Dynamic Competition in Online Retailing: Implications of Network Effects

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Problem Definition: Live-streaming e-commerce, a thriving branch of online retailing, has a unique feature that customers are constantly engaged in social interactions with peers, which could shape their purchase decisions through the network effect – they are more likely to purchase if many others have done so. In a competitive environment, we take a novel perspective to distinguish two types of network effects: The specific network effect helps an individual retailer expand its own future market, whereas the non-specific network effect generates a new market that is shared by competing retailers. Our goal is to investigate the impact of different types of network effects on the dynamic competition in online retailing. *Methodology/Results:* We formulate a two-period duopoly competition game between newsvendor-type online retailers with asymmetric costs, where stock-out based substitution occurs within periods and the network effect regulates market dynamics across periods. We then solve for the equilibrium outcomes and compare them in different scenarios. There are three major findings. First, under the non-specific network effect, the high-cost retailer orders less in the first period compared to under the specific network effect. Such an ordering strategy can be interpreted as free-riding the low-cost retailer's stockpiling and benefit from the common market expansion at a lower inventory cost. Second, the comparison of the retailers' profits between scenarios can be non-intuitive. The low-cost [high-cost] retailer, which is in the advantageous [disadvantageous] position, may achieve a higher profit under the non-specific [specific] network effect, which tends to soften [intensify] the duopoly competition. Third, with pricing flexibility, the low-cost retailer can counteract the high-cost retailer's free-riding by adopting a price markdown strategy, where its price decreases along the two periods. *Managerial Implications:* Our findings offer an important managerial insight for online retailers: They need to be cautious about the type of network effects in presence, because they would induce different market diffusion processes and lead to contrasting equilibrium results for the dynamic duopoly competition. Moreover, retailers' inventory and pricing decisions must be paired with the type of network effects to achieve the best performance.

1. Introduction

Ever since online retailing was born, it has been constantly redefined by business and technology innovations. In recent years, with the rapid growth of the social media platforms (e.g., Facebook, YouTube, and WeChat), online retailing has found its way to a revolutionary mode, called *social commerce*, which combines e-commerce with online social medias. This new mode of online retailing has seen great success and even greater potentials (Statista 2023b). Among various forms for social commerce, live-streaming e-commerce is the most popular and successful one. In China, where live-streaming e-commerce was pioneered, it has thrived to become a five-trillion-RMB retail market by 2023 (Statista 2023a). Usually, online retailers will hire social media influencers to sell their products on live-streaming platforms, and during the sales process, the influencers introduce the features of the products, communicate with consumers, and answer their questions. The resulting sales can be enormous, especially when the hired influencer is a celebrity and the selling periods coincide with some special event. For example, it is reported that Austin Li, a star e-commerce live-streamer in China, sold products worth $1.7 billion in Alibaba's Singles Day in 2021 (Tan 2021). Bridging entertainment with online shopping, such live video selling creates a highly engaging purchase experience for all customers, which substantially increases customer acquisition and sales conversion for the online retailers.

A couple of interesting features about live-streaming e-commerce are noteworthy. First, many famous influencers are hired only short-term by competing online retailers, e.g., few days before and after e-commerce holidays such as Single Day and Cyber Monday. As a result, the selling season is relatively short, and the main body of the target market, which consists of the influencers' followers, cannot last. Thus, the online retailers are effectively competing newsvendors selling substitutable short life-cycle products. Second, since live video sales involve substantial interactions between the influencer and the customers and, more importantly, among the customers themselves, the market diffusion process for the online retailer can be significantly propelled. Indeed, a customer can derive additional utility (e.g., psychological satisfaction) from making purchases if many others are doing the same. Hence, the social contagions and interactions of existing customers can greatly attract new customers and expand the potential market for retailers. Here, we refer to such a common effect as the *network effect*.

To further characterize the impact of network effects on the competing online retailers' individual market diffusion process, we take a novel perspective and distinguish two types of network effects. First, when customers derive additional utility from patronizing a specific retailer, we have the *Specific Network Effect*. In this case, the induced new demand is based on the specific retailer's existing sales and the market expansion is exclusively for that retailer. This could happen when the product renders network externalities (e.g., Economides 1996, Katz and Shapiro 1994) or, in our setting, when the hired live-streamer is a famous influencer or the retailer has a strong brand name (e.g., Wongkitrungrueng and Assarut 2020, Katona et al. 2011). Second, when customers' additional utility can be derived from any one of the competing retailers, we have the *Non-specific Network Effect*. This effect can be observed when the social interaction of customers with the products, especially in the form of word-of-mouth, is not brand specific. Such cross-brand communication and word-of-mouth spillover have also been well documented in the marketing literature (see, e.g., Peres and Van den Bulte 2014, Libai et al. 2009). In fact, with similar products on the market, new customers may not have a strong preference for a specific retailer. Therefore, as the rising tide lifts all ships, existing sales of a particular product can contribute to the expansion of the common market for retailers selling substitutable products.

Specific or non-specific, the network effects create a dependence between the current decisions and future demand for retailers, because the current decisions affect current sales, which decide the market expansion and, thus, future demand. Moreover, recognizing the different types of network effects, retailers may react differently to resolve the trade-off between current profits and future demand, leading to contrasting equilibrium outcomes and managerial implications. Therefore, the central objective of this paper is to understand the impact of the two types of network effects on the competition of online retailers. To that end, we develop a two-period duopoly competition model, in which two newsvendor-type retailers with asymmetric costs compete in inventory (in the form of stock-out-based substitution) in each period, and the market diffusion across periods is regulated by the network effect. As a salient feature in our model, we explicitly consider the two types of network effects and their implications on the dynamic online retailer competition. Then, we seek to address the following key research questions: In the scenario with each type of network effects, how do the network effect's strength and the competition intensity interplay with each other to determine the equilibrium? How does each retailer's equilibrium profit change under different types of network effects? When allowing pricing flexibility for the cost-efficient retailer, how to characterize its pricing strategy and how does it affect the previous results?

To start with, we focus on the competing retailers' inventory decisions by assuming that they post the same price to the customers. As a result, they appear the same to the potential customers and thus receive the same initial demand. However, since one retailer is assumed to have cost advantage against the other, we find that the stock-out based substitution, if exists, can only happen from the high-cost retailer to the low-cost retailer (i.e., unmet demand at the high-cost retailer switch to the low-cost retailer for fulfillment). Furthermore, such a unilateral demand spillover prevails under both types of network effects and is the key to understand the retailers' first-period order quantities, which reflect the trade-off between current profit and future demand. Specifically, we find that the low-cost retailer's order quantity is always positively correlated with the substitution rate, which represents the competition intensity. As for the high-cost retailer, its ordering decision is independent of the competition level under the specific network effect, but the correlation becomes negative under the non-specific network effect. In addition, the high-cost retailer's order quantity is lower under the non-specific network effect than under the specific network effect. This result gives rise to an interesting implication: Since both retailers' sales help expand the common market under the non-specific network effect, the high-cost retailer may intentionally order a smaller quantity to send unmet demand to the rival, so that it may enjoy an enlarged market at a lower inventory cost – a *free-riding* behavior.

Then, turning to the retailers' financial performance, we find and compare their equilibrium profits under the two types of network effects. Since the specific network effect tends to intensify the competition, intuition may suggest that the low-cost retailer will be better off under the specific network effect than the non-specific network effect; and for the high-cost retailer, the reverse is true. However, we obtain a counter-intuitive result: Each retailer's profit may be higher or lower in either case, depending on the system parameters such as cost asymmetry, strength of network effects, and competition intensity. For the low-cost retailer, when the cost difference and the substitution rate are both large and network effects are strong, its equilibrium profit is higher under the non-specific, rather than the specific, network effect. This is because, in this case, collaboratively expanding the market with the rival has more benefits than engaging in intensified competition and inducing new customers individually. For the high-cost retailer, with a small cost difference, a small substitution rate, and strong network effects, its profit is lower under the non-specific network effect. The underlying reason is that the free-riding behavior cannot achieve the expected benefit and it is better for the retailer to generate future demand by itself, which can be done under the specific network effect. Our result has an important managerial implication: In a duopoly relationship, the more advantageous firm may not always prefer an even more competitive environment, such as under the specific network effect, and the less advantageous firm may not always like the competition to be reduced, such as under the non-specific network effect. Firms' best interests are usually met at the balance between competition and collaboration.

Lastly, we extend our model to accommodate the pricing decision. We allow pricing flexibility only for the low-cost retailer, because online retailing is such a competitive industry that near-zero profit margin is not uncommon in practice. As such, the high-cost retailer's price is exogenously fixed at a certain level, and the low-cost retailer may set prices to change its share of the common market. Assuming a committed pricing scheme, we solve the extended model and derive insights into the retailer's pricing strategy. Results reveal that, when the network effect is strong enough, the retailer's price is increasing in time under the specific network effect and decreasing under the non-specific network effect. For the former case, it is intuitive that the retailer wants to take advantage of the specific network effect in the first period, but has no incentive to lower its price in the second period because the expanded market is exclusively its own. For the latter case, however, it is interesting to note that the low-cost retailer's price is lower in the second period than in the first, which can be seen as a counteraction against the high-cost retailer's free-riding behavior under the non-specific network effect: The low-cost retailer contributes more to the common market expansion and is therefore just getting its fair share using a lower retail price. Finally, we compare retailers' profits under the different types of network effects and observe that, with slightly modified conditions, the previous results continue to hold.

The rest of this paper is organized as follows. We position this paper in the related literature in Section 2. Section 3 describes the essential elements of the model and presents the preliminary results. We show our main results in Section 4. In Section 5, we extend our model to allow pricing flexibility. Lastly, Section 6 concludes this paper. All proofs are relegated to the Appendix.

2. Literature Review

Caro et al. (2020) have recently provided a comprehensive research survey on retail operations and discussed its new trends; our work belongs to this vast literature at large. While the landscape of retailing industry has been constantly changed by fast technology development and novel business models, live-streaming e-commerce is currently the most trendy way of retailing. Academic research follows this trend as well. There is a growing stream of studies on the operations and economics of live-streaming e-commerce; see, e.g., Wongkitrungrueng and Assarut (2020), Qi et al. (2022), Hou et al. (2022), and Chen et al. (2020). These papers focus on some important aspects such as customer trust and engagement, the role of influencers, and the mechanism design for the platforms, and investigate how these aspects affect various stakeholders. In the similar vein, we identify the two types of network effects motivated by the live-streaming e-commerce setting and examine their impact on the retailers' dynamic competition. Thus, our work expands the width of the literature stream on online retailing and enriches its content.

The analytical model in our paper is based on the competitive newsvendors framework, which has long been studied in the inventory literature; see, e.g., Parlar (1988), Lippman and McCardle (1997), and Netessine and Rudi (2003). The inventory competition due to stock-out based substitution has been used to study firms' strategic stocking decisions in various settings, some of which share similar flavor to our work. For instances, we assume the competing retailers have asymmetric costs; Jiang et al. (2011) and Güler et al. (2018) focus on asymmetric information regarding demand and cost, respectively. Our work is motivated by online retailing; Straubert and Sucky (2023) study a problem from online marketplaces. Our results yield a discussion on the competition-collaboration relationship between the retailers; Dong et al. (2023) develop a co-opetitive newsvendor model in a supply chain setting. Departing from all the above prior works with the single-period assumption, we investigate a multi-period newsvendor competition. In this regard, our paper is more closely related to dynamic inventory competition model (see, e.g., Liu et al. 2007, Hall and Porteus 2000, Nagarajan and Rajagopalan 2008, 2009, Olsen and Parker 2008, 2014). Aside from having different scopes and research agendas, our paper differs from those related works in an important way: The market dynamic is determined by the product substitution/availability in their works, whereas it is the network effect that dictates the market diffusion process in ours. Overall, we contribute to the competing newsvendor literature by incorporating the notion of network effects into the problem. Being closely related to the inventory decisions on the one hand and regulating the system dynamics on the other, network effect is a critical driver in the online retailing setting and should not be ignored.

A salient feature of our paper is the consideration of network effects, which has always been an intriguing topic for scholars in economics, marketing, and operations management. In its original definition, network effect mainly refers to the positive externalities of the network products (see, e.g., Economides 1996, Katz and Shapiro 1994); Shy (2011) provides a short survey on this topic. More broadly, network effect can be understood as the potential demand increase due to the realized sales volumes, and the reason for the demand attraction is not necessarily network externality, but may be anything that gives customers additional satisfaction. Viewed in this way, network effect has been extensively studied, especially in the retail operations settings. For example, Katona et al. (2011) examine people's adoption decisions in an online social network, where the network effect is manifested via individual connections between members. Wang and Wang (2017) endogenize network effects in a static consumer choice model and analyze the corresponding assortment optimization; and Feng and Wang (2021) extend it to a dynamic model. Chen and Chen (2021) study duopoly competition with quality differentiation and network effects. Feng and Hu (2024) consider firms' entry and quality decisions under network effects.

While the specific network effect is similar to those studied in the above papers, the non-specific network effect can only occur in a competitive environment such as ours, and it emphasizes the notion of free-riding or spillover – when firms' sales jointly expand their common market, they may benefit from one another. Moreover, the non-specific network effect is often observed when customers' interactions on their social networks take the form of word-of-mouth (see, e.g. Godes and Mayzlin 2004, Geng et al. 2022) or when customers engage in interpersonal communication not specifying product brand (see, e.g., Krishnan and Vakratsas 2012, Libai et al. 2009). Hence, although in different names, the non-specific network effect has been studied by prior works. For example, Hu et al. (2020) study the innovation spillover when an innovator outsources its products to a contract manufacturer, which may also be a competitor in the end market. Peres and Van den Bulte (2014) show that the positive word-of-mouth spillover from rivals may make a new product's reseller unprofitable when the product is exclusively sold. Haviv et al. (2020) empirically quantify the positive intertemporal spillover effect between sellers on console video game platforms.

To sum, our contribution to the literature of network effects is twofold. First, we analyze the impact of network effects on a dynamic newsvendor competition in an online retailing setting. Second, we explicitly examine the two distinct types of network effects and compare their respective implications on the retailers' operational decisions and financial performances.

3. Model and Analysis

We study a two-period duopoly competition where the market dynamic is dictated by the network effect among customers. In this section, we first give the model setup and problem formulation, then distinguish the two types of network effects and describe how they work, and finally solve the dynamic game in different scenarios.

3.1. General Model Framework

3.1.1. Firms. Consider two online retailers, A and B, selling substitutable perishable products to a market with a random size. The two retailers are assumed to be asymmetric *only* in unit ordering cost, but otherwise identical. Without loss of generality, let retailer A's cost c_A be smaller than retailer B's cost c_B. We further assume that the two retailers post the same, exogenously given, retail price to the customers. The above assumptions capture two commonly observed features of online retailers: First, as many competing online retailers are from geographically different locations, their cost asymmetry may be due to different transportation expenses. Second, online retailing is highly competitive and the retail price for similar products are usually the same. Being the more cost-efficient firm, retailer A may have more pricing flexibility than its rival; e.g., retailer A could be more able to afford a price markdown in order to snatch market. We do not consider such possibility in the main model, but will defer relevant discussions to Section 5. Until then, we assume the posted retail price is always $p > c_B > c_A$ for both firms.

In each period, facing a random demand and possible stock-out based substitution, the two retailers are essentially competing newsvendors, just as modeled by Lippman and McCardle (1997). Specifically, in period t, retailer i ($i = A, B$) decides the order quantity $y_{i,t}$ first and then receives a random initial demand allocation $M_{i,t}$ (to be detailed in a moment). Any leftovers after each period ends are salvaged through other non-profitable channels; we normalize the salvage value to zero. When a shortage occurs at one retailer, part of its excess demand will spillover to the other retailer. As such, the effective demand for retailer i is given by

$$D_{i,t} = M_{i,t} + \theta(M_{j,t} - y_{j,t})^+.$$

Here, $\theta \in [0,1]$ is an exogenous parameter measuring the degree of substitution due to stock-out. Larger θ value means higher substitution rate, which implies more intensified inventory competition between the retailers. Finally, in period t, retailer i's realized sales is $R_{i,t} = \min\{D_{i,t}, y_{i,t}\}$ and its profit is $\pi_{i,t} = pR_{i,t} - c_i y_{i,t}$.

3.1.2. Demand and Market Dynamics. Next, we detail the assumptions about the demand process and how the market evolves across periods. First, in every period t, there is a seed demand X_t that characterizes the size of the customer base, which serves as the common market for both retailers. The seed demand $\{X_t\}$ form an independent and identically distributed sequence of random variables, with distribution $F(\cdot)$ and density $f(\cdot)$. This portion of the demand can be seen as the relatively stable market of patrons with a fixed distribution. Second, newly attracted customers arrive every period and become part of the demand. This market diffusion process is dictated by the network effect, which is operated on new customers through the old customers who purchased before. In particular, we assume that the new market induced by the network effect in the current period is based on the realized sales of the retailers from the previous period. Therefore, for retailer i in period t, its initial allocation of the demand, $M_{i,t}$, consists of two parts:

$$M_{i,t} = \alpha_{i,t} X_t + Z_{i,t}. \tag{1}$$

The first part represents a split of the seed demand. We employ the deterministic splitting rule from Lippman and McCardle (1997). Specifically, $\alpha_{i,t}$ is the split ratio, which satisfies $\alpha_{A,t} + \alpha_{B,t} = 1$. Moreover, we assume that the splitting is based on the posted price, which is the foremost influential factor to customers' purchase decision. In our main model, since the two retailers always post the same price, they will get an equal share of the common stable market, i.e., $\alpha_{i,t} = 1/2$. We will assume other forms of $\alpha_{i,t}$ in Section 5 when the pricing flexibility is considered.

The second part of retailer i's initial allocation, $Z_{i,t}$, represents retailer i's demand induced by the network effect. In general, the induced demand is a non-decreasing function of firms' sales in the previous period, i.e., $Z_{i,t} = z_i(R_{A,t-1}, R_{B,t-1})$. That is, more past sales tend to generate more potential demand. In the duopoly online retailing context, we identify two types of network effects, which give rise to different functional forms of $Z_{i,t}$. They are delineated later in Section 3.2.

3.1.3. Problem Formulation. The aforementioned major components of our model yield a dynamic competitive newsvendor problem facing the two online retailers: They stock up to compete for demand from a common market in each period, and the market diffusion across periods is regulated by the network effect. We formulate this problem as a two-period two-person Nash game with complete information.

Recall that our primary research goal is to investigate the role of the network effect and how it affects the dynamic newsvendor competition. Since the market dynamics in our model is solely determined by the network effect, the impact of the network effect can fully manifest as soon as the period moves forward once; therefore, concentrating on a two-period model suffices to serve the purpose. Besides, compared to a general multi-period model, the two-period model admits tractable analysis, which allows derivation of clean and useful managerial insights.

