta_t)\} \cdot \mathbf{1}_{\{I_t + |K_t| > \check{\alpha}_t^m(\theta_t)\}}$.

(iii) For each t *and* $i \in \mathcal{M}$, $\check{\alpha}_t^i(\theta_t)$ *and* $\check{x}_t^{i*}(I_t, K_t, \theta_t)$ *are decreasing in* θ_t^j *for any* $j \in \mathcal{M}$.

Part (i) follows from the same argument as the proof of Theorem 1. The proof of **part (ii)** is identical to that of Theorem 2. The proof of **part (iii)** follows from that of Theorem 6(iii). Hence, we omit the proof of **Theorem 10**. ■

Proof of Theorem 8

Part (i). We prove $\partial_{I_t}\check{V}_t(I_t,K_t,\theta_t) \geq \partial_{I_t}V_t(I_t,K_t,\theta_t)$ and $\check{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$ by backward induction. It suffices to show that if $\partial_{I_s}\check{V}_s(I_s,K_s,\theta_s) \geq \partial_{I_s}V_s(I_s,K_s,\theta_s)$ for $s = t-1$, then (a) $\check{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$ and (b) $\partial_{I_t}\check{V}_t(I_t,K_t,\theta_t) \geq \partial_{I_t}V_t(I_t,K_t,\theta_t)$. $\partial_{I_0}\check{V}_0(I_0,K_0,\theta_0) = \partial_{I_0}V_0(I_0,K_0,\theta_0) = 0$, so the initial condition is satisfied.

Since $y_t^*(J_t,D_t,\theta_t) \leq D_t$ for any D_t, $\check{V}_s(I_s,K_s,\theta_s)$ and $V_s(I_s,K_s,\theta_s)$ are concave in I_s, and $\partial_{I_s}\check{V}_s(I_s,K_s,\theta_s) \geq \partial_{I_s}V_s(I_s,K_s,\theta_s)$, it follows immediately that $\check{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$ are concave, and $\partial_{J_t}\check{W}_t(J_t,\theta_t) \geq \partial_{J_t}W_t(J_t,\theta_t)$. Together with (6) and (29), we have $\check{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$. To complete the induction, by Lemma 3(v), $\partial_{I_t}\check{V}_t(I_t,K_t,\theta_t) \geq \partial_{I_t}V_t(I_t,K_t,\theta_t)$ follows from (2), (28), the concavity of $\check{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$, and $\partial_{J_t}\check{W}_t(J_t,\theta_t) \geq \partial_{J_t}W_t(J_t,\theta_t)$. For any $i \in \mathcal{M}$, $\check{x}_t^{i*}(I_t,K_t,\theta_t) \geq x_t^{i*}(I_t,K_t,\theta_t)$ follows from $\check{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$.

Part (ii). This part follows directly from equation (18) that the term $\sum_{j=1}^n \tilde{r}_t^j D_t^j$ is separable from the decision variable, x_t, in the objective function.

Part (iii). We show by backward induction. Let two systems be equivalent except that $\bar{r}_{t_0}^{j_0} > r_{t_0}^{j_0}$ for some $t_0 \in \mathcal{T}$ and $j_0 \in \mathcal{M}$. It suffices to show that if $\bar{r}_s^j \geq r_s^j$ and $\partial_{I_s}\bar{V}_s(I_s,K_s,\theta_s) \geq \partial_{I_s}V_s(I_s,K_s,\theta_s)$ for any $j \in \mathcal{N}$ and $s = t-1$, then (a) $\bar{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$ and θ_t; and (b) $\partial_{I_t}\bar{V}_t(I_t,K_t,\theta_t) \geq \partial_{I_t}V_t(I_t,K_t,\theta_t)$. For $t = t_0$, the initial condition is clearly satisfied. $\partial_{I_s}\bar{V}_s(I_s,K_s,\theta_s) \geq \partial_{I_s}V_s(I_s,K_s,\theta_s)$ yields that $\partial_{I_s}\bar{V}_s(I_s|\theta_t) \geq \partial_{I_s}V_s(I_s|\theta_t)$. Thus, by Lemma 3(v), the concavity of $\bar{V}_s(\cdot|\theta_t)$ and $V_s(\cdot|\theta_t)$ and $\bar{r}_t^j \geq r_t^j$ imply that $\partial_{J_t}\bar{W}_t(J_t,D_t,\theta_t) \geq \partial_{J_t}W_t(J_t,D_t,\theta_t)$ and, hence, $\partial_{J_t}\bar{W}_t(J_t,\theta_t) \geq \partial_{J_t}W_t(J_t,\theta_t)$. The concavity of $\bar{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$ and $\partial_{J_t}\bar{W}_t(J_t,\theta_t) \geq \partial_{J_t}W_t(J_t,\theta_t)$, together with (6), yield that $\bar{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$ for any $i \in \mathcal{M}$. Invoking Lemma 3(v) again, since $\bar{W}_t(\cdot,\theta_t)$ and $W_t(\cdot,\theta_t)$ are concave and $\partial_{J_t}\bar{W}_t(J_t,\theta_t) \geq \partial_{J_t}W_t(J_t,\theta_t)$, we have $\partial_{I_t}\bar{V}_t(I_t,K_t,\theta_t) \geq \partial_{I_t}V_t(I_t,K_t,\theta_t)$. This completes the induction. For any $i \in \mathcal{M}$, $\bar{x}_t^{i*}(I_t,K_t,\theta_t) \geq x_t^{i*}(I_t,K_t,\theta_t)$ follows from $\bar{\alpha}_t^i(\theta_t) \geq \alpha_t^i(\theta_t)$. ■