The sequence of events in each period t is as follows. (1) Both retailers observe the previous sales of each firm, $(R_{A,t-1}, R_{B,t-1})$, and thus obtain the distribution of the initial allocation $M_{i,t}$; if $t = 1$, we set $R_{i,0} = 0$. (2) The retailers simultaneously decide the order quantity $y_{i,t}$ based on the observed previous sales. (3) The seed demand X_t is realized, and so is the initial allocation. For each retailer, if there is an excess demand, a fixed proportion (θ) of the unsatisfied customers will attempt to purchase from the other retailer; otherwise, any unused products are salvaged with zero value. (4) The sales of each retailer, $R_{i,t}$, is realized and the profit collected. The objective of each retailer is to maximize its total (discounted) profit of all periods. As such, retailer i solves the following problem simultaneously with retailer j:

$$(\mathcal{P}) \qquad \max_{y_{i,1}, y_{i,2}} \mathbb{E}_{X_1,X_2}\left[(pR_{i,1} - c_i y_{i,1}) + \rho\,(pR_{i,2} - c_i y_{i,2})\right],$$

subject to

$$R_{i,t} = \min\{D_{i,t}, y_{i,t}\}, D_{i,t} = M_{i,t} + \theta(M_{j,t} - y_{j,t})^+, \text{ and } M_{i,t} = \alpha_{i,t} X_t + Z_{i,t}; t = 1, 2.$$

The parameter $0 < \rho \le 1$ is the discount factor. The equilibrium concept adopted is feedback Nash equilibrium. We treat the sales $R_{i,t}$ as state variables in each period, and solve for the order quantities $y_{i,t}$ via backward induction.

3.2. Two Types of Network Effects

The previous subsection formulates a general problem in the sense that the network effect is not defined in details. In this subsection, we identify two types of network effects, discuss their differences, and describe how each of them decides the retailers' initial demand allocations.

A common practice of online retailing nowadays is to hire social-media influencers to sell the products in their live-streaming rooms. Owing to the strong social contagion nature of the live-streaming platforms, this selling modality can easily attract a large number of customers and drive up sales. Indeed, a high transaction volume in an influencer's live-streaming room can effectively draw new customers and generate new demand. Such market expansion is due to the network effect pertaining to the customers' social interactions. That is, a customer can derive higher utility from purchasing the product when more peers make the purchases as well. Note that the extra utility may be from the network externality of product usage, or it may be from the pure psychological satisfaction[1]. Hence, the network effect studied in our paper is a more general concept than that for the network products (e.g., Economides 1996, Katz and Shapiro 1994). While the network effect will always expand the potential market, the true impact on the competing retailers' individual customer base needs a more careful examination. We distinguish the two types of network effects in our duopoly online retailing setting.

3.2.1. Specific Network Effects. First, the network effect could be specifically attributed to one retailer; hence the name *specific network effect*. In this case, past sales of retailer i ($i = A, B$) induce new customers exclusively for retailer i. Practically, there are two possible situations in which the specific network effect comes into being. Customer's extra utility/satisfaction is specific to either the brand name or the influencer who sells for the retailer. In the former situation, the retailers usually possess highly differentiating brand names (not to be confused with rivals) or their products have certain exclusive features (e.g., video game console). Hence, when new customers are attracted to the market by the size of old users, they have already chosen the specific retailer to patronize. In the latter situation, the retailers hire famous influencers or even key opinion leaders to live-streaming their product sales. Here, the new customers generated by the network effect are mainly followers of the specific influencer.

We use "Scenario S" to refer to the case with the specific network effect and give the notations a superscript "S" whenever necessary. In Scenario S, the newly induced demand in period t for retailer i can be written as $Z_{i,t}^S = \gamma R_{i,t-1}^S$, a linear function of its previous sales. The parameter $\gamma > 0$

[1] An example is the herding behavior, which is commonly observed in live-streaming ecommerce. New customers are more likely to purchase if there is already a high sales volume.

represents the strength of the network effect (e.g., it could measure how famous the influencers are). Finally, the market dynamic equation (1) can be further written as

$$M_{i,t}^S = \frac{1}{2} X_t + \gamma R_{i,t-1}^S. \tag{2}$$

One direct observation from (2) is that, each retailer's initial allocation only depends on the seed demand and its *own* sales from the previous period.

3.2.2. Non-specific Network Effects. Second, the network effect may not be specific regarding either retailer; we call it the *non-specific network effect* and we refer to the induced case as Scenario N. In practice, the non-specific network effect may be observed when the brand name of a specific retailer is not the root driver of the network effect. Since the two retailers are selling substitutable products, the new customers may not be able to tell them apart. As a result, the customers derive extra utility/satisfaction mainly because they are using similar, but not necessarily the same, products as their peers. In the context of live-streaming e-commerce, the non-specific network effect is especially common – with noisy interpersonal communications/recommendations from various social media platforms, it is hard for customers to perfectly discern one product from the other; see, for example, Libai et al. (2009) and Krishnan and Vakratsas (2012) for discussions on the cross-brand word-of-mouth spillover that explains such a phenomenon. Hence, when fresh customers are drawn to the potential market, they could end up purchasing from either retailer.

Given the nature of the non-specific network effect, its generated demand is modeled as a linear function of the retailers' *total* past sales, denoted by $R_{t-1}^N = R_{A,t-1}^N + R_{B,t-1}^N$ (similar to before, superscript "N" indicates Scenario N). Then, γR_{t-1}^N is the newly induced market that is to be shared by both firms; the parameter γ again denotes the strength of the network effect. We further assume that the two retailers split this market in the same way as they split the seed demand; i.e., $Z_{i,t}^N = \gamma R_{t-1}^N/2$. Therefore, we can write the initial allocation equation in this scenario as

$$M_{i,t}^N = \frac{1}{2}\left(X_t + \gamma R_{t-1}^N\right). \tag{3}$$

Therefore, newly attracted customers do not have ex ante preference towards the retailers and they will simply choose from the two according to the same splitting rule of the seed demand. Finally, it is worth mentioning that, unlike Scenario S, retailer i's initial allocation in (3) depends on not only its own but also its rival's previous sales.

3.3. Equilibrium Analysis

In this subsection, we solve the dynamic duopoly competition in different scenarios and provide preliminary analysis to the equilibrium results. Before analyzing Scenarios S and N, we first look at a benchmark scenario, in which the network effect in absent. In each scenario, we solve for the retailers' equilibrium order quantities and discuss some immediate observations.

3.3.1. Benchmark Scenario: No Network Effect. Without network effects, the market dynamic across the two periods is gone, and the model is reduced to repeated static games. Hence, it suffices to solve the single-period competitive newsvendor model, which has been studied by Lippman and McCardle (1997). Thus, we can obtain the retailers' equilibrium order quantities, y_A^0 and y_B^0, which are the same in each period.

LEMMA 1. *Suppose there is no network effect. A unique Nash equilibrium exists:* $y_A^0 = \frac{1+\theta}{2}\zeta_A^0 - \frac{\theta}{2}\zeta_B^0$ *and* $y_B^0 = \frac{1}{2}\zeta_B^0$*, where* $\zeta_i^0 = F^{-1}(1 - \frac{c_i}{p})$ $(i = A, B)$*.*

Note that ζ_i^0 is exactly the classic newsvendor's critical z-score for retailer i, and $\zeta_A^0 > \zeta_B^0$ because $c_A < c_B$. Moreover, we have $y_A^0 > y_B^0$ for all θ. That is, both the competition intensity and the cost asymmetry together boost the order quantity of retailer A, the low-cost firm. Retailer B, on the other hand, is in a disadvantageous position in the duopoly relationship.

3.3.2. Scenario S: Specific Network Effects. Next, we include network effects in our analysis. Under the specific network effect, the market dynamic equation is given by (2). In this scenario, it is each retailer's individual past sales that determines the expansion of its own market, so they take advantage of the network effect independently.

LEMMA 2. *Suppose the specific network effect is present. A unique Nash equilibrium exists: In the first period,* $y_{A,1}^S = \frac{1+\theta}{2}\zeta_A^S - \frac{\theta}{2}\zeta_B^S$ *and* $y_{B,1}^S = \frac{1}{2}\zeta_B^S$*, where* $\zeta_i^S = F^{-1}(1 - \frac{c_i}{p+\rho\gamma(p-c_i)})$*; in the second period, given the two retailers' sales,* $R_{A,1}^S$ *and* $R_{B,1}^S$*,* $y_{A,2}^S = y_A^0 + \gamma R_{A,1}^S$ *and* $y_{B,2}^S = y_B^0 + \gamma R_{B,1}^S$*. Moreover, the retailers' equilibrium sales and profits are specified in appendix.*

The retailers' first-period order quantity has the same form as in the benchmark case, but with different critical z-scores. Moreover, we have $\zeta_A^S > \zeta_B^S$ due to the cost asymmetry $c_A < c_B$. Therefore, $y_{A,1}^S > y_{B,1}^S$. For the second-period order quantity $y_{i,2}^S$, it must first satisfy the induced demand $\gamma R_{i,1}^S$ (contingent on the previous sales), and the rest is the same as the benchmark scenario, due to the end-of-horizon effect. Furthermore, we remark that the network effect has essentially risen the underage cost in the first period – the stock-out risk means not only losing current customers, but also potential future customers, to the rival. Hence, the critical z-score increases, i.e., $\zeta_i^S > \zeta_i^0$ (see proof in appendix). This is especially true for the high-cost retailer B, and we can see that retailer B orders more in the first period compared to the benchmark case.

3.3.3. Scenario N: Non-specific Network Effects. Now, we turn to the non-specific network effect. In Scenario N, the market expansion characterized by equation (3) is based on the joint sales of both retailers, and thus the induced new demand is pooled into the seed demand and shared by the two firms. Hence, the initial allocation of a retailer in the second period is positively correlated with how much the other retailer sold in the first period. In this regard, the retailers enjoy the benefit of the network effect in a collaborative manner. In addition, since retailers' sales affect each other's customers acquisition, their order quantities may also intertwine.

Lemma 3. *Suppose the non-specific network effect is present. A unique Nash equilibrium exists: In the first period,* $y^N_{A,1} = \frac{1+\theta}{2}\zeta^N_A - \frac{\theta}{2}\zeta^N_B$ *and* $y^N_{B,1} = \frac{1}{2}\zeta^N_B$*, where*

$$\zeta^N_A = F^{-1}\left(1 - \frac{c_A}{p + \frac{1}{2}\rho\gamma(p - c_A)}\right) \text{ and } \zeta^N_B = F^{-1}\left(1 - \frac{c_B - \frac{1}{2}\rho\gamma\theta(p - c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p + \frac{1}{2}\rho\gamma(1-\theta)(p - c_B)}\right);$$

in the second period, given the two retailers' total sales $R^N_1 := R^N_{A,1} + R^N_{B,1}$, $y^N_{A,2} = y^0_A + \frac{1}{2}\gamma R^N_1$ *and* $y^N_{B,2} = y^0_B + \frac{1}{2}\gamma R^N_1$*. Moreover, the retailers' equilibrium sales and profits are specified in appendix.*

Despite the similarities in solution format between the two scenarios, as shown in Lemma 3, we highlight a couple of interesting differences here. First, for retailer A, we have $\zeta^N_A < \zeta^S_A$ (see proof in appendix). Thus, although the non-specific network effect also tends to increase the underage cost, it does not increase it as much as the specific network effect does. After all, the increased sales expand the common, not individual, market.

Second, for retailer B, its critical z-score and order quantity become quite involved. Particularly, retailer B's objective function contains a part proportional to retailer A's first-period sales, and this part is decreasing in retailer B's order quantity at equilibrium. Indeed, when retailer B orders more, there is less demand spillover to retailer A, negatively affecting retailer A's sales and the market expansion, which essentially increases retailer B's overage cost. Therefore, since the rival's sales is helpful to its profit, retailer B may somehow "free-ride" retailer A's order quantity – it saves on ordering cost and still enjoys the benefit of the network effect, an enlarged market.

4. Main Results

Given the equilibrium outcomes obtained above, in this section, we present the main results regarding the comparison between different scenarios. While the benchmark scenario provides a basic characterization of the duopoly game without network effects, we will mainly compare Scenarios S and N and examine the retailers' ordering strategies and financial performances in the presence of the two types of network effects respectively. Although network effects in general will benefit both retailers, we are interested in the relative improvement of their profits with the different types of network effects. The specific network effect is competitive in nature, whereas the non-specific network effect has a cooperative feature. Therefore, it is interesting to investigate in each scenario whether the duopoly competition is intensified or softened and, more importantly, whether each retailer enjoys more or less benefit from the network effect induced demand.

In the following, we look into the equilibrium order quantities, sales volumes, and profits of the retailers. Since the second (i.e., the last) period would be controlled by the end-of-horizon effect, we focus on the first-period equilibrium to highlight the impact of the network effect across periods. Three parameters are of particular interests, namely, the network effect strength γ (measuring the benefit of the network effect), the stock-out based substitution factor θ (measuring the degree of competition), and the cost difference $\Delta c := c_B - c_A$ (measuring the asymmetry level of the retailers). Our results and discussions will anchor on the three parameters. Lastly, to maintain tractability of the analysis and provide clean results with clear managerial insights, we will set the discount factor $\rho = 1$ and assume uniform distribution for the seed demand for the rest of the paper; that is, let $X_t \sim U[\mu - \sigma, \mu + \sigma]$ for $t = 1, 2$. We have numerically tested the robustness of this assumption and our results will remain qualitatively the same if other distribution is assumed.

4.1. Order Quantities

We start by examining the retailers' equilibrium order quantities from two aspects. First, we identify the unilateral substitution effect in demand across the two firms. Second, we compare the order quantities of each retailer between scenarios, which showcases the different impacts caused by the specific and the non-specific network effect.

LEMMA 4. (*a*) *In the benchmark scenario, retailer A always has a higher stock level than retailer B, i.e.,* $y_A^0 > y_B^0$.

(*b*) *Consider Scenario k* ($k = S, N$)*. In the first period, retailer A orders more than retailer B, i.e.,* $y_{A,1}^k > y_{B,1}^k$*. In the second period, retailer A still has a higher stock level than retailer B after they both satisfy the demand generated by the network effect, i.e.,* $y_{A,2}^k - Z_{A,2}^k > y_{B,2}^k - Z_{B,2}^k$.

Lemma 4 states that in all scenarios, retailer A always has more supplies than retailer B. Indeed, the order quantities depend on the critical z-scores, which further depend on the system parameters, especially on the order cost c_i. Therefore, the dominance of retailer A's stock level is mainly due to the cost advantage it has over retailer B. Posting the same retail price, the retailer with less ordering cost tend to order more. This feature is unaltered even in the presence of network effects: According to Lemma 4(b), retailer A orders more in the first period, and in the second period, it has more leftover supplies after satisfying the network effect induced demand. In fact, Lemmas 1-3 indicate that $y^k_{i,2} - Z^k_{i,2} = y^0_i$, i.e., the duopoly competition in the second period after fulfilling the demand $Z_{i,2}$ becomes the same as the single-period game.

One important implication of Lemma 4 is that there exists a unilateral demand spillover, from retailer B to retailer A. Specifically, since the two retailers face the same demand in the first period, i.e., $X_1/2$, stock-out based substitution can only occur when retailer B fails to satisfy all the demand and the retailer A has excess supplies. Similarly, in the second period for Scenarios S and N, the retailers' stocks are first used to fill the demand from newly induced customers; then, the remaining stocks are facing the same demand $X_2/2$. Hence, the demand spillover, if exists, should only be possible that part of retailer B's customers turn to seek purchase from retailer A. Therefore, at the equilibrium in any situation, we can only observe such one-sided substitution.

Knowing that the demand spillover is always from retailer B to retailer A, we can better understand the incentives of each retailer's ordering behavior. In fact, the characteristics of the duopoly game are largely affected by the one-sided demand substitution. Retailer A is likely to increase its order, compared to the non-competitive case, in order to accommodate the excess demand from retailer B. This asymmetric demand spillover has its root in the cost asymmetry, and retailer A, who is in the advantageous position, tends to benefit from the substitution; i.e., the larger the degree of competition θ is, the more beneficial the demand spillover is to retailer A. For retailer B, on the other hand, the higher ordering cost is to its disadvantage. However, as we will see later, the impact of the unilateral demand spillover on retailer B is not necessarily all negative when the network effect is brought into the equation.

Next, we study the equilibrium order quantity for each retailer. Our focus is on the retailers' ordering behavior in the first period to highlight the impact of market dynamics caused by the network effect. The proposition below summarizes the results concerning retailer A.

Proposition 1. *Retailer A's equilibrium first-period order quantity in different scenarios, namely y^0_A, $y^S_{A,1}$, and $y^N_{A,1}$, are all increasing in θ. Moreover, when γ is large, the following statements hold.*

(*a*) *There exists a $\underline{\theta}^{S0} \in [0,1]$ such that $y^S_{A,1} \le y^0_A$ if $c_A < \frac{1}{2}c_B$ and $\theta \in [\underline{\theta}^{S0}, 1]$; moreover, there exist $\underline{\theta}^{N0}, \overline{\theta}^{N0} \in [0,1]$ such that $y^N_{A,1} \le y^0_A$ if $\theta \in [\underline{\theta}^{N0}, \overline{\theta}^{N0}]$.*

(*b*) *There exists a $\underline{\theta}^{SN} \in [0,1]$ such that $y^S_{A,1} \ge y^N_{A,1}$ for $\theta \in [0, \underline{\theta}^{SN}]$ and $y^S_{A,1} \le y^N_{A,1}$ for $\theta \in [\underline{\theta}^{SN}, 1]$.*

Proposition 1 shows that retailer A's order quantity increases in the substitution rate θ in all three scenarios. Recall that the stock-out based substitution is one-sided, with the excess demand spilling from retailer B to retailer A. Hence, as θ increases, retailer A will face more potential spillover demand, leading to a larger order quantity in equilibrium. Interestingly, however, the functional dependence on θ is not the same in different scenarios. As shown in the proof (see appendix for details), y_A^0 and $y_{A,1}^S$ is linear, whereas $y_{A,1}^N$ is convex in θ. Therefore, compared to the benchmark scenario, the specific network effect does not change the underlying logic of the duopoly competition: Due to the cost advantage, retailer A enjoys the unilateral demand spillover and the benefit is proportionate to the competition intensity. This fact will not change in the presence of the specific network effect because the substitution independently affects each retailer's individual market expansion. However, when the non-specific network effect is present, the demand spillover in the first period has multiple implications. It increases retailer A's current demand on the one hand, and it expands both retailers' future market on the other. The later sends incentives to retailer B to strategically spill demand over to the rival, which causes retailer A's equilibrium first-period order quantity $y_{A,1}^N$ to have a more complex functional relationship with the competition intensity θ. As such, the non-specific network effect tends to make the duopoly game more involved.

Proposition 1(a) indicates that retailer A may or may not order more in Scenario S/N than what it would order in the benchmark scenario. Note that the conditions include γ being large, meaning that the network effect should be strong enough. Since retailer A always enjoys the one-sided substitution, with a strong network effect (no matter which type), it does not have to order a lot if the demand spillover is not small; by contrast, if θ is small, then retailer A's underage cost will rise facing a strong network effect, and thus its order quantity will be larger than the benchmark case. Hence, strong network effect and intensified competition together lead to a lower first-period order quantity for retailer A compared to benchmark. With the specific network effect, this is true after θ exceeds a threshold, because both $y_{A,1}^S$ and y_A^0 are linear function of θ. However, with the non-specific network effect, the positive effect of a large substitution rate disappears when θ is too large. This is due to the convexity of $y_{A,1}^N$; moreover, it reveals that retailer A's order incentive could be enhanced by the competition intensity when the non-specific network effect is strong enough.

Finally, Scenarios S and N are compared in Proposition 1(b) regarding retailer A's first-period order quantity, which we find may cross each other depending on the system parameters. Basically, retailer A orders more in Scenario S when the substitution rate is small, but it orders more in Scenario N otherwise. Compared to the specific network effect, when the non-specific network effect is present, retailer A's first-period sales are used to expand the total market, only half of which contributes to its own future market. Hence, retailer A faces a smaller underage cost in Scenario N than in Scenario S, especially when there is only limited demand spillover. As a result, $y_{A,1}^N \leq y_{A,1}^S$ when θ is small. On the other hand, when θ is large, the overage cost for retailer A also becomes small.

Besides, seeing a large substitution rate and the presence of the non-specific network effect, retailer B may strategically send its demand to retailer A, hoping the total future market gets enlarged (as shown later in Proposition 2). This will further lower retailer A's overage cost. However, retailer B does not have such an incentive in Scenario S, where each retailer is responsible for its own market expansion. Therefore, with a large demand spillover rate, we have $y^N_{A,1} \geq y^S_{A,1}$.

Now, we turn to study retailer B's ordering decision in each scenario. The comparison results are given in the next proposition.

Proposition 2. *Consider retailer B's equilibrium first-period order quantity in different scenarios. The following statements hold.*

(a) $y^N_{B,1}$ *is decreasing in* θ*, whereas* y^0_B *and* $y^S_{B,1}$ *are independent of* θ*.*

(b) $y^0_B < y^N_{B,1} < y^S_{B,1}$*.*

(c) As a function of γ*,* $y^S_{B,1} - y^N_{B,1}$ *is first increasing then decreasing (to zero) in* γ*.*

First, Proposition 2(a) shows that how retailer B's order quantity depends on θ is in contrast to the results concerning retailer A. Specifically, only in Scenario N will the substitution rate affect retailer B's order, and the correlation is negative; in other scenarios, however, the substitution rate is irrelevant to retailer B's order decision. Due to the unilateral demand spillover, retailer B is not concerned with the possible stock-out based substitution when there is no network effect, or when the network effect only operates on individual sales. After all, in these scenarios, larger or smaller θ will not affect retailer B's overage or underage costs. In Scenario N, however, the demand spillover does have an indirect impact on retailer B's second-period market size. By spilling the excess demand over to retailer A, the shared market induced by the non-specific network effect may benefit retailer B. Hence, the larger the demand spillover is, the more incentive retailer B has to order less – so it can enjoy the expanded future market with a smaller cost. This explains why $y^N_{B,1}$ is decreasing in θ.

Second, regardless of the system parameters, we always have the same ordered relationship of retailer B's order quantities across scenarios, as given in Proposition 2(b). To understand the comparison $y^0_B < y^N_{B,1}$ and $y^0_B < y^S_{B,1}$, it is intuitive that, specific or non-specific, the network effect always increases the underage cost of retailer B, and thus it tends to increase the order quantity in the first period. In this way, the sales volume could be higher and the second-period market may be larger due to the network effect. For the comparison $y^N_{B,1} < y^S_{B,1}$, i.e., retailer B always orders more in Scenario S than in Scenario N, the underlying reason is twofold. On the one hand, retailer B faces a larger underage cost under the specific, rather than the non-specific, network effect. On the other hand, as discussed previously, the non-specific network effect depresses retailer B's order incentive: It will limit its order quantity to utilize the (total) market expansion at a lower cost. Therefore, under the above two driving forces, retailer B's first-period order quantity in Scenario N is dominated by that in Scenario S.

In summary, although both types of network effects increase retailer B's order quantity, the increase is smaller under the non-specific network effect. Indeed, when the future market expansion is from both retailers' past sales and is going to be shared, the less cost-efficient firm may want to take advantage of such collaborative feature of the network effect and strategically shift excess demand to the rival who is more capable of stocking up to meet the demand. In this sense, retailer B's behavior of limiting its order quantity to send customers to retailer A can be seen as "free-riding" the rival's more cost-efficient stockpiling. The difference $y^S_{B,1} - y^N_{B,1}$ reflects and measures such a free-riding behavior. Proposition 2(c) characterizes how the order difference depends on the strength of the network effect: $y^S_{B,1} - y^N_{B,1}$ is first increasing and then decreasing in γ. Hence, when the network effect is relatively weak, any incremental growth in the strength enlarges retailer B's free-riding behavior; that is, retailer B's order increases less under the non-specific network effect than it does under the specific network effect. However, when the network effect is already strong, as it gets stronger, retailer B's order will increase in similar magnitudes in both Scenarios S and N, rendering the free-riding behavior diminishing in scale, even approaching zero.

4.2. Sales

Before investigating the retailers' equilibrium profits, we use this subsection to first examine their first-period expected sales in different scenarios. The reason why we focus on the sales in the first period is threefold. First, in our two-period dynamic model, the first-period sales are the only effective state variables and thus deserve attention. Second, together with the order quantities, the sales can offer useful insights into retailers' profit generating performance. Third, most importantly, it is through the retailers' sales in the first period that the network effect influences the market dynamics, so studying sales helps us understand how retailers' strategies are formed.

We start with the individual retailer's first-period sales under different types of network effects.

Proposition 3. *Consider retailer i's equilibrium first-period sales in Scenario k, i.e.,* $R^k_{i,1}$ *for* $i = A, B$ *and* $k = S, N$. *The following statements hold.*

(a) $\mathbb{E}[R^S_{A,1}]$ *and* $\mathbb{E}[R^N_{A,1}]$ *are increasing in* θ. *Moreover, when* γ *is large, there exists a* $\underline{\theta}' \in [0,1]$ *such that* $\mathbb{E}[R^S_{A,1}] \geq \mathbb{E}[R^N_{A,1}]$ *for* $\theta \in [0, \underline{\theta}']$ *and* $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ *for* $\theta \in [\underline{\theta}', 1]$.

(b) $\mathbb{E}[R^N_{B,1}]$ *is decreasing in* θ, *whereas* $\mathbb{E}[R^S_{B,1}]$ *is independent of* θ. *Moreover,* $\mathbb{E}[R^N_{B,1}] < \mathbb{E}[R^S_{B,1}]$.

It is noteworthy that the monotonicity and the relative magnitude of each retailer's sales in Scenarios S and N are consistent with those of their order quantities; see Propositions 1&2. For retailer B, due to the unilateral demand spillover, it is clear that its sales $R^k_{B,1} = \min\{X_1/2, y^k_{B,1}\}$ has the same property as its order $y^k_{B,1}$ in Scenario k. For retailer A, however, it is not straightforward to have the above result. In particular, the threshold in Proposition 3(a) is different from that in Proposition 1(b). Yet, qualitatively, we obtain a similar result that retailer A sells more [less] in Scenario S when the substitution rate θ is small [large]. Therefore, our finding indicates that the retailers' sales are mostly determined by its own order, even in the presence of stock-out based substitution.

Under the non-specific network effect, the market expansion is based on the total, rather than individual, sales. Hence, for Scenario N, we scrutinize the retailers' total sales in the first period.

Proposition 4. *Consider retailers' total equilibrium first-period sales in Scenario N, i.e., R^N_1. There exists a $\gamma^R > 0$ such that, if $\gamma > \gamma^R$, $\mathbb{E}[R^N_1]$ is first decreasing and then increasing in θ.*

The above result does not directly follow from Proposition 3, because, with the non-specific network effect, the two retailers' sales change in opposite directions as θ increases: Retailer A's sales is boosted whereas retailer B's sales diminish. As a result, as shown in Proposition 4, the impact of the substitution rate on the total sales R^N_1 is non-monotone if the non-specific network effect is strong enough. Basically, as θ increases, each retailer receives an incentive to change its order quantity. For retailer A, it tends to order more so that the sales go up to drive the next period demand via the network effect, and this incentive gets stronger when θ is larger (more demand spillover increases its underage cost). For retailer B, it decides to order less to free-ride the rival's sales and take advantage of the expanded common market; and this incentive is diminishing as θ becomes large (more demand spillover makes the free-riding more effective so it does not have to sacrifice too much first-period demand). Consequently, the total sales first decreases and then increases in θ. In summary, the non-monotone impact of substitution rate on the two retailers' total first-period sales offers a perspective for us to understand how the non-specific network effect influences the retailers' profits, as we will see in the next subsections.

4.3. Profits

In this subsection, we investigate the impact of different types of network effects on the retailers' equilibrium profits. The focus is on the comparison between Scenarios S and N, so that insights could be shed on how the results of the duopoly competition are driven by the network effect with contrasting natures. Our findings pivot on three parameters, namely, the stock-out based substitution rate θ, the retailers' cost difference Δc, and the strength of the network effect γ. In the duopoly competition, due to the cost asymmetry, retailer A enjoys an advantage against retailer B; moreover, this advantage is naturally enhanced when θ or Δc increases. Recall that the specific network effect tends to ignite fierce competition while the non-specific network effect fosters collaboration between firms. Hence, one may intuit that in Scenario S, the competition is intensified by the specific network effect, and retailer A will have more advantage and thus better outcome; by contrast, in Scenario N, the non-specific network effect will soften the competition, which is beneficial to retailer B. However, our results counter the above intuition by showing that there are exceptions in both cases. That is, under certain conditions, retailer A [B] may be better [worse] off in Scenario N than in Scenario S. In the following, Π_i^k is retailer i's total profit in Scenario k $(i = A, B; k = S, N)$.

For retailer A, the next proposition gives the monotonicity property of its profit and characterizes the situation where Π_A^S and Π_A^N cross each other.

Proposition 5. *Consider retailer A's equilibrium profit under different types of network effects. The following statements hold.*

(a) Π_A^S *is increasing in* θ.

(b) *There exist three thresholds,* $\Delta^A \in (0, p - c_A)$, $\gamma^A > 0$, *and* $\theta^A \in [0, 1]$, *such that, if* $\Delta c \geq \Delta^A$ *and* $\gamma \geq \gamma^A$, *then we have* $\Pi_A^S \geq \Pi_A^N$ *for* $\theta \in [0, \theta^A]$ *and* $\Pi_A^S \leq \Pi_A^N$ *for* $\theta \in [\theta^A, 1]$.

Since retailer A is in the advantageous position in the duopoly competition, the substitution rate θ, which represents the intensity of the competition, is positively correlated with retailer A's total profit, regardless which type of the network effect is in presence. The unilateral demand spillover to the more cost-efficient retailer is the main reason behind this result.

When comparing retailer A's equilibrium profit in Scenarios S and N, we identify a counter-intuitive case where $\Pi_A^S \leq \Pi_A^N$. In particular, this case occurs when θ, Δc, and γ are all relatively large. As we mentioned, with a strong non-specific network effect, retailer B tends to order less in the expectation that retailer A gets more demand and the future market expands more. This effect is enhanced when the substitution rate θ and the cost difference are also large – because retailer B's "free-riding" behavior becomes even more efficient. This behavior in turn benefits retailer A's sales and profit (not only the first-period profit but also the second-period because the market is expanded more). On the other hand, under a strong specific network effect, retailer B is taking a rather different action, which is to order more quantity to drive up its own sales so that its own future market can be enlarged more. As a consequence, retailer A will get less substitution demand and its profit performance will not be as good as in Scenario N.

Another angle of viewing this result is from the competition and collaboration relationship between the two retailers. The conditions on parameters θ and Δc (i.e., they are both large) imply that the duopoly competition is intensified and retailer A is in a very advantageous position. With large γ, the retailers also face a strong network effect, and in Scenario S [N], a strong network effect means more [less] competition. Hence, when the competition is already intense, an even escalated competition level (e.g., due to the specific network effect) is not necessarily always beneficial to the advantageous firm; rather, introducing some collaborative element (such as the non-specific network effect) may be a better strategy.

Next, we look at retailer B's profit in different scenarios. Similarly, we examine how its profit depends on the system parameters and how the profit changes across scenarios. Our findings are in parallel with those for retailer A.

Proposition 6. *Consider retailer B's equilibrium profit under different types of network effects. The following statements hold.*

(*a*) Π_B^S *is independent of* θ*, whereas* Π_B^N *is increasing in* θ*.*

(*b*) *There exist three thresholds,* $\Delta^B > 0$*,* $\gamma^B > 0$*, and* $\theta^B \in [0,1]$*, such that, if* $0 < \Delta c \le \Delta^B$ *and* $\gamma \ge \gamma^B$*, then we have* $\Pi_B^S \ge \Pi_B^N$ *for* $\theta \in [0, \theta^B]$ *and* $\Pi_B^S \le \Pi_B^N$ *for* $\theta \in [\theta^B, 1]$*.*

Here, retailer B's profit is independent of the substitution rate in Scenario S but increases in θ in Scenario N. Under the specific network effect, the two retailers are relatively independent in terms of utilizing the network effect to generate new demand. Due to the one-sided substitution, retailer B never receives excess demand from retailer A, and therefore its profit does not interact with the substitution rate. Under the non-specific network effect, however, the demand spillover from retailer B to retailer A turns out to have an impact back on retailer B, because the future market induced by the network effect is for both retailers. Furthermore, we show that this positive impact of θ on Π_B^N dominates the negative impact related to disadvantageous position in the competition, resulting in the increasing relationship given in Proposition 6(a).

In Scenario N, retailer B has the incentive to order less and "free-ride" retailer A's first-period sales to get a larger second-period market. By contrast, in Scenario S, retailer B has to depend on itself, so it tends to order more in the first-period (and thus has to sacrifice its first-period profit) for the sake of network effects induced market expansion. In fact, retailer B always obtains higher first-period profit in Scenario N. However, the benefit of the non-specific network effect for retailer B does not seem to always extend to the second period. The comparison between Π_B^S and Π_B^N again provides a counter-intuitive situation where retailer B is worse off in Scenario N. In this case, the substitution rate is relatively small, the two retailers have a small cost difference, and the network effect is strong. The underlying logic goes as follows. Seeing a strong non-specific network effect, retailer B wants to send many customers to retailer A by ordering small quantity, but the demand spillover is only moderate due to a small substitution rate. As a result, retailer B's second-period profit is hurt. On the other hand, with a strong specific network effect, retailer B could achieve a high second-period profit when it orders large quantity to obtain high sales in the first period, especially when the competition is soft (both substitution rate and the cost difference are small). Therefore, retailer B's total profit turns to be higher in Scenario S than in Scenario N.

From the viewpoint of the duopoly competitive/collaborative relationship, the above results can be interpreted as follows. When θ and Δc are small, the duopoly competition is relaxed. Particularly, retailer B's disadvantage with respect to the unilateral demand spillover becomes alleviated and the retailers' cost gap tends to close up. In this case, will retailer B continues to benefit further when the competition intensity is reduced even more? Not necessarily. In fact, the more collaborative-oriented non-specific network effect could actually hurt retailer B's profit, compared to when the competition-inducing specific network effect is present.

The above comparison results generate interesting insights into competing firms' relationship. Facing multiple levers to adjust the degrees of competition and collaboration between rivalries, a healthy duopoly relationship should not go to extremities. Adding competitiveness may not always benefit the advantageous firm when the competition is already intense; and, facing already softened competition, the disadvantageous firm may not always prefer collaborative initiatives. The proper level of competition is likely to be a mix of both.

5. Pricing Flexibility for Retailer A

Up to now, we have assumed that the two retailers post the same price p in both periods, and this retail price is exogenous. This is the reason why the split of the seed demand is always half-half – the customers are assumed to make their initial choice of purchase by just looking at the posted price. In other words, although the retailers are not symmetric in nature due to the cost difference, they appear the same to customers and each attract half of the market initially. The primary goal of this section, therefore, is to include pricing strategy, an important marketing lever, into the problem and study how the network effect affects the pricing. Price-setting newsvendors' problem, in monopoly or in competitive settings, has been intensively studied in the literature (see, e.g., Petruzzi and Dada 1999, Zhao and Atkins 2008, Salinger and Ampudia 2011). Departing from the prior works, our paper focuses on the impact of the different types of network effects in a dynamic duopoly game. Hence, to concentrate on the focal features, we will consider the pricing flexibility for retailer A only. Specifically, we make the following assumptions in this section.

Assumption 1. *Retailer A has a certain level of pricing flexibility, whereas retailer B does not. That is, retailer A can set its retail price but retailer B's price is fixed at p.*

We justify the above assumption from two aspects. First, since retailer A is advantageous against retailer B in term of cost, it naturally has more rooms to price to its own benefit. In our online retailing contexts, firms often face intense competition, and near-zero profit margin is not uncommon across the industry. Hence, once the two retailers engage in a price war, retailer B will be the first one to reach the minimum price for survival. From there on, retailer A still has some pricing flexibility, but retailer B does not anymore. Second, endowing retailer A the pricing flexibility complements our study. From previous sections, we identify a case that retailer B can "free-ride" retailer A's sales under the non-specific network effect to enjoy a larger future market. Having the same posted price, retailer A has no way to counteract retailer B's actions. Therefore, in this section, we particularly investigate retailer A's pricing strategy as a counteraction against retailer B's order decision under different types of network effects. Following the common practice in online retailing, retailer A's pricing decision is assumed to take the form of price discount: In period t, there is a price markdown $e_t \geq 0$ from retailer A, and thus its effective price is $p_{A,t} = p - e_t \leq p$.

Assumption 2. *Retailer A decides its pricing strategy $\{e_t : t = 1, 2\}$ before the two-period inventory competition and commits to it throughout the game.*

This assumption simply states that retailer A's markdown pricing strategy is a longer-term decision than the inventory ordering decision. Indeed, newsvendor model is for perishable products or products with short life cycle, and the classic model does not even consider the pricing decision. Hence, in a newsvendor model, the pricing decision by default is at the long-term strategic level, rather than at the short-term operational level. Practically, there are real-life evidences that support what we highlight here, namely, firms typically adjust prices less often than making ordering decisions. According to the study conducted by Gautier et al. (2022), only 12.3% of products (out of 166 products) in Europe see price changes within a month, and most of the changes are due to sales and promotions; moreover, the study shows that similar result holds for different countries. Therefore, in our model, the price markdown e_t $(t = 1, 2)$ is decided by retailer A up front and is committed to during the ensuing dynamic inventory competition. We further assume $e_t \leq \Delta c$ so that retailer A still has the advantage in margin after offering discount (as a result, the stock-out based substitution is still one-sided).

Assumption 3. *Given retailer A's price markdown e_t, the splitting rule of the common market is given by $\alpha_{A,t} = 1/2 + \beta e_t$, and $\alpha_{B,t} = 1 - \alpha_{A,t}$ for $t = 1, 2$, where $\beta > 0$ is an exogenous parameter.*

Our assumption concerning the demand allocation captures the natural intuition that deeper price markdown draws more customers. It is a direct generalization of the splitting rule in the base model where both retailers set the same price. In fact, one way to interpret the assumed splitting rule is to replace β with β'/p; then, it means that the extra market share gained by offering promotion is proportional to the percentage price discount (i.e., the discount rate e_t/p). Furthermore, the functional form of the demand allocation has a sound theoretical foundation: From consumer choice perspective, the splitting of the common market, $\alpha_{i,t}$, can be derived from the Hotelling model with chosen parameters (assuming full market coverage and setting the transportation cost as $(2\beta)^{-1}$). Finally, it is worth noting that, under the specific network effect, the price-dependent splitting rule only applies to the seed demand, because the network effect induced market is not for share. Under the non-specific network effect, however, it is the seed market plus the newly generated market that is split based on the factor $\alpha_{i,t}$.

Incorporating the above assumptions, we modify the model formulated in Section 3 by adding a Stage 0 before the two-period game, which is retailer A's strategic decision on the markdowns in the two periods. Therefore, given the pricing strategy, we may solve the dynamic duopoly game in the same way as before, yielding retailers' equilibrium inventory decisions and retailer A's profit as a function of price markdowns, i.e., $\Pi_A^k(e_1, e_2)$ for Scenario $k = S, N$. Note that the initial allocation equations are given by (1), not by (2) or (3) anymore. In addition, we keep assuming the seed demand follows a uniform distribution and the second-period profit is not discounted, i.e., $\rho = 1$. Then, the modified model can be directly solved by optimizing retailer A's profit:

$$\max_{0 \le e_1, e_2 \le \Delta c} \Pi_A^k(e_1, e_2).$$

In the following, we attempt to answer three questions. How does the pricing strategy affect the equilibrium results of the dynamic duopoly game? How to characterize retailer A's optimal markdown pricing strategy in different scenarios? How does retailer A's optimal profit under different types of network effects compare with each other?

5.1. The Impact of the Price Markdowns on Order Quantities

Given a pricing strategy, after solving the two-period competitive newsvendor game, the equilibrium outcomes are all functions of (e_1, e_2). This subsection focuses on the retailers' equilibrium first-period order quantities and conducts comparative statics analysis to identify some interesting monotonicity properties in different scenarios.

Proposition 7. *Consider retailer i's equilibrium first-period order quantity* $y_{i,1}^k$ *as a function of* (e_1, e_2) *in Scenario* k $(i = A, B,\ k = S, N)$.

(*a*) *There exists a* $\gamma_1^D > 0$ *such that if* $\gamma > \gamma_1^D$, *then* $y_{A,1}^k$ *increases in* e_1 *and* $y_{B,1}^k$ *decreases in* e_1.

(*b*) *In Scenario S,* $y_{A,1}^S$ *decreases in* e_2 *whereas* $y_{B,1}^S$ *is independent of* e_2.

(*c*) *In Scenario N, there exist three thresholds,* $\theta^D \in [0,1]$, $\beta^D > 0$, *and* $\gamma_2^D > 0$, *such that* $y_{A,1}^N$ *decreases in* e_2 *and* $y_{B,1}^N$ *increases in* e_2 *if* $\theta^D \le \theta \le 1$, $0 < \beta < \beta^D$, *and* $\gamma > \gamma_2^D$.

Proposition 7(a) characterizes the monotonicity of $y_{i,1}^k$ regarding the first-period price markdown e_1. Interestingly, the two types of network effects give rise to a similar monotonicity property: When the network effect is strong enough, retailer A's [B's] first-period order quantity is positively [negatively] affected by e_1. The underlying reason for this result is the dominating role of retailers' initial demand allocation in the first period. Specific or non-specific, a strong network effect provides great incentives for retailer A to concentrate on increasing the sales and expand its future market. Thus, as e_1 increases, retailer A enjoys a larger market share in the first period and it must order more to guarantee larger sales. On the other hand, retailer A's reduced price will divert more customers away, leaving a smaller market for retailer B. As such, ordering less in the first period is simply retailer B's response to the declined demand.

Proposition 7(b)&(c) reveals that the monotonicity properties of retailers' order quantities regarding the second-period price markdown e_2 is not the same anymore in different scenarios. In Scenario S, larger e_2 means that retailer A gets less unit margin in the second period, which renders a lower underage cost via the specific network effect; as such, $y_{A,1}^S$ becomes smaller. However, since retailer B is in a disadvantageous position in the duopoly competition and it never gets substitution demand from the rival, its first-period order is independent of retailer A's pricing strategy in the second period.

In Scenario N, due to the interplay of retailers' ordering strategies under the non-specific network effect, the monotonicity properties become more involved. While retailer A's order quantity may still decrease in e_2, just as in Scenario S, it is a completely different case for retailer B. In particular, Proposition 7(c) reveals an interesting situation where retailer A utilizes its pricing flexibility to make retailer B order more in the first period, influencing its free-riding behavior. In this case, retailer B's first-period order quantity, which is independent in e_2 under the specific network effect, becomes increasing in e_2. There are two drivers for this result. First, a larger second-period price markdown leads to a lower underage cost for retailer A in the first period (via a strong network effect), causing its order quantity to decrease. Hence, retailer B has to order more by itself to take advantage of the network effect. Second, larger e_2 decreases retailer B's market allocation, hurting its second-period profit. This sends incentive to retailer B to gain more profit in the first period by ordering more and securing more sales. Note that, while the first driver originates from the collaborative nature of the non-specific network effect, the second driver reflects retailer A's counteraction against retailer B's free-riding behavior.

5.2. Optimal Pricing Strategy

Next, we examine retailer A's optimization problem over its pricing strategy under different types of network effects. Since the objective function $\Pi_A^k(e_1, e_2)$ is complex, it is analytically difficult, if not impossible, to obtain the close form solution. However, based on the properties of the profit functions, we are able to compare retailer A's optimal prices between periods under the specific or the non-specific network effect. Furthermore, we present an interesting finding that retailer A may adopt contrastingly different pricing strategies in Scenarios S versus N.

Proposition 8. *Consider retailer A's optimal pricing strategy under the two types of network effects. There exists a $\gamma_D \geq 0$ such that the following statements hold for $\gamma > \gamma_D$:*

(a) *In Scenario S, we have $e_1^S \geq e_2^S = 0$; therefore $\alpha_{A,1}^S \geq \alpha_{A,2}^S = \frac{1}{2}$ and $p_{A,1}^S \leq p_{A,2}^S = p$.*

(b) *In Scenario N, we have $e_2^N \geq e_1^N \geq 0$; therefore $\alpha_{A,2}^N \geq \alpha_{A,1}^N \geq \frac{1}{2}$ and $p_{A,2}^N \leq p_{A,1}^N \leq p$.*

Proposition 8 reveals that retailer A's optimal pricing strategy may exhibit opposite monotonicity when the network effect is specific vis-à-vis non-specific. If the strength of the network effect is large, then retailer A's prices are increasing across the two periods in Scenario S but decreasing in Scenario N. Let us first understand the underlying drivers of this result. In Scenario S, since the network effect is based on individual sales, when γ is large enough, it makes sense for retailer A to set relatively low first-period price to attract customers and expand its own market. In the second period, retailer A will raise its price, $p_{A,2}^S = p$, and simply not give in profit because there is no future market to consider. Overall, setting increasing prices in this scenario fits the exploration-exploitation intuition, where firms first use low price to "explore" (expand market via the network effect) and then "exploit" the market by pricing high.

In Scenario N, interestingly, retailer A's pricing strategy is against the above intuition. The root cause for retailer A's prices to decrease across periods is that the non-specific network effect induces a common market to share. When the non-specific network effect is strong, the newly generated market can be very large. However, as we mentioned before, for this to work, retailer A invests more in stockpiling and contributes more sales than retailer B, who tends to "free-ride" this benefit. Hence, for retailer A to get its fair share of the expanded market in the second period, it must price low to attract more customers. The lowered price can be seen as retailer A using a marketing lever to counteract retailer B's operational decisions (which could potentially harm retailer A).

Viewing the result from a higher level, we obtain a nice perspective towards the balanced competitive relationship between the duopoly. Under a strong specific network effect, the competition is intensified. Thus, the price competition is getting softened along time; in fact, retailer A in the second period voluntarily forgoes its pricing flexibility, which is a relief of the pressure on retailer B due to the price competition. Under the non-specific network effect, the collaborative aspect of the firms is emphasized. When this effect is strong, the duopoly relationship is tilted towards the non-competitive direction. As a result, retailer A's pricing gets more aggressive as it attempts to seize more market from retailer B. Therefore, similar to the insights derived in Section 4, even with pricing decisions, we see that the competitive-collaborative relationship should not go to extremities but needs to be balanced.

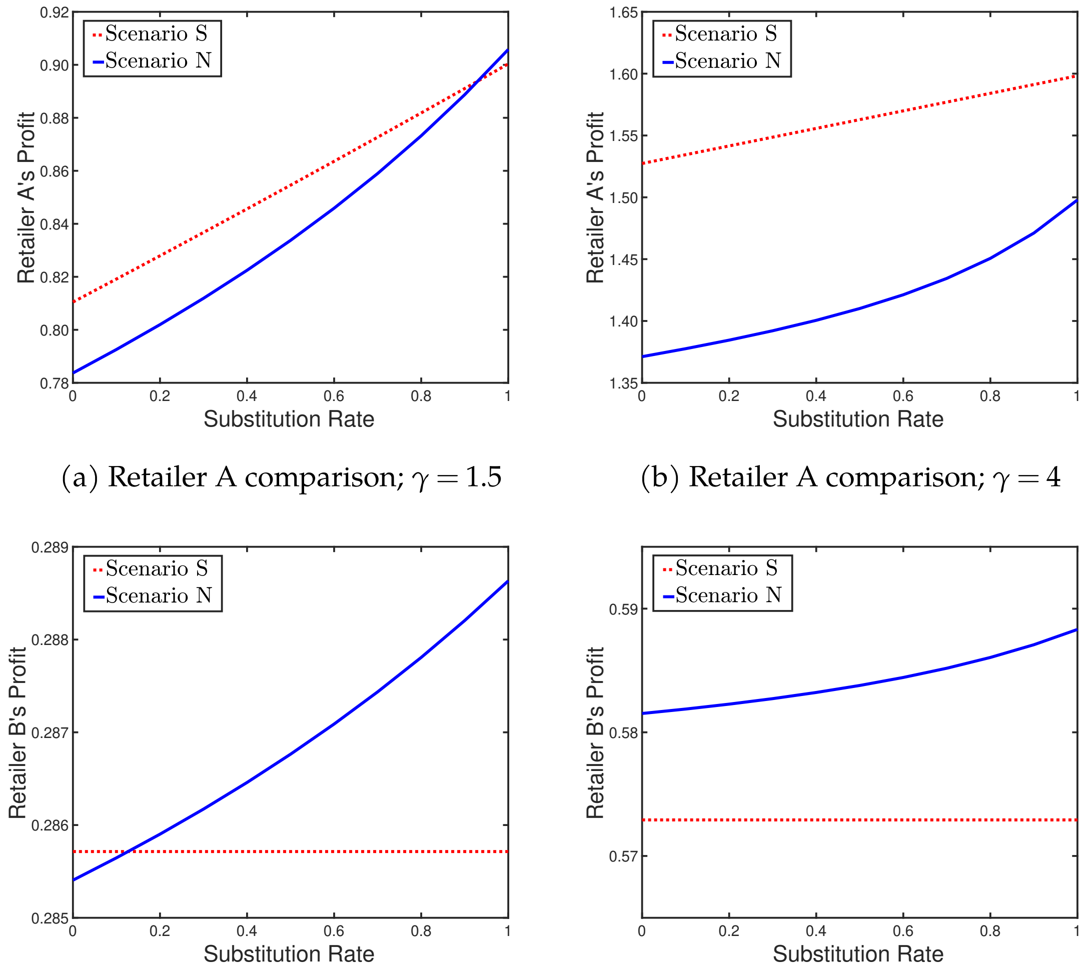
5.3. Comparing Optimal Profit Between Scenarios

As parallel results with those in Section 4.3, retailers' optimal profits are compared between Scenarios S and N. Recall that, in the main model with exogenous identical price, each retailer's profit may cross under the different types of network effects. That is, retailer A may gain more profit in Scenario N whereas retailer B may achieve higher gains in Scenario S. Will this result hold when retailer A possesses pricing flexibility? The main purpose of this subsection is to answer this question. Since solving retailer A's price optimization problem is intractable analytically, we resort to numerical studies and make observations based on extensive experiments. Overall, when retailer A optimally chooses the price markdowns, we can still observe the previous results; however, in this case, we require slightly more restrictive conditions on the network effect strength.

We present a graphical illustration to visualize our findings; see Figure 1. From these graphs, we gather observations regarding the retailers' total profit (after price optimization) comparison in different scenarios, with the network effect being weak and strong, respectively. First, Figure 1(a)&(c) replicate the results in Propositions 5&6 for γ of median value. For retailer A, as θ becomes large (approaching to 1), retailer A performs better in Scenario N than in Scenario S. For retailer B, its profit is higher in Scenario S when θ is not large. Note that our choice of parameters reflects the conditions in those propositions: The strength of network effects is not too small, and the cost difference is large for retailer A's comparison but small for retailer B's.

Second, Figure 1(b)&(d) exemplify a violation of our previous propositions. Indeed, all other parameters being the same, when γ gets even larger, the retailers' profits in the two scenarios will not cross each other anymore; instead, retailer A [B] always achieves higher profit in Scenario S [N]. Hence, when retailer A is allowed to set prices, if the network effect is very strong, then retailers' profit under one type of network effects can dominate that under the other. This means

Figure 1 **Retailers' final equilibrium profits comparison between Scenarios with different network effect strength.**



(a) Retailer A comparison; $\gamma = 1.5$

(b) Retailer A comparison; $\gamma = 4$

(c) Retailer B comparison; $\gamma = 1.5$

(d) Retailer B comparison; $\gamma = 4$

Parameters: We fix $p = 1$, $\beta = 1/4$, and $X_t \sim U[0,1], t = 1,2$. For retailer A's comparison, we choose $c_A = 0.05$, $c_B = 0.5$; and for retailer B's comparison, we choose $c_A = 0.46$, $c_B = 0.5$.

that, for Propositions 5&6 to hold true here, their conditions need to be modified such that γ is upper bounded. Moreover, the underlying reason behind the change is the same for both propositions: With pricing flexibility, retailer A will simply offer aggressive price markdowns, which lets it win more market share on one hand and take advantage of the very *strong* network effect on the other, both driving up retailer A's profit. Note that, under the non-specific network effect, although retailer B would also benefit from the strong network effect, it still cannot obtain more profit because retailer A's aggressive pricing will take away too much market.

6. Concluding Remarks

Live-streaming e-commerce is a highly competitive market where customers, unlike traditional retail, constantly engage in social interactions that shape their purchase decisions. A typical observation in such a context is that past sales can expand the future market – this is referred to as the network effect. We take a novel approach to identify two types of network effects in a competitive environment and distinguish the different ways they regulate the market diffusion process. Under the specific network effect (Scenario S), an individual retailer's market expansion is solely determined by its own past sales. Under the non-specific network effect (Scenario N), however, competing retailers' total past sales together generate new customers who are added to the common market. As such, the two types of network effects inject different dynamics to the system and will lead to contrasting results. To study them in details, we formulate and solve a two-period duopoly competition between newsvendor-type online retailers with asymmetric costs. Three major findings and their managerial implications are highlighted and discussed below.

First, under the non-specific network effect, the high-cost retailer (retailer B) may free-ride the sales of the low-cost retailer (retailer A) and benefit from the common market expansion without paying too much on inventory. Indeed, to take advantage of the expansion of the common market, retailer B can divert excess demand to its rival rather than trying to meet the demand by itself with a higher cost. An implication of this result is that, facing a competitive rival with a lower operating cost and when the non-specific network effect is in presence, an online retailer with a higher cost can utilize the rival's cost efficiency in inventory stockpiling and enjoy an enlarged future total market. For retailer A, its order quantities exhibit different changes between the two scenarios, and the intensity of the duopoly competition is a deciding factor.

Second, the comparison of the retailers' profits between scenarios may sometimes go against the intuition that, the low-cost [high-cost] retailer will do better under the specific [non-specific] network effect. Such a seemingly plausible conjecture could be false, and we identify the conditions under which the opposite is true. Interestingly, the conditions have a consistent implication regarding the duopoly relationship: If the low-cost retailer already retains highly competitive advantages against its rival, then compared to the non-specific network effect, it will not achieve a higher profit under the specific network effect, which escalates the competition; if the high-cost retailer is in a not-so-disadvantageous position, then it can perform better under the specific network effect than the non-specific network effect, which entails a collaborative nature. Thus, online retailers may meet their best interests with a balanced competitive-collaborative relationship.

Third, with pricing flexibility, the low-cost retailer can counteract the high-cost retailer's free-riding by adopting an appropriate price markdown strategy. Here, we find that, under the non-specific network effect, retailer A's price is decreasing across periods. The low price in the second period signifies retailer A's intention to seize a larger portion of the common market, which is expanded (mostly) due to its past sales. In other words, the low-cost retailer is trying to get a fair share of the jointly induced market. By contrast, under the specific network effect, retailer A sets increasing prices because it focuses more on exploiting the increased market in the second period. Moreover, we find that our previous results can largely hold (with slightly more restrictive conditions) when pricing flexibility is allowed. One practical implication is that, based on the interplay between network effects and pricing strategies, the low-cost retailer can leverage its pricing flexibility to its advantage and benefit more from the network effect.

To conclude, we point out some interesting avenues for future research, which could extend this paper in multiple directions. (1) In practice, both types of network effects may co-exist and their joint impact deserves further studies. The co-existence may arise from customers' heterogeneous means of social interactions; it may also come into being because of retailers' different marketing strategies (e.g., whether to clearly distinguish the brand names or not). (2) The online retailers may sell non-perishable products and therefore inventory may be carried over across periods. In this case, the system dynamics must include the changes in inventory position, and the equilibrium order quantities are expected to be more complicated. (3) Viewing through the lens of supply chain management, our model can be extended to include the upstream manufacturers in the distribution channels. The impact of network effects will ripple along the supply chains and, as a result, trigger strategic reactions from different firms.

References

Caro, F., A. G. Kök, V. Martínez-de Albéniz. 2020. The future of retail operations. *Manufacturing & Service Operations Management* **22**(1) 47–58.

Chen, N., Y. J. Chen. 2021. Duopoly competition with network effects in discrete choice models. *Operations Research* **69**(2) 545–559.

Chen, Y., G. Gallego, P. Gao, Y. Li. 2020. Position auctions with endogenous product information: Why live-streaming advertising is thriving. *Working Paper* .

Dong, B., Y. Ren, C. McIntosh. 2023. A co-opetitive newsvendor model with product substitution and a wholesale price contract. *European Journal of Operational Research* **311**(2) 502–514.

Economides, N. 1996. The economics of networks. *International Journal of Industrial Organization* **14**(6) 673–699.

Feng, Q., Z. Wang. 2021. Dynamic multinomial logit choice model with network effect. *Available at SSRN 3939717* .

Feng, Y., M. Hu. 2024. Market entry and competition under network effects. *Operations Research* .

Gautier, E., C. Conflitti, R. P. Faber, B. Fabo, L. Fadejeva, V. Jouvanceau, J. O. Menz, T. Messner, P. Petroulas, P. Roldan-Blanco, et al. 2022. New facts on consumer price rigidity in the euro area. *Available at SSRN 4161422* .

Geng, X., X. Guo, G. Xiao. 2022. Impact of social interactions on duopoly competition with quality considerations. *Management Science* **68**(2) 941–959.

Godes, D., D. Mayzlin. 2004. Using online conversations to study word-of-mouth communication. *Marketing Science* **23**(4) 545–560.

Güler, K., E. Körpeoğlu, A. Şen. 2018. Newsvendor competition under asymmetric cost information. *European Journal of Operational Research* **271**(2) 561–576.

Hall, J., E. Porteus. 2000. Customer service competition in capacitated systems. *Manufacturing & Service Operations Management* **2**(2) 144–165.

Haviv, A., Y. Huang, N. Li. 2020. Intertemporal demand spillover effects on video game platforms. *Management Science* **66**(10) 4788–4807.

Hou, J., H. Shen, F. Xu. 2022. A model of livestream selling with online influencers. *Working Paper* .

Hu, B., Y. Mai, S. Pekeč. 2020. Managing innovation spillover in outsourcing. *Production and Operations Management* **29**(10) 2252–2267.

Jiang, H., S. Netessine, S. Savin. 2011. Robust newsvendor competition under asymmetric information. *Operations Research* **59**(1) 254–261.

Katona, Z., P. P. Zubcsek, M. Sarvary. 2011. Network effects and personal influences: The diffusion of an online social network. *Journal of Marketing Research* **48**(3) 425–443.

Katz, M. L., C. Shapiro. 1994. Systems competition and network effects. *Journal of Economic Perspectives* **8**(2) 93–115.

Krishnan, T. V., D. Vakratsas. 2012. The multiple roles of interpersonal communication in new product growth. *International Journal of Research in Marketing* **29**(3) 292–305.

Libai, B., E. Muller, R. Peres. 2009. The role of within-brand and cross-brand communications in competitive growth. *Journal of Marketing* **73**(3) 19–34.

Lippman, S. A., K. F. McCardle. 1997. The competitive newsboy. *Operations Research* **45**(1) 54–65.

Liu, L., W. Shang, S. Wu. 2007. Dynamic competitive newsvendors with service-sensitive demands. *Manufacturing & Service Operations Management* **9**(1) 84–93.

Nagarajan, M., S. Rajagopalan. 2008. Inventory models for substitutable products: Optimal policies and heuristics. *Management Science* **54**(8) 1453–1466.

Nagarajan, M., S. Rajagopalan. 2009. A multiperiod model of inventory competition. *Operations Research* **57**(3) 785–790.

Netessine, S., N. Rudi. 2003. Centralized and competitive inventory models with demand substitution. *Operations Research* **51**(2) 329–335.

Olsen, T. L., R. P. Parker. 2008. Inventory management under market size dynamics. *Management Science* **54**(10) 1805–1821.

Olsen, T. L., R. P. Parker. 2014. On markov equilibria in dynamic inventory competition. *Operations Research* **62**(2) 332–344.

Parlar, M. 1988. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics* **35**(3) 397–409.

Peres, R., C. Van den Bulte. 2014. When to take or forgo new product exclusivity: Balancing protection from competition against word-of-mouth spillover. *Journal of Marketing* **78**(2) 83–100.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Operations Research* **47**(2) 183–194.

Qi, A., S. Sethi, L. Wei, J. Zhang. 2022. Top or regular influencer? contracting in live-streaming platform selling. *Available at SSRN: 3668390* .

Salinger, M., M. Ampudia. 2011. Simple economics of the price-setting newsvendor problem. *Management Science* **57**(11) 1996–1998.

Shy, O. 2011. A short survey of network economics. *Review of Industrial Organization* **38**(2) 119–149.

Statista. 2023a. Market size of live streaming e-commerce in china from 2019 to 2023 with estimates until 2026. URL `https://www.statista.com/statistics/1127635/china-market-size-of-live-commerce/`. Accessed on April 29, 2023.

Statista. 2023b. Social commerce gross merchandise value (gmv) in the united states from 2023 to 2028. URL `https://www.statista.com/statistics/277045/us-social-commerce-revenue-forecast/`. April 29, 2023.

Straubert, C., E. Sucky. 2023. Inventory competition on electronic marketplaces–a competitive newsvendor problem with a unilateral sales commission fee. *European Journal of Operational Research* **309**(2) 656–670.

Tan, H. 2021. China's lipstick king sold an astonishing $1.7 billion in goods in 12 hours — and that was just in a promotion for the country's biggest shopping day. URL `https://www.businessinsider.com/china-lipstick-king-sold-17-billion-stuff-in-12-hours-2021-10`. *Business Insider*, April 29, 2023.

Wang, R., Z. Wang. 2017. Consumer choice models with endogenous network effects. *Management Science* **63**(11) 3944–3960.

Wongkitrungrueng, A., N. Assarut. 2020. The role of live streaming in building consumer trust and engagement with social commerce sellers. *Journal of Business Research* **117** 543–556.

Zhao, X., D. R. Atkins. 2008. Newsvendors under simultaneous price and inventory competition. *Manufacturing & Service Operations Management* **10**(3) 539–546.

Online Appendices to "Dynamic Competition in Online Retailing: The Implications of Network Effects"

Appendix A: Proof of Statements

A.1. Proof of Statements in Section 3

Proof of Lemma 1: In benchmark scenario, retailer A maximizes its single-period profit in each period t:

$$\max_{y_{A,t}} p\mathbb{E}\left[y_{A,t}\wedge D_{A,t}\right]-c_A y_{A,t}=(p-c_A)y_{A,t}-p\mathbb{E}\left[y_{A,t}-D_{A,t}\right]^+,$$

where $D_{A,t}=\frac{1}{2}X_t+\theta\left[\frac{1}{2}X_t-y_{B,t}\right]^+$. Retailer A's profit is concave in $y_{A,t}$ and its derivative w.r.t $y_{A,t}$ is $p-c_A-p\mathbf{Prob}\left(D_{A,t}\le y_{A,t}\right)$, where

$$\mathbf{Prob}\left(D_{A,t}\le y_{A,t}\right)=\int_0^{2y_{B,t}}\mathbb{1}_{\{z\le 2y_{A,t}\}}f(z)dz+\int_{2y_{B,t}}^{+\infty}\mathbb{1}_{\left\{z\le\frac{2}{1+\theta}y_{A,t}+\frac{2\theta}{1+\theta}y_{B,t}\right\}}f(z)dz. \quad (4)$$

If $y_{A,t}\ge y_{B,t}$, $2y_{B,t}\le\frac{2}{1+\theta}y_{A,t}+\frac{2\theta}{1+\theta}y_{B,t}\le 2y_{A,t}$ for $\theta\in[0,1]$. Thus, by (4), we have $\mathbf{Prob}\left(D_{A,t}\le y_{A,t}\right)=F\left(\frac{2}{1+\theta}y_{A,t}+\frac{2\theta}{1+\theta}y_{B,t}\right)$. If $y_{A,t}<y_{B,t}$, $2y_{A,t}\le\frac{2}{1+\theta}y_{A,t}+\frac{2\theta}{1+\theta}y_{B,t}\le 2y_{B,t}$ for $\theta\in[0,1]$. Thus, by (4), we have $\mathbf{Prob}\left(D_{A,t}\le y_{A,t}\right)=F\left(2y_{A,t}\right)$. Therefore, the best response of retailer A is

$$y^*_{A,t}(y_{B,t})=\begin{cases}\frac{1+\theta}{2}\zeta_A^0-\theta y_{B,t}, & \text{if } y_{B,t}\le\frac{1}{2}\zeta_A^0,\\ \frac{1}{2}\zeta_A^0, & \text{if } y_{B,t}>\frac{1}{2}\zeta_A^0,\end{cases} \quad (5)$$

where $\zeta_A^0:=F^{-1}(1-\frac{c_A}{p})$. On the other hand, retailer B's problem is as follows:

$$\max_{y_{B,t}} p\mathbb{E}\left[y_{B,t}\wedge D_{B,t}\right]-c_B y_{B,t}=(p-c_B)y_{B,t}-p\mathbb{E}\left[y_{B,t}-D_{B,t}\right]^+,$$

where $D_{B,t}=\frac{1}{2}X_t+\theta\left[\frac{1}{2}X_t-y_{A,t}\right]^+$. By analogous arguments, the best response of retailer B is

$$y^*_{B,t}(y_{A,t})=\begin{cases}\frac{1+\theta}{2}\zeta_B^0-\theta y_{A,t}, & \text{if } y_{A,t}\le\frac{1}{2}\zeta_B^0,\\ \frac{1}{2}\zeta_B^0, & \text{if } y_{A,t}>\frac{1}{2}\zeta_B^0,\end{cases} \quad (6)$$

where $\zeta_B^0:=F^{-1}(1-\frac{c_B}{p})$.

Since $c_A<c_B$, we have $\zeta_A^0>\zeta_B^0$. By combining (5) and (6), there exists a unique Nash equilibrium $(y^0_{A,t},y^0_{B,t})$, where $y^0_{A,t}\equiv y^0_A:=\frac{1+\theta}{2}\zeta_A^0-\frac{\theta}{2}\zeta_B^0$ and $y^0_{B,t}\equiv y^0_B:=\frac{1}{2}\zeta^0_B$ for $t=1,2$.

Last, by substituting $y^0_{A,t}$ and $y^0_{B,t}$ into retailers' profit functions, retailers A's and B's equilibrium profit in benchmark scenario is $\pi^0_{A,t}$ and $\pi^0_{B,t}$, respectively: $\pi^0_{A,t}=\pi^0_A:=(p-c_A)y^0_A-p\mathbb{E}[L^0_A]$ and $\pi^0_{B,t}=\pi^0_B:=(p-c_A)y^0_B-p\mathbb{E}[L^0_B]$, where $L^0_A:=\frac{1}{2}\left[(1+\theta)\zeta^0_A-\theta\zeta^0_B-X_t-\theta(X_t-\zeta^0_B)^+\right]^+$ and $L^0_B:=\frac{1}{2}\left[\zeta^0_B-X_t-\theta\left\{X_t-(1+\theta)\zeta^0_A+\theta\zeta^0_B\right\}^+\right]^+$, $t=1,2$. Furthermore, retailer i's equilibrium sales in each period are $R^0_{i,t}=R^0_i:=y^0_i-L^0_i$, $i=A,B$ and $t=1,2$. Thus, we have proved Lemma 1. *Q.E.D.*

Proof of Lemma 2: We first study the subgame in the second period. Given the first-period seed demand X_1 and retailers' first-period sales, retailer A maximizes its second-period profit:

$$\max_{y_{A,2}}\mathbb{E}\left[p(y_{A,2}\wedge D_{A,2})-c_A y_{A,2}|X_1\right]$$

$$= \mathbb{E}\left[(p-c_A)y_{A,2} - p\left(y_{A,2} - \frac{1}{2}X_2 - \gamma R_{A,1} - \theta\left(\frac{1}{2}X_2 + \gamma R_{B,1} - y_{B,2}\right)^+\right)^+ \middle| X_1\right]$$
$$= \mathbb{E}\left[(p-c_A)\overline{y}_{A,2} - p\left(\overline{y}_{A,2} - \frac{1}{2}X_2 - \theta\left(\frac{1}{2}X_2 - \overline{y}_{B,2}\right)^+\right)^+ + (p-c_A)\gamma R_{A,1} \middle| X_1\right] \tag{7}$$

where $D_{A,2} = \frac{1}{2}X_2 + \gamma R_{A,1} + \theta\left[\frac{1}{2}X_2 + \gamma R_{B,1} - y_{B,2}\right]^+$, $\overline{y}_{A,2} := y_{A,2} - \gamma R_{A,1}$, and $\overline{y}_{B,2} := y_{B,2} - \gamma R_{B,1}$. Given firm i's first-period sales, $\overline{y}_{i,2}$ is uniquely determined by $y_{i,2}$. Thus, we use $\overline{y}_{A,2}$ and $\overline{y}_{B,2}$ as firm A's and firm B's second-period decision, respectively. Note that the profit function in (7) is concave in $\overline{y}_{A,2}$. By analogous arguments in the proof of Lemma 1, retailer A's best response in the second period is:

$$\overline{y}^*_{A,2}(\overline{y}_{B,2}) = \begin{cases} \frac{1+\theta}{2}\zeta^0_A - \theta\overline{y}_{B,2}, & \text{if } \overline{y}_{B,2} \le \frac{1}{2}\zeta^0_A, \\ \frac{1}{2}\zeta^0_A, & \text{if } \overline{y}_{B,2} > \frac{1}{2}\zeta^0_A. \end{cases} \tag{8}$$

On the other hand, given X_1 and retailers' first-period sales, retailer B maximizes its second-period profit:

$$\max_{y_{B,2}} \mathbb{E}\left[p(y_{B,2} \wedge D_{B,2}) - c_B y_{B,2} | X_1\right]$$
$$= \mathbb{E}\left[(p-c_B)y_{B,2} - p\left(y_{B,2} - \frac{1}{2}X_2 - \gamma R_{B,1} - \theta\left(\frac{1}{2}X_2 + \gamma R_{A,1} - y_{A,2}\right)^+\right)^+ \middle| X_1\right]$$
$$= \mathbb{E}\left[(p-c_B)\overline{y}_{B,2} - p\left(\overline{y}_{B,2} - \frac{1}{2}X_2 - \theta\left(\frac{1}{2}X_2 - \overline{y}_{A,2}\right)^+\right)^+ + (p-c_B)\gamma R_{B,1} \middle| X_1\right], \tag{9}$$

where $D_{B,2} = \frac{1}{2}X_2 + \gamma R_{B,1} + \theta\left[\frac{1}{2}X_2 + \gamma R_{A,1} - y_{A,2}\right]^+$. Retailer B's profit function in (9) is concave in $\overline{y}_{B,2}$. By analogous arguments in the proof of Lemma 1, retailer B's best response in the second period is:

$$\overline{y}^*_{B,2}(\overline{y}_{A,2}) = \begin{cases} \frac{1+\theta}{2}\zeta^0_B - \theta\overline{y}_{A,2}, & \text{if } \overline{y}_{A,2} \le \frac{1}{2}\zeta^0_B, \\ \frac{1}{2}\zeta^0_B, & \text{if } \overline{y}_{A,2} > \frac{1}{2}\zeta^0_B, \end{cases} \tag{10}$$

By combining (8) and (10), there exists a unique Nash equilibrium in the second period, $(\overline{y}^S_{A,2}, \overline{y}^S_{B,2})$, where $\overline{y}^S_{A,2} = y^0_A$ and $\overline{y}^S_{B,2} = y^0_B$. That is, retailers' equilibrium order quantities are $y^S_{A,2} = y^0_A + \gamma R_{A,1}$ and $y^S_{B,2} = y^0_B + \gamma R_{B,1}$. Furthermore, by substituting $\overline{y}^S_{A,2}$ and $\overline{y}^S_{B,2}$ into (7) and (9), retailer A's second-period profit equals $\pi^0_A + (p-c_A)\gamma\mathbb{E}\left[R_{A,1}|X_1\right]$ and retailer B's second period profit equals $\pi^0_B + (p-c_B)\gamma\mathbb{E}\left[R_{B,1}|X_1\right]$.

Next, we study two retailers' competition in the first period. Firm A's optimizes its order quantity to maximize its profit in two periods:

$$\max_{y_{A,1}} p\mathbb{E}\left[y_{A,1} \wedge D_{A,1}\right] - c_A y_{A,1} + \rho(p-c_A)\gamma\mathbb{E}\left[y_{A,1} \wedge D_{A,1}\right] + \rho\pi^0_A$$
$$= \left[p - c_A + \rho(p-c_A)\gamma\right]y_{A,1} - \left[p + \rho(p-c_A)\gamma\right]\mathbb{E}\left[y_{A,1} - D_{A,1}\right]^+ + \rho\pi^0_A, \tag{11}$$

where $D_{A,1} = \frac{1}{2}X_1 + \theta\left[\frac{1}{2}X_1 - y_{B,1}\right]^+$. Note that profit function in (11) is concave in $y_{A,1}$ and it derivative w.r.t. $y_{A,1}$ is $(p-c_A) + \rho\gamma(p-c_A) - [p + \rho\gamma(p-c_A)]\textbf{Prob}\left(D_{A,1} \le y_{A,1}\right)$, where $\textbf{Prob}\left(D_{A,1} \le y_{A,1}\right)$ follows (4). Thus, by analogous arguments in the proof of Lemma 1, the best response of retailer A is

$$y^*_{A,1}(y_{B,1}) = \begin{cases} \frac{1+\theta}{2}\zeta^S_A - \theta y_{B,1}, & \text{if } y_{B,1} \le \frac{1}{2}\zeta^S_A, \\ \frac{1}{2}\zeta^S_A, & \text{if } y_{B,1} > \frac{1}{2}\zeta^S_A, \end{cases} \tag{12}$$

where $\zeta_A^S := F^{-1}(1 - \frac{c_A}{p+\rho\gamma(p-c_A)})$. On the other hand, retailer B optimizes its order quantity to maximize its total profit in two periods:

$$\begin{aligned}&\max_{y_{B,1}} p\mathbb{E}\left[y_{B,1} \wedge D_{B,1}\right] - c_B y_{B,1} + \rho(p-c_B)\gamma\mathbb{E}\left[y_{B,1} \wedge D_{B,1}\right] + \rho\pi_B^0 \\ =& \ [p - c_B + \rho(p-c_B)\gamma]y_{B,1} - [p + \rho(p-c_B)\gamma]\mathbb{E}\left[y_{B,1} - D_{B,1}\right]^+ + \rho\pi_B^0,\end{aligned} \tag{13}$$

where $D_{B,1} = \frac{1}{2}X_1 + \theta\left[\frac{1}{2}X_1 - y_{A,1}\right]^+$. Note that the profit function in (13) is concave in $y_{B,1}$. By analogous arguments, retailer B's best response is

$$y_{B,1}^*(y_{A,1}) = \begin{cases} \frac{1+\theta}{2}\zeta_B^S - \theta y_{A,1}, & \text{if } y_{A,1} \le \frac{1}{2}\zeta_B^S, \\ \frac{1}{2}\zeta_B^S, & \text{if } y_{A,1} > \frac{1}{2}\zeta_B^S, \end{cases} \tag{14}$$

where $\zeta_B^S := F^{-1}(1 - \frac{c_B}{p+\rho\gamma(p-c_B)})$. Note that $\zeta_i^S > \zeta_i^0$ when $\gamma > 0$ since $\rho > 0$ and $p > c_i$, $i = A, B$.

Since $c_A < c_B$, we have $\frac{c_A}{p+\rho\gamma(p-c_A)} < \frac{c_B}{p+\rho\gamma(p-c_B)}$ and thus $\zeta_A^S > \zeta_B^S$. By combining (12) and (14), there exists a unique Nash equilibrium $(y_{A,1}^S, y_{B,1}^S)$, where $y_{A,1}^S = \frac{1+\theta}{2}\zeta_A^S - \frac{\theta}{2}\zeta_B^S$, $y_{B,1}^S = \frac{1}{2}\zeta_B^S$, and $\zeta_i^S := F^{-1}(1 - \frac{c_i}{p+\rho\gamma(p-c_i)})$, $i = A, B$.

Last, by substituting $y_{A,1}^S$ and $y_{B,1}^S$ into (11) and (13), we have retailers' equilibrium profits as follows: $\Pi_A^S = [p - c_A + \rho(p-c_A)\gamma]y_{A,1}^S - [p + \rho(p-c_A)\gamma]\mathbb{E}[L_{A,1}^S] + \rho\pi_A^0$ and $\Pi_B^S = [p - c_B + \rho(p - c_B)\gamma]y_{B,1}^S - [p + \rho(p-c_B)\gamma]\mathbb{E}[L_{B,1}^S] + \rho\pi_B^0$, where $L_{A,1}^S := \frac{1}{2}\left[(1+\theta)\zeta_A^S - \theta\zeta_B^S - X_1 - \theta(X_1 - \zeta_B^S)^+\right]^+$ and $L_{B,1}^S := \frac{1}{2}\left[\zeta_B^S - X_1 - \theta\left\{X_1 - (1+\theta)\zeta_A^S + \theta\zeta_B^S\right\}^+\right]^+$. Furthermore, retailers' equilibrium first-period sales are $R_{A,1}^S := y_{A,1}^S - L_{A,1}^S$ and $R_{B,1}^S := y_{B,1}^S - L_{B,1}^S$. Retailers' second-period order quantities are $y_{A,2}^S = y_A^0 + \gamma R_{A,1}^S$ and $y_{B,2}^S = y_B^0 + \gamma R_{B,1}^S$. Thus, we have proved Lemma 2. *Q.E.D.*

Proof of Lemma 3: We first study the subgame in the second period. Retailers' demand is $D_{A,2} = \frac{1}{2}X_2 + \frac{1}{2}\gamma R_1 + \theta\left[\frac{1}{2}X_2 + \frac{1}{2}\gamma R_1 - y_{B,2}\right]^+$ and $D_{B,2} = \frac{1}{2}X_2 + \frac{1}{2}\gamma R_1 + \theta\left[\frac{1}{2}X_2 + \frac{1}{2}\gamma R_1 - y_{A,2}\right]^+$. Let $\overline{\overline{y}}_{A,2} := y_{A,2} - \frac{1}{2}\gamma R_1$ and $\overline{\overline{y}}_{B,2} := y_{B,2} - \frac{1}{2}\gamma R_1$, by analogous arguments in the proof of Lemma 2, we have retailer A's second-period best response, which has the same form as (8). Moreover, by analogous arguments in the proof of Lemma 2, we have retailer B's second-period best response, which has the same form as (10).

By combining two retailers' best responses, there exists a unique Nash equilibrium in the second period, $(\overline{\overline{y}}_{A,2}^N, \overline{\overline{y}}_{B,2}^N)$, where $\overline{\overline{y}}_{A,2}^N = y_A^0$ and $\overline{\overline{y}}_{B,2}^N = y_B^0$. That is, retailer A's equilibrium order quantity is $y_{A,2}^N = y_A^0 + \frac{1}{2}\gamma R_1$ and $y_{B,2}^N = y_B^0 + \frac{1}{2}\gamma R_1$. Furthermore, by substituting $\overline{\overline{y}}_{A,2}^N$ and $\overline{\overline{y}}_{B,2}^N$, retailer A's second-period profit equals $\pi_A^0 + \frac{1}{2}(p - c_A)\gamma\mathbb{E}\left[R_1|X_1\right]$ and retailer B's second-period profit equals $\pi_B^0 + \frac{1}{2}(p - c_B)\gamma\mathbb{E}\left[R_1|X_1\right]$.

Next, we study two firms' competition in the first period. Retailer A's maximizes its profit in two periods:

$$\begin{aligned}&\max_{y_{A,1}} p\mathbb{E}\left[y_{A,1} \wedge D_{A,1}\right] - c_A y_{A,1} + \rho\frac{1}{2}(p-c_A)\gamma\mathbb{E}\left[(y_{A,1} \wedge D_{A,1}) + (y_{B,1} \wedge D_{B,1})\right] + \rho\pi_A^0 \\ =& \ [p - c_A + \frac{1}{2}\rho(p-c_A)\gamma]y_{A,1} - [p + \frac{1}{2}\rho(p-c_A)\gamma]\mathbb{E}\left[y_{A,1} - D_{A,1}\right]^+ \\ &+ \frac{1}{2}\rho(p-c_A)\gamma y_{B,1} - \frac{1}{2}\rho(p-c_A)\gamma\mathbb{E}\left[y_{B,1} - D_{B,1}\right]^+ + \rho\pi_A^0,\end{aligned} \tag{15}$$

where $D_{i,1} = \frac{1}{2}X_1 + \theta\left[\frac{1}{2}X_1 - y_{j,1}\right]^+$, $j \neq i$ and $i, j = A, B$. The first-order derivative of profit function in (15) w.r.t. $y_{A,1}$ is $p-c_A+\frac{1}{2}\rho(p-c_A)\gamma-[p+\frac{1}{2}\rho(p-c_A)\gamma]\mathbf{Prob}(D_{A,1}\le y_{A,1})-\frac{1}{2}\rho(p-c_A)\gamma\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}}$, where $\mathbf{Prob}(D_{A,1}\le y_{A,1})$ is same to (4) and

$$\mathbb{E}\left(y_{B,1}-D_{B,1}\right)^+=\int_0^{2y_{A,1}}(y_{B,1}-\frac{1}{2}z)^+f(z)dz+\frac{1+\theta}{2}\int_{2y_{A,1}}^{+\infty}\left(\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}-z\right)^+f(z)dz. \quad (16)$$

If $y_{A,1}\ge y_{B,1}$, $\frac{2}{1+\theta}y_{A,1}+\frac{2\theta}{1+\theta}y_{B,1}\in[2y_{B,1},2y_{A,1}]$ and $\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}\in[2y_{B,1},2y_{A,1}]$ for $\theta\in[0,1]$. Thus, $\mathbf{Prob}\left(D_{A,1}\le y_{A,1}\right)=F\left(\frac{2}{1+\theta}y_{A,1}+\frac{2\theta}{1+\theta}y_{B,1}\right)$ and $\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}}=\frac{\partial}{\partial y_{A,1}}\int_0^{2y_{B,1}}(y_{B,1}-\frac{1}{2}z)f(z)dz=0$. If $y_{A,1}<y_{B,1}$, $\frac{2}{1+\theta}y_{A,1}+\frac{2\theta}{1+\theta}y_{B,1}\in[2y_{A,1},2y_{B,1}]$ and $\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}\in[2y_{A,1},2y_{B,1}]$ for $\theta\in[0,1]$. Thus, $\mathbb{P}\left(D_{A,1}\le y_{A,1}\right)=F\left(2y_{A,1}\right)$ and $\frac{\partial\mathbb{E}[y_{B,1}-D_{B,1}]^+}{\partial y_{A,1}}=\theta\left[F\left(\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}\right)-F\left(2y_{A,1}\right)\right]$. Therefore, if $y_{A,1}\ge y_{B,1}$, the first-order derivative of retailer A's profit function in (15) is $p-c_A+\frac{1}{2}\rho(p-c_A)\gamma-[p+\frac{1}{2}\rho(p-c_A)\gamma]F\left(\frac{2}{1+\theta}y_{A,1}+\frac{2\theta}{1+\theta}y_{B,1}\right)$, and its second-order derivative is $-\frac{2}{1+\theta}[p+\frac{1}{2}\rho(p-c_A)\gamma]f\left(\frac{2}{1+\theta}y_{A,1}+\frac{2\theta}{1+\theta}y_{B,1}\right)$. If $y_{A,1}<y_{B,1}$, the first-order derivative of retailer A's profit function in (15) is $p-c_A+\frac{1}{2}\rho(p-c_A)\gamma-[p+\frac{1}{2}\rho(p-c_A)\gamma(1-\theta)]F\left(2y_{A,1}\right)-\frac{1}{2}\rho(p-c_A)\gamma\theta F\left(\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}\right)$, and its second-order derivative is $-2[p+\frac{1}{2}\rho(p-c_A)\gamma(1-\theta)]f\left(2y_{A,1}\right)-\rho(p-c_A)\gamma\frac{\theta^2}{1+\theta}f\left(\frac{2}{1+\theta}y_{B,1}+\frac{2\theta}{1+\theta}y_{A,1}\right)$. Therefore, given $y_{B,1}\ge 0$, retailer A's profit function in (15) has two concave pieces for $y_{A,1}\ge y_{B,1}$ and $y_{A,1}<y_{B,1}$. Furthermore, the first-order derivative of retailer A's profit function is continuous at $y_{A,1}=y_{B,1}$. Thus, retailer A's profit function is concave in $y_{A,1}$ for $y_{A,1}\ge 0$. Therefore, retailer A's best response is as follows: If $y_{B,1}\le\frac{1}{2}\zeta_A^N$, where $\zeta_A^N:=F^{-1}\left(1-\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}\right)$, $y_{A,1}^*(y_{B,1})=\frac{1+\theta}{2}\zeta_A^N-\theta y_{B,1}$; If $y_{B,1}>\frac{1}{2}\zeta_A^N$, the following equation,

$$p-c_A+\frac{1}{2}\rho(p-c_A)\gamma=[p+\frac{1}{2}\rho(p-c_A)\gamma(1-\theta)]F\left(2y_{A,1}^*(y_{B,1})\right)+\frac{1}{2}\rho(p-c_A)\gamma\theta F\left(\frac{2y_{B,1}+2\theta y_{A,1}^*(y_{B,1})}{1+\theta}\right),$$

implicitly characterizes $y_{A,1}^*(y_{B,1})$. By the above equation, $y_{A,1}^*(y_{B,1})$ is decreasing in $y_{B,1}$ and goes to $\frac{1}{2}F^{-1}\left(1-\frac{c_A}{p+\frac{1}{2}\rho(p-c_A)\gamma(1-\theta)}\right)$ as $y_{B,1}\to+\infty$.

On the other hand, retailer B maximizes its profits in two periods:

$$\begin{aligned}&\max_{y_{B,1}} p\mathbb{E}\left[y_{B,1}\wedge D_{B,1}\right]-c_By_{B,1}+\frac{1}{2}\rho(p-c_B)\gamma\mathbb{E}\left[(y_{A,1}\wedge D_{A,1})+(y_{B,1}\wedge D_{B,1})\right]+\rho\pi_B^0\\ =&\ [p-c_B+\frac{1}{2}\rho(p-c_B)\gamma]y_{B,1}-[p+\frac{1}{2}\rho(p-c_B)\gamma]\mathbb{E}\left[y_{B,1}-D_{B,1}\right]^+\\ &+\frac{1}{2}\rho(p-c_B)\gamma y_{A,1}-\frac{1}{2}\rho(p-c_B)\gamma\mathbb{E}\left[y_{A,1}-D_{A,1}\right]^++\rho\pi_B^0.\end{aligned} \quad (17)$$

By analogous arguments for retailer A's problem, the best response of retailer B is as follows: If $y_{A,1}<\frac{1}{2}F^{-1}\left(1-\frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)}\right)$, $y_{B,1}^*(y_{A,1})=\frac{1+\theta}{2}F^{-1}\left(1-\frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)}\right)-\theta y_{A,1}$; If $y_{A,1}\ge\frac{1}{2}F^{-1}\left(1-\frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)}\right)$, the following equation

$$p-c_B+\frac{1}{2}\rho(p-c_B)\gamma=[p+\frac{1}{2}\rho(p-c_B)\gamma(1-\theta)]F\left(2y_{B,1}^*(y_{A,1})\right)+\frac{1}{2}\rho(p-c_B)\gamma\theta F\left(\frac{2y_{A,1}+2\theta y_{B,1}^*(y_{A,1})}{1+\theta}\right),$$

implicitly characterizes $y_{B,1}^*(y_{A,1})$. By the above equation, $y_{B,1}^*(y_{A,1})$ is decreasing in $y_{A,1}$ and goes to $\frac{1}{2}F^{-1}\left(1-\frac{c_B}{p+\frac{1}{2}\rho(p-c_B)\gamma(1-\theta)}\right)$ as $y_{A,1}\to+\infty$.

We show that there exists a unique Nash equilibrium of the two-period game, $(y_{A,1}^N,y_{B,1}^N)$, which is in area $\{(y_{A,1},y_{B,1}):y_{A,1}\ge y_{B,1}\}$: We first show that there exists no equilibrium in area $\{(y_{A,1},y_{B,1}):y_{A,1}<y_{B,1}\}$,

i.e., two firms' best responses do not intersect each other in the area. We assume, to the contrary, there exists an equilibrium in area $\{(y_{A,1}, y_{B,1}) : y_{A,1} < y_{B,1}\}$. Thus, it must satisfy the following system:

$$\begin{cases} p - c_A + \frac{1}{2}\rho(p-c_A)\gamma = [p + \frac{1}{2}\rho(p-c_A)\gamma(1-\theta)]F(2y_{A,1}) + \frac{1}{2}\rho(p-c_A)\gamma\theta F\left(\frac{2}{1+\theta}y_{B,1} + \frac{2\theta}{1+\theta}y_{A,1}\right) \\ p - c_B + \frac{1}{2}\rho(p-c_B)\gamma = [p + \frac{1}{2}\rho(p-c_B)\gamma]F\left(\frac{2}{1+\theta}y_{B,1} + \frac{2\theta}{1+\theta}y_{A,1}\right). \end{cases} \tag{18}$$

Since $y_{A,1} < y_{B,1}$, $\frac{2}{1+\theta}y_{B,1} + \frac{2\theta}{1+\theta}y_{A,1} \geq y_{B,1} + y_{A,1} > 2y_{A,1}$ for $\theta \in [0,1]$. Thus, $F\left(\frac{2}{1+\theta}y_{B,1} + \frac{2\theta}{1+\theta}y_{A,1}\right) > F(2y_{A,1})$ for $\theta \in [0,1]$. Thus, by the first equation in (18), we have $p - c_A + \frac{1}{2}\rho(p-c_A)\gamma < [p + \frac{1}{2}\rho(p-c_B)\gamma]F\left(\frac{2}{1+\theta}y_{B,1} + \frac{2\theta}{1+\theta}y_{A,1}\right)$, which contradicts with the second equation in (18). Therefore, there exists no equilibrium in area $\{(y_{A,1}, y_{B,1}) : y_{A,1} < y_{B,1}\}$. Next, by the above arguments, both retailers' best responses are decreasing in $y_{B,1}$ in area $\{(y_{A,1}, y_{B,1}) : y_{A,1} \geq y_{B,1}\}$, where retailer A's best response drops from $\frac{1+\theta}{2}\zeta_A^N$ to $\frac{1}{2}\zeta_A^N$ on interval $[0, \frac{1}{2}\zeta_A^N]$ and retailer B's best response drops from infinity to $\frac{1}{2}F^{-1}\left(1 - \frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)}\right)$ on interval $\left(\frac{1}{2}F^{-1}\left(1 - \frac{c_B}{p+\frac{1}{2}\rho(p-c_B)\gamma(1-\theta)}\right), \frac{1}{2}F^{-1}\left(1 - \frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)}\right)\right]$. Therefore, there exists a unique equilibrium, $(y_{A,1}^N, y_{B,1}^N)$, which solves the following system:

$$\begin{cases} p - c_A + \frac{1}{2}\rho(p-c_A)\gamma = [p + \frac{1}{2}\rho(p-c_A)\gamma]F\left(\frac{2}{1+\theta}y_{A,1} + \frac{2\theta}{1+\theta}y_{B,1}\right) \\ p - c_B + \frac{1}{2}\rho(p-c_B)\gamma = [p + \frac{1}{2}\rho(p-c_B)\gamma(1-\theta)]F(2y_{B,1}) + \frac{1}{2}\rho(p-c_B)\gamma\theta F\left(\frac{2}{1+\theta}y_{A,1} + \frac{2\theta}{1+\theta}y_{B,1}\right). \end{cases}$$

By the above system, we have $y_{A,1}^N = \frac{1+\theta}{2}\zeta_A^N - \frac{\theta}{2}\zeta_B^N$ and $y_{B,1}^N = \frac{1}{2}\zeta_B^N$, where

$$\zeta_A^N = F^{-1}\left(1 - \frac{c_A}{p + \frac{1}{2}\rho\gamma(p-c_A)}\right) \text{ and } \zeta_B^N = F^{-1}\left(1 - \frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p + \frac{1}{2}\rho\gamma(1-\theta)(p-c_B)}\right). \tag{19}$$

Moreover, by the definition, we have $\zeta_A^N < \zeta_A^S$.

Last, by substituting $y_{A,1}^N$ and $y_{B,1}^N$ into (15) and (17), retailers' equilibrium profits are $\Pi_A^N := [p - c_A + \frac{1}{2}\rho(p-c_A)\gamma]y_{A,1}^N - [p + \frac{1}{2}\rho(p-c_A)\gamma]\mathbb{E}[L_{A,1}^N] + \frac{1}{2}\rho(p-c_A)\gamma y_{B,1}^N - \frac{1}{2}\rho(p-c_A)\gamma\mathbb{E}[L_{B,1}^N] + \rho\pi_A^0$ and $\Pi_B^N := [p - c_B + \frac{1}{2}\rho(p-c_B)\gamma]y_{B,1}^N - [p + \frac{1}{2}\rho(p-c_B)\gamma]\mathbb{E}[L_{B,1}^N] + \frac{1}{2}\rho(p-c_B)\gamma y_{A,1}^N - \frac{1}{2}\rho(p-c_B)\gamma\mathbb{E}[L_{A,1}^N] + \rho\pi_B^0$, where $L_{A,1}^N := \frac{1}{2}[(1+\theta)\zeta_A^N - \theta\zeta_B^N - X_1 - \theta(X_1 - \zeta_B^N)^+]^+$ and $L_{B,1}^N := \frac{1}{2}[\zeta_B^N - X_1 - \theta\left\{X_1 - (1+\theta)\zeta_A^N + \theta\zeta_B^N\right\}^+]^+$. Furthermore, retailers' equilibrium first-period sales are $R_{A,1}^N := y_{A,1}^N - L_{A,1}^N$, $R_{B,1}^N := y_{B,1}^N - L_{B,1}^N$, and $R_1^N := R_{A,1}^N + R_{B,1}^N$. Retailers' second-period equilibrium order quantities are $y_{A,2}^N = y_A^0 + \frac{1}{2}\gamma R_1^N$ and $y_{B,2}^S = y_B^0 + \frac{1}{2}\gamma R_1^N$. Thus, we have proved Lemma 3. *Q.E.D.*

A.2. Proof of Statements in Section 4

Proof of Lemma 4: (a). By Lemma 1, since $c_A < c_B$ and thus $\zeta_A^0 > \zeta_B^0$, we have $y_A^0 - y_B^0 = \frac{1+\theta}{2}(\zeta_A^0 - \zeta_B^0) > 0$, $t = 1, 2$, for $\theta \in [0,1]$.

(b). We first study Scenario S. By Lemma 2, since $c_A < c_B$ and $\zeta_A^S > \zeta_B^S$, we have $y_{A,1}^S - y_{B,1}^S = \frac{1+\theta}{2}(\zeta_A^S - \zeta_B^S) > 0$. Furthermore, by Lemma 2 and part (a), $y_{A,2}^S - Z_{A,2}^S = y_A^0 > y_B^0 = y_{B,2}^S - Z_{B,2}^S$, for $\theta \in [0,1]$.

Second, we study Scenario N. Since

$$\frac{\partial}{\partial\theta}\frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p + \frac{1}{2}\rho\gamma(1-\theta)(p-c_B)} = \frac{2\gamma p\rho(\gamma\rho+2)(c_B-c_A)(p-c_B)}{(p(\gamma\rho+2) - c_A\gamma\rho)[p(2+\gamma\rho(1-\theta)) - c_B\gamma\rho(1-\theta)]^2} \geq 0 \tag{20}$$

for $\theta \in [0,1]$, we have

$$\frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p+\frac{1}{2}\rho\gamma(1-\theta)(p-c_B)} \geq \frac{c_B}{p+\frac{1}{2}\rho\gamma(p-c_B)} > \frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)},$$

where the first inequality holds by (20) and the second holds by $c_A < c_B$. Thus, by their definition in Lemma 4, we have $\zeta_A^N > \zeta_B^N$. By Lemma 3, $y_{A,1}^N - y_{B,1}^N = \frac{1+\theta}{2}(\zeta_A^N - \zeta_B^N) > 0$. Furthermore, by Lemma 3 and part (a), $y_{A,2}^N - Z_{A,2}^N = y_A^0 > y_B^0 = y_{B,2}^N - Z_{B,2}^N$, for $\theta \in [0,1]$. *Q.E.D.*

Proof of Proposition 1: By Lemma 1, $y_{A,1}^0 = \frac{1+\theta}{2}\zeta_A^0 - \frac{\theta}{2}\zeta_B^0 = \frac{1}{2}\zeta_A^0 + \frac{\theta}{2}(\zeta_A^0 - \zeta_B^0)$. Since $c_A < c_B$, $\zeta_A^0 > \zeta_B^0$ and thus $y_{A,1}^0$ is increasing in $\theta \in [0,1]$. By Lemma 2, $y_{A,1}^S = \frac{1+\theta}{2}\zeta_A^S - \frac{\theta}{2}\zeta_B^S = \frac{1}{2}\zeta_A^S + \frac{\theta}{2}(\zeta_A^S - \zeta_B^S)$. Since $c_A < c_B$, $\zeta_A^S > \zeta_B^S$ and thus $y_{A,1}^S$ is increasing in $\theta \in [0,1]$. By Lemma 3, $y_{A,1}^N = \frac{1+\theta}{2}\zeta_A^N - \frac{\theta}{2}\zeta_B^N = \frac{1}{2}\zeta_A^N + \frac{\theta}{2}(\zeta_A^N - \zeta_B^N)$. By the proof of Lemma 4, we have $\zeta_A^N > \zeta_B^N$. Furthermore, we have

$$\begin{aligned}\frac{\partial \zeta_B^N}{\partial\theta} &= \frac{\partial}{\partial\theta}F^{-1}\left(1 - \frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p+\frac{1}{2}\rho\gamma(1-\theta)(p-c_B)}\right) \\ &= -\frac{1}{f\left(1 - \frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p+\frac{1}{2}\rho\gamma(1-\theta)(p-c_B)}\right)}\frac{\partial}{\partial\theta}\frac{c_B - \frac{1}{2}\rho\gamma\theta(p-c_B)\frac{c_A}{p+\frac{1}{2}\rho\gamma(p-c_A)}}{p+\frac{1}{2}\rho\gamma(1-\theta)(p-c_B)} \leq 0, \end{aligned} \tag{21}$$

where the inequality holds by (20). Thus, we have $\frac{\partial y_{A,1}^N}{\partial\theta} = \frac{1}{2}(\zeta_A^N - \zeta_B^N) - \frac{\theta}{2}\frac{\partial\zeta_B^N}{\partial\theta} > 0$, where the inequality holds by $\zeta_A^N > \zeta_B^N$ and (21). Thus, $y_{A,1}^N$ is increasing in θ.

(a). First, we compare $y_{A,1}^0$ and $y_{A,1}^S$. Since $X_1 \sim U[\mu-\sigma, \mu+\sigma]$, by Lemma 1 and Lemma 2, we have the closed forms of $y_{A,1}^0$ and $y_{A,1}^S$. Note that $y_{A,1}^0$ and $y_{A,1}^S$ are linearly increasing in θ. If $\theta = 0$, we have $y_{A,1}^S - y_{A,1}^0 = \frac{\gamma c_A(p-c_A)\sigma}{p(\gamma p - \gamma c_A + p)} > 0$. If $\theta = 1$, we have $y_{A,1}^S - y_{A,1}^0 = \frac{\gamma\sigma}{p(-\gamma c_A+\gamma p+p)(-\gamma c_B+\gamma p+p)}g_A^{S0}(\gamma)$, where $g_A^{S0}(\gamma) = (2c_A - c_B)(p-c_A)(p-c_B)\gamma + p(-2c_A^2 - c_B(p-c_B) + 2c_A p)$. Note that, if $c_A < \frac{1}{2}c_B$, since γ is large, then $(2c_A - c_B)(p-c_A)(p-c_B) < 0$ and $g_A^{S0}(\gamma) < 0$. Thus, there exists a $\underline{\theta}^{S0}$ such that $y_{A,1}^S \leq y_{A,1}^0$ for $\theta \in [\underline{\theta}^{S0}, 1]$ if $c_A < \frac{1}{2}c_B$.

Second, we compare $y_{A,1}^0$ and $y_{A,1}^N$. Since $X_1 \sim U[\mu-\sigma, \mu+\sigma]$, by Lemma 3, we have the closed forms of $y_{A,1}^N$. Note that $y_{A,1}^N$ is convexly increasing in θ since $\frac{\partial^2 y_{A,1}^N}{\partial\theta^2} = \frac{4\gamma(\gamma+2)p\sigma(c_B-c_A)(p-c_B)[(\gamma+2)p-\gamma c_B]}{((\gamma+2)p-\gamma c_A)[p(\gamma(1-\theta)+2)-\gamma c_B(1-\theta)]^3} \geq 0$. If $\theta = 0$, we have $y_{A,1}^N - y_{A,1}^0 = \frac{\gamma c_A\sigma(p-c_A)}{p((\gamma+2)p-\gamma c_A)} > 0$. If $\theta = 1$, we have $y_{A,1}^N - y_{A,1}^0 = \frac{\gamma c_A\sigma(-2c_A+c_B+p)}{p((\gamma+2)p-\gamma c_A)} > 0$. For $\theta \in (0,1)$, we have $y_{A,1}^N - y_{A,1}^0 = \frac{\gamma\sigma}{p((\gamma+2)p-\gamma c_A)(p(\gamma(1-\theta)+2)+\gamma c_B(1-\theta))}g_A^{N0}(\theta)$, where $g_A^{N0}(\theta) = (c_B - c_A)(p-c_B)((\gamma+2)p - \gamma c_A)\theta^2 + [-2c_A^2 p + c_A(-\gamma c_B^2 + \gamma c_B p + 2p^2) + (\gamma+2)c_B p(c_B - p)]\theta + c_A(p-c_A)((\gamma+2)p - \gamma c_B)$. Note that $(c_B - c_A)(p-c_B)((\gamma+2)p-\gamma c_A) > 0$. Furthermore, when γ is large, the quadratic function $g_A^{N0}(\theta)$ must have two roots on $[0,1]$, i.e., $\underline{\theta}^{N0}, \overline{\theta}^{N0} \in [0,1]$. Thus, $g_A^{N0}(\theta) \leq 0$ (and thus, $y_{A,1}^N \leq y_{A,1}^0$) for $\theta \in [\underline{\theta}^{N0}, \overline{\theta}^{N0}]$.

(b). We already show that $y_{A,1}^S$ is linearly increasing in θ and $y_{A,1}^N$ is convexly increasing in θ. Furthermore, if $\theta = 0$, we have $y_{A,1}^S - y_{A,1}^N = \frac{\gamma c_A\sigma(p-c_A)}{(-\gamma c_A+\gamma p+p)((\gamma+2)p-\gamma c_A)} > 0$. If $\theta = 1$, we have $y_{A,1}^S - y_{A,1}^N = \frac{\sigma\gamma}{(-\gamma c_A+\gamma p+p)((\gamma+2)p-\gamma c_A)(-\gamma c_B+\gamma p+p)}g_A^{SN}(\gamma)$, where $g_A^{SN}(\gamma) := -(c_B - c_A)(p-c_A)(p-c_B)\gamma^2 - (p - c_B)(3c_A^2 + 3c_B p - 2c_A(c_B + 2p))\gamma + p(-2c_A^2 - c_A c_B - 2c_B(p-c_B) + 3c_A p)$ and has the same sign as $y_{A,1}^S - y_{A,1}^N$. Since γ is large, $g_A^{SN}(\gamma) < 0$ and thus $y_{A,1}^S < y_{A,1}^N$ at $\theta = 1$. By the above arguments, there exists a

$\underline{\theta}^{SN} \in [0,1]$ such that $y^S_{A,1} \geq y^N_{A,1}$ for $\theta \in [0, \underline{\theta}^{SN}]$ and $y^S_{A,1} \leq y^N_{A,1}$ for $\theta \in [\underline{\theta}^{SN}, 1]$. *Q.E.D.*

Proof of Proposition 2: (a). By Lemma 1 and Lemma 2, $y^0_{B,1} = \frac{1}{2}\zeta^0_B$ and $y^S_{B,1} = \frac{1}{2}\zeta^S_B$ are both independent of θ. By Lemma 3, $y^N_{B,1} = \frac{1}{2}\zeta^N_B$. By (21), we have $\frac{\partial y^N_{B,1}}{\partial\theta} = \frac{1}{2}\frac{\partial \zeta^N_B}{\partial\theta} \leq 0$. Thus, $y^N_{B,1}$ is decreasing in θ.

(b). When $\gamma > 0$, $y^0_{B,1} < y^N_{B,1} < y^S_{B,1}$ holds if and only if

$$\frac{c_B}{p} > \frac{c_B - \frac{1}{2}\rho\gamma\theta(p - c_B)\frac{c_A}{p + \frac{1}{2}\rho\gamma(p - c_A)}}{p + \frac{1}{2}\rho\gamma(1-\theta)(p - c_B)} > \frac{c_B}{p + \rho\gamma(p - c_B)} \tag{22}$$

for $\theta \in [0,1]$. By (20), we have

$$\frac{c_B - \frac{1}{2}\rho\gamma(p - c_B)\frac{c_A}{p + \frac{1}{2}\rho\gamma(p - c_A)}}{p} \geq \frac{c_B - \frac{1}{2}\rho\gamma\theta(p - c_B)\frac{c_A}{p + \frac{1}{2}\rho\gamma(p - c_A)}}{p + \frac{1}{2}\rho\gamma(1-\theta)(p - c_B)} \geq \frac{c_B}{p + \frac{1}{2}\rho\gamma(p - c_B)}. \tag{23}$$

Furthermore, since

$$\frac{c_B}{p} > \frac{c_B - \frac{1}{2}\rho\gamma(p - c_B)\frac{c_A}{p + \frac{1}{2}\rho\gamma(p - c_A)}}{p} \text{ and } \frac{c_B}{p + \frac{1}{2}\rho\gamma(p - c_B)} > \frac{c_B}{p + \rho\gamma(p - c_B)},$$

(22) holds for all $\theta \in [0,1]$.

(c). By Lemma 2 and Lemma 3, we have $\frac{\partial}{\partial\theta}(y^S_{B,1} - y^N_{B,1}) = (p - c_B)g^{SN}_B(\gamma)/[(-\gamma c_A + \gamma p + 2p)^2(\gamma c_B - \gamma p - p)^2(\gamma c_B\theta - \gamma c_B - \gamma\theta p + \gamma p + 2p)^2]$, where $g^{SN}_B(\gamma) = \sum_{j=0}^4 a^{SN}_j \gamma^j$ with $a^{SN}_4 \leq 0$, $a^{SN}_3 \leq 0$, $a^{SN}_1 \geq 0$, and $a^{SN}_0 \geq 0$. By Descartes' Rule of Signs, $g^{SN}_B(\gamma)$ has only one positive real root and decreases in γ for $\gamma \geq 0$. So, $g^{SN}_B(\gamma)$ is first positive then negative, as γ increases. Hence, $y^S_{B,1} - y^N_{B,1}$ first increases then decreases in γ. Furthermore, $y^S_{B,1} - y^N_{B,1} = \frac{\gamma\sigma(c_B - p)(\gamma c_A c_B(\theta-1) - 2(\gamma+1)c_A\theta p + (\gamma+2)c_B(\theta+1)p)}{((\gamma+2)p - \gamma c_A)(-\gamma c_B + \gamma p + p)(p(\gamma(\theta-1)-2) - \gamma c_B(\theta-1))}$, which goes to zero as $\gamma \to +\infty$. *Q.E.D.*

Proof of Proposition 3: (a). By Lemma 2, we have the closed form of $\mathbb{E}[R^S_{A,1}]$. Since $\frac{\partial \mathbb{E}[R^S_{A,1}]}{\partial\theta} = \frac{(\gamma+1)p\sigma(c_B - c_A)(c_A(-2\gamma c_B + \gamma p + p) + (\gamma+1)c_B p)}{2(-\gamma c_A + \gamma p + p)^2(-\gamma c_B + \gamma p + p)^2} > 0$, $\mathbb{E}[R^S_{A,1}]$ is increasing in θ. Moreover, by Lemma 3, we have the closed form of $\mathbb{E}[R^N_{A,1}]$. Similarly, $\frac{\partial \mathbb{E}[R^N_{A,1}]}{\partial\theta}$ can also be shown to be positive and thus $\mathbb{E}[R^N_{A,1}]$ is increasing in θ. If $\theta = 0$, we have $\mathbb{E}[R^S_{A,1}] - \mathbb{E}[R^N_{A,1}] = \frac{\gamma c_A^2\sigma(p - c_A)(-3\gamma c_A + (3\gamma+4)p)}{2(-\gamma c_A + \gamma p + p)^2(\gamma c_A - (\gamma+2)p)^2} > 0$. If $\theta = 1$, we have $\mathbb{E}[R^S_{A,1}] - \mathbb{E}[R^N_{A,1}] = \frac{\gamma\sigma}{2(-\gamma c_A + \gamma p + p)^2(-\gamma c_A + \gamma p + 2p)^2(-\gamma c_B + \gamma p + p)^2} g^{SN}_R(\gamma)$, where $g^{SN}_R(\gamma)$ is a quartic function of γ with $\frac{\partial^4 g^{SN}_R(\gamma)}{\partial\gamma^4} = -(c_A - c_B)^2(c_A - p)^2(c_B - p)^2 < 0$. Thus, since γ is large, $g^{SN}_R(\gamma) < 0$ and thus $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ at $\theta = 1$. Therefore, there exists a $\underline{\theta}' \in [0,1]$ such that $\mathbb{E}[R^S_{A,1}] \geq \mathbb{E}[R^N_{A,1}]$ for $\theta \in [0, \underline{\theta}']$ and $\mathbb{E}[R^S_{A,1}] < \mathbb{E}[R^N_{A,1}]$ for $\theta \in [\underline{\theta}', 1]$.

(b). Since $R^k_{B,1} = \min\{X_1/2, y^k_{B,1}\}$, part (b) follows Proposition 2. *Q.E.D.*

Proof of Proposition 4: By Lemma 3, we have the closed form of $\mathbb{E}[R^N_1]$. Moreover, $\frac{\partial}{\partial\theta}\mathbb{E}[R^N_1] = \frac{2(c_B - c_A)(2+\gamma)p^2\sigma}{(\gamma c_A - (\gamma+2)p)^2(p(\gamma(1-\theta)+2) - \gamma c_B(1-\theta))^3} g^N_R(\theta)$, where $g^N_R(\theta) = \gamma(p - c_B)((\gamma+6)c_A - (\gamma+2)c_B)\theta + c_A[(\gamma+2)^2 p - \gamma(\gamma+6)c_B] + (\gamma+2)c_B(\gamma c_B - (\gamma-2)p)$, is a linear function of θ, and has the same sign as $\frac{\partial}{\partial\theta}\mathbb{E}[R^N_1]$. Note that, if $\gamma > \gamma^R := \sqrt{\frac{9c_A^2c_B^2 - 8c_A^2c_Bp - 6c_Ac_B^3 + 4c_Ac_B^2p + c_B^4 - 4c_B^3p + 4c_B^2p^2}{(c_A - c_B)^2(c_B - p)^2}} + \frac{-3c_Ac_B + 2c_Ap + c_B^2}{(c_A - c_B)(c_B - p)}$, $g^N_R(0) < 0$. Moreover,

$g_R^{SN}(1) = 2p((\gamma+2)c_B - (\gamma-2)c_A) > 0$ for all $\gamma > 0$. Thus, if $\gamma > \gamma^R$, $\frac{\partial}{\partial\theta}\mathbb{E}[R_1^N]$ is first negative then positive and $\mathbb{E}[R_1^N]$ first decreases then increases in θ for $\theta \in [0,1]$. *Q.E.D.*

Proof of Proposition 5: (a). By Lemma 2&3, we have the closed forms of Π_A^S and Π_A^N. Furthermore, we have $\frac{\partial \Pi_A^S}{\partial\theta} = \frac{\sigma(c_A-c_B)^2}{2p(-\gamma c_A+\gamma p+p)(-\gamma c_B+\gamma p+p)^2} h_A^S(\gamma)$, where $h_A^S(\gamma)$ is cubic function of γ and $h_A^S(\gamma) = \gamma^3(p-c_A)(p-c_B)^2 + \gamma^2\left[-2p^2(c_A+2c_B) + c_B p(2c_A+c_B) + 4p^3\right] + \gamma\left[-p^2(c_A+2c_B) + 5p^3\right] + 2p^3$. Note that $\frac{\partial^3 h_A^S(\gamma)}{\partial\gamma^3} = 6(p-c_A)(p-c_B)^2 > 0$, $h_A^S(0) = 2p^3 > 0$, and the two roots of $\frac{\partial h_A^S(\gamma)}{\partial\gamma} = 0$ are both smaller than 0. Thus, $h_A^S(\gamma) \geq 0$ for $\gamma \geq 0$. So $\frac{\partial \Pi_A^S}{\partial\theta} \geq 0$ and Π_A^S is increasing in θ.

(b). Note that

$$\Pi_A^S - \Pi_A^N = \frac{\gamma\sigma h_A(\theta,\gamma)}{2(-\gamma c_A+\gamma p+p)(-\gamma c_A+\gamma p+2p)^2(-\gamma c_B+\gamma p+p)^2(-\gamma c_B\theta+\gamma c_B+\gamma\theta p-\gamma p-2p)^2},$$

where $h_A(\theta,\gamma)$ is a cubic function of θ and has the same sign as $\Pi_A^S - \Pi_A^N$.

We first show that $\Pi_A^S > \Pi_A^N$ at $\theta = 0$. If $\theta = 0$, $h_A(0,\gamma)$ has the same sign as $[c_A^2\left(c_B^2 + 2c_B p - p^2\right) - 4c_A c_B^2 p + 2c_B^2 p^2]\gamma^2 + 2p(2c_A^2(c_B - p) - 3c_A c_B^2 + 3c_B^2 p)\gamma + 4p^2\left(c_B^2 - c_A^2\right)$. Note that, the polynomial is positive for $\gamma \geq 0$, so we have $h_A(0,\gamma) > 0$. On the other hand, at $\theta = 1$, $h_A(1,\gamma)$ is a quartic function of γ with $\frac{\partial^4 h_A(1,\gamma)}{\partial\gamma^4} = 24(c_A - c_B)(c_A - p)(c_B - p)^2\left(2c_A^2 - 3c_A p + c_B p\right)$. Thus, if $c_A < \frac{3}{4}p - \frac{1}{4}\sqrt{9p^2 - 8c_B p}$, we have $\frac{\partial^4 h_A(1,\gamma)}{\partial\gamma^4} < 0$. Thus, there exists a $\Delta^A \in (0, p - c_A)$ and $\gamma^A > 0$ such that $h_A(1,\gamma) < 0$ if $\Delta c \geq \Delta^A$ and $\gamma \geq \gamma^A$, where γ^A is the maximum of zero and the largest real root of $h_A(1,\gamma) = 0$.

Last, note that $\frac{\partial^3 h_A(\theta,\gamma)}{\partial\theta^3} = 6\gamma(\gamma+1)^2 p^2 (c_A - c_B)^2 (c_B - p)^2 (\gamma c_A - (\gamma+2)p)^2 \geq 0$ for $\gamma \geq 0$. If $\Delta c \geq \Delta^A$ and $\gamma \geq \gamma^A$, by the above arguments, $h_A(1,\gamma) \leq 0 < h_A(0,\gamma)$. Since $h_A(\theta,\gamma)$ is a cubic function of θ, there exists only one root of $h_A(\theta,\gamma) = 0$ on $[0,1]$. Let θ^A be the root. Thus, we have $h_A(\theta,\gamma) \geq 0$ on $[0,\theta^A]$ and $h_A(\theta,\gamma) \leq 0$ on $[\theta^A, 1]$. Since $h_A(\theta,\gamma)$ and $\Pi_A^S - \Pi_A^N$ have the same sign, we have proved Proposition 5(b). *Q.E.D.*

Proof of Proposition 6: (a). By Lemma 2, we have the closed form of Π_B^S. Furthermore, since $\mathbb{E}[L_B^k] = \int_0^{\zeta_B^k}(\zeta_B^k - z) f(z) dz$, $k = 0, S$, Π_B^S is independent of θ. By Lemma 3, we have the closed form of Π_B^N. Since $\frac{\partial \Pi_B^N}{\partial\theta} = \gamma(\gamma+2)p\sigma(c_B - c_A)(p - c_B)[(\gamma+2)c_A p + (\gamma+2)c_B p - 2\gamma c_A c_B]/[(\gamma c_A - (\gamma+2)p)^2[p(\gamma(\theta-1) - 2) - \gamma c_B(\theta-1)]^2] > 0$ if $\gamma > 0$. So Π_B^N is increasing in θ.

(b). Note that $\Pi_B^S - \Pi_B^N = (p - c_B)\gamma h_B(\theta,\gamma)/[2(-\gamma c_A + \gamma p + 2p)^2(-\gamma c_B + \gamma p + p)(\gamma c_B\theta - \gamma c_B - \gamma\theta p + \gamma p + 2p)]$, where $h_B(\theta,\gamma)$ is a linear function of θ and has the same sign as $\Pi_B^S - \Pi_B^N$. When $\theta = 0$, $h_B(0,\gamma) = [c_A^2\left(c_B^2 - 4c_B p + 2p^2\right) + 2c_A c_B^2 p - c_B^2 p^2]\gamma^2 + 2p\left[-3c_A^2(c_B - p) + 2c_A c_B^2 - 2c_B^2 p\right]\gamma + 4p^2\left(c_A^2 - c_B^2\right)$, where $c_A^2\left(c_B^2 - 4c_B p + 2p^2\right) + 2c_A c_B^2 p - c_B^2 p^2 \geq 0$ if $\Delta c \leq \Delta^B := \frac{c_A^3 - c_A p^2}{c_A^2 + 2c_A p - p^2} + \sqrt{2}\sqrt{\frac{c_A^4 p^2 - 2c_A^3 p^3 + c_A^2 p^4}{(c_A^2 + 2c_A p - p^2)^2}}$. Thus, there exists a $\gamma_0^B > 0$ such that $h_B(0,\gamma) \geq 0$ if $0 < \Delta c \leq \Delta^B$ and $\gamma \geq \gamma_0^B$. When $\theta = 1$, $h_B(1,\gamma) = 2p[-2c_A^2 c_B + c_A^2 p + 2c_A c_B^2 - c_B^2 p]\gamma^2 + 2p[-5c_A^2 c_B + 5c_A^2 p + 4c_A c_B^2 - 4c_B^2 p]\gamma + 2p\left(4c_A^2 p - 4c_B^2 p\right)$, where $-2c_A^2 c_B + c_A^2 p + 2c_A c_B^2 - c_B^2 p < 0$. Thus, there exists a $\gamma_1^B > 0$ such that $h_B(1,\gamma) < 0$ if $\gamma \geq \gamma_1^B$. Along with part (a), if $0 < \Delta c \leq \Delta^B$ and $\gamma \geq \gamma^B := \max\{\gamma_0^B, \gamma_1^B\}$, there exists a θ^B such that $\Pi_B^S \geq \Pi_B^N$ for $\theta \in [0, \theta^B]$ and $\Pi_B^S \leq \Pi_B^N$ for $\theta \in [\theta^B, 1]$. *Q.E.D.*

A.3. Proof of Statements in Section 5

We first consider the two-period inventory game between retailers A&B. Given retailer A's discounts and market split ratios in two periods, Lemma 5 [Lemma 6] characterizes the two retailers' equilibrium order quantities and profits in Scenario S [N].

LEMMA 5. *Suppose the specific network effect is present. Given retailer A's discounts e_t and split ratios $\alpha_{A,t}, t = 1,2$, a unique Nash equilibrium exists: In the first period, $y^S_{A,1} = [\alpha_{A,1} + \theta(1-\alpha_{A,1})]\widehat{\zeta}^S_{A,1} - \theta(1-\alpha_{A,1})\widehat{\zeta}^S_{B,1}$ and $y^S_{B,1} = (1-\alpha_{A,1})\widehat{\zeta}^S_{B,1}$, where*

$$\widehat{\zeta}^S_{A,1} = F^{-1}\left(1 - \frac{c_A}{p - e_1 + \rho\gamma(p - e_2 - c_A)}\right) \text{ and } \widehat{\zeta}^S_{B,1} = F^{-1}\left(1 - \frac{c_B}{p + \rho\gamma(p - c_B)}\right);$$

in the second period, given the two retailers' sales, $R^S_{A,1}$ and $R^S_{B,1}$, $y^S_{A,2} = [\alpha_{A,2} + \theta(1-\alpha_{A,2})]\widehat{\zeta}^S_{A,2} - \theta(1-\alpha_{A,2})\widehat{\zeta}^S_{B,2} + \gamma R^S_{A,1}$ and $y^S_{B,2} = (1-\alpha_{A,2})\widehat{\zeta}^S_{B,2} + \gamma R^S_{B,1}$, where $\widehat{\zeta}^S_{A,2} = F^{-1}\left(1 - \frac{c_A}{p-e_2}\right)$ and $\widehat{\zeta}^S_{B,2} = \zeta^0_B$. Furthermore, retailer A's and B's total profits are

$$\begin{aligned}\Pi^S_A(e_1,e_2) \;=\; & [p - e_1 - c_A + \rho(p - e_2 - c_A)\gamma]y^S_{A,1} - [p - e_1 + \rho(p - e_2 - c_A)\gamma]\mathbb{E}[\widehat{L}^S_{A,1}] \\ & +\rho\left\{[p - e_2 - c_A]y^S_{A,2} - [p - e_1]\mathbb{E}[\widehat{L}^S_{A,2}]\right\} \text{ and } \qquad (24)\\ \Pi^S_B(e_1,e_2) \;=\; & [p - c_B + \rho(p - c_B)\gamma]y^S_{B,1} - [p - c_B + \rho(p - c_B)\gamma]\mathbb{E}[\widehat{L}^S_{B,1}] + \rho\{[p - c_B]y^S_{B,2} - p\mathbb{E}[\widehat{L}^S_{B,2}]\} \qquad (25)\end{aligned}$$

where $\widehat{L}^S_{A,t} = \left[[\alpha_{A,t} + \theta(1-\alpha_{A,t})]\widehat{\zeta}^S_{A,t} - \theta(1-\alpha_{A,t})\widehat{\zeta}^S_{B,t} - \alpha_{A,t}X_t - \theta(1-\alpha_{A,t})(X_t - \widehat{\zeta}^S_{B,t})^+\right]^+$ and $\widehat{L}^S_{B,t} = \left[(1-\alpha_{A,t})(\widehat{\zeta}^S_{B,t} - X_t) - \theta\left\{\alpha_{A,t}X_t - [\alpha_{A,t} + \theta(1-\alpha_{A,t})]\widehat{\zeta}^S_{A,t} + (1-\alpha_{A,t})\theta\widehat{\zeta}^S_{B,t}\right\}^+\right]^+$, $t = 1,2$.

Proof of Lemma 5: We first study the subgame in the second period. Let $\overline{y}_{A,2} := y_{A,2} - \gamma R_{A,1}$ and $\overline{y}_{B,2} := y_{B,2} - \gamma R_{B,1}$. By analogous arguments in the proof of Lemma 2, retailer A's best response in the second period is

$$\overline{y}^*_{A,2}(\overline{y}_{B,2}) = \begin{cases}[\alpha_{A,2} + \theta(1-\alpha_{A,2})]\widehat{\zeta}^S_{A,2} - \theta\overline{y}_{B,2}, & \text{if } \overline{y}_{B,2} \le (1-\alpha_{A,2})\widehat{\zeta}^S_{A,2},\\ \alpha_{A,2}\widehat{\zeta}^S_{A,2}, & \text{if } \overline{y}_{B,2} > (1-\alpha_{A,2})\widehat{\zeta}^S_{A,2}.\end{cases} \qquad (26)$$

On the other hand, retailer B's best response in the second period is

$$\overline{y}^*_{B,2}(\overline{y}_{A,2}) = \begin{cases}[1 - \alpha_{A,2} + \theta\alpha_{A,2}]\,\widehat{\zeta}^S_{B,2} - \theta\overline{y}_{A,2}, & \text{if } \overline{y}_{A,2} \le \alpha_{A,2}\widehat{\zeta}^S_{B,2},\\ (1-\alpha_{A,2})\widehat{\zeta}^S_{B,2}, & \text{if } \overline{y}_{A,2} > \alpha_{A,2}\widehat{\zeta}^S_{B,2}.\end{cases} \qquad (27)$$

Furthermore, we show $\widehat{\zeta}^S_{B,2} < \widehat{\zeta}^S_{A,2}$: By assumption, $e_2 \le c_B - c_A = c_B\left(1 - \frac{c_A}{c_B}\right) < p\left(1 - \frac{c_A}{c_B}\right)$, and thus, $p\frac{c_A}{c_B} < p - e_2$. Thus, $\frac{c_A}{p-e_2} < \frac{c_B}{p}$ and $\widehat{\zeta}^S_{A,2} > \widehat{\zeta}^S_{B,2}$. Since $\widehat{\zeta}^S_{B,2} < \widehat{\zeta}^S_{A,2}$, by combining (26) and (27), there exists a unique Nash equilibrium in the second period, $(\overline{y}^S_{A,2}, \overline{y}^S_{B,2})$, where $\overline{y}^S_{A,2} = [\alpha_{A,2} + \theta(1-\alpha_{A,2})]\widehat{\zeta}^S_{A,2} - \theta(1-\alpha_{A,2})\widehat{\zeta}^S_{B,2}$ and $\overline{y}^S_{B,2} = (1-\alpha_{A,2})\widehat{\zeta}^S_{B,2}$. Thus, $y^S_{A,2}$ and $y^S_{B,2}$ follow the forms in Lemma 5.

Then, we study the game in the first period. By analogous arguments, retailer A's best response in the first period is

$$y^*_{A,1}(y_{B,1}) = \begin{cases}[\alpha_{A,1} + \theta(1-\alpha_{A,1})]\widehat{\zeta}^S_{A,1} - \theta y_{B,1}, & \text{if } y_{B,1} \le (1-\alpha_{A,1})\widehat{\zeta}^S_{A,1},\\ \alpha_{A,1}\widehat{\zeta}^S_{A,1}, & \text{if } y_{B,1} > (1-\alpha_{A,1})\widehat{\zeta}^S_{A,1}.\end{cases} \qquad (28)$$

On the other hand, retailer B's best response in the first period is

$$y^*_{B,1}(y_{A,1}) = \begin{cases}[1 - \alpha_{A,1} + \theta\alpha_{A,1}]\,\widehat{\zeta}^S_{B,1} - \theta y_{A,1}, & \text{if } y_{A,1} \le \alpha_{A,1}\widehat{\zeta}^S_{B,1},\\ (1-\alpha_{A,1})\widehat{\zeta}^S_{B,1}, & \text{if } y_{A,1} > \alpha_{A,1}\widehat{\zeta}^S_{B,1},\end{cases} \qquad (29)$$

We show that $\widehat{\zeta}^S_{B,1} < \widehat{\zeta}^S_{A,1}$: $e_1 + \rho e_2\gamma \le c_B(1+\rho\gamma)(1-\frac{c_A}{c_B}) < p(1+\rho\gamma)(1-\frac{c_A}{c_B})$. Thus, $p - e_1 + \rho\gamma(p-e_2-c_A) > p(1+\rho\gamma) - \rho\gamma c_A - (1+\rho\gamma)p(1-\frac{c_A}{c_B}) = \frac{c_A}{c_B}(p+\rho\gamma(p-c_B))$, leading to $\frac{c_A}{p_{A,1}+\rho\gamma(p_{A,2}-c_A)} < \frac{c_B}{p+\rho\gamma(p-c_B)}$. So, $\widehat{\zeta}^S_{B,1} < \widehat{\zeta}^S_{A,1}$. Therefore, by combining (28) and (29), there exists a unique Nash equilibrium in the first period, $(y^{DS}_{A,1}, y^{DS}_{B,1})$, where $y^{DS}_{A,1} = [\alpha_{A,1} + \theta(1-\alpha_{A,1})]\widehat{\zeta}^S_{A,1} - \theta(1-\alpha_{A,1})\widehat{\zeta}^S_{B,1}$ and $y^{DS}_{B,1} = (1-\alpha_{A,1})\widehat{\zeta}^S_{B,1}$. Furthermore, by substituting $y^S_{A,t}$ and $y^S_{B,t}$, $t = 1,2$, we have retailers' profits (24) and (25). *Q.E.D.*

Lemma 6. *Suppose the non-specific network effect is present. Given retailer A's discounts e_t and split ratios $\alpha_{A,t}$, $t = 1,2$, a unique Nash equilibrium exists: In the first period, $y^N_{A,1} = [\alpha_{A,1} + \theta(1-\alpha_{A,1})]\widehat{\zeta}^N_{A,1} - \theta(1-\alpha_{A,1})\widehat{\zeta}^N_{B,1}$ and $y^N_{B,1} = (1-\alpha_{A,1})\widehat{\zeta}^N_{B,1}$, where*

$$\begin{aligned}
\widehat{\zeta}^N_{A,1} &= F^{-1}\left(1 - \frac{c_A}{p - e_1 + \rho\alpha_{A,2}\gamma(p - e_2 - c_A)}\right) \text{ and} \\
\widehat{\zeta}^N_{B,1} &= F^{-1}\left(1 - \frac{c_B - \rho(1-\alpha_{A,2})\gamma\theta(p-c_B)\frac{c_A}{p-e_1+\rho\alpha_{A,2}\gamma(p-e_2-c_A)}}{p + \rho(1-\alpha_{A,2})\gamma(1-\theta)(p-c_B)}\right);
\end{aligned}$$

in the second period, given the two retailers' total sales, $R^N_1 := R^N_{A,1} + R^N_{B,1}$, $y^N_{A,2} = [\alpha_{A,2} + \theta(1-\alpha_{A,2})]\widehat{\zeta}^N_{A,2} - \theta(1-\alpha_{A,2})\widehat{\zeta}^N_{B,2} + \gamma\alpha_{A,2}R^N_1$ and $y^N_{B,2} = (1-\alpha_{A,2})\widehat{\zeta}^N_{B,2} + \gamma(1-\alpha_{A,2})R^N_1$, where $\widehat{\zeta}^N_{A,2} = \widehat{\zeta}^S_{A,2}$ and $\widehat{\zeta}^N_{B,2} = \zeta^0_B$. Furthermore, retailer A's and B's profits are

$$\begin{aligned}
\Pi^N_A(e_1,e_2) &= [p - e_1 - c_A + \rho(p-e_2-c_A)\alpha_{A,2}\gamma]y^N_{A,1} - [p - e_1 + \rho(p-e_2-c_A)\alpha_{A,2}\gamma]\mathbb{E}[\widehat{L}^N_{A,1}] \\
&\quad + \rho(p-e_2-c_A)\alpha_{A,2}\gamma y^N_{B,1} - \rho(p-e_2-c_A)\alpha_{A,2}\gamma\mathbb{E}[\widehat{L}^N_{B,1}] \\
&\quad + \rho\left\{[p-e_2-c_A]y^N_{A,2} - [p-e_1]\mathbb{E}[\widehat{L}^N_{A,2}]\right\} \text{ and} \qquad (30) \\
\Pi^N_B(e_1,e_2) &= [p - c_B + \rho(p-c_B)(1-\alpha_{A,2})\gamma]y^N_{B,1} - [p - c_B + \rho(p-c_B)(1-\alpha_{A,2})\gamma]\mathbb{E}[\widehat{L}^N_{B,1}] \\
&\quad + \rho(p-c_B)(1-\alpha_{A,2})\gamma y^N_{A,1} - \rho(p-c_B)(1-\alpha_{A,2})\gamma\mathbb{E}[\widehat{L}^N_{A,1}] \\
&\quad + \rho\{[p-c_B]y^S_{B,2} - p\mathbb{E}[L^S_{B,2}]\}. \qquad (31)
\end{aligned}$$

where $\widehat{L}^N_{A,t} = \left[[\alpha_{A,t} + \theta(1-\alpha_{A,t})]\widehat{\zeta}^N_{A,t} - \theta(1-\alpha_{A,t})\widehat{\zeta}^N_{B,t} - \alpha_{A,t}X_t - \theta(1-\alpha_{A,t})(X_t - \widehat{\zeta}^N_{B,t})^+\right]^+$ and $\widehat{L}^N_{B,t} = \left[(1-\alpha_{A,t})(\widehat{\zeta}^N_{B,t} - X_t) - \theta\left\{\alpha_{A,t}X_t - [\alpha_{A,t} + \theta(1-\alpha_{A,t})]\widehat{\zeta}^N_{A,t} + (1-\alpha_{A,t})\theta\widehat{\zeta}^N_{B,t}\right\}^+\right]^+$, $t = 1,2$.

Proof of Lemma 6: We first study the subgame in the second period. Let $\overline{\overline{y}}_{A,2} := y_{A,2} - \alpha_{A,2}\gamma R_1$, and $\overline{\overline{y}}_{B,2} := y_{B,2} - (1-\alpha_{A,2})\gamma R_1$. By analogous arguments in the proof of Lemma 5, retailer A's best response in the second period has the same form as (26), except we replace $\widehat{\zeta}^S_{A,2}$ with $\widehat{\zeta}^N_{A,2}$. On the other hand, retailer B's best response in the second period has the same form as (27), except we replace $\widehat{\zeta}^S_{B,2}$ with $\widehat{\zeta}^N_{B,2}$. Thus, by two firms' best responses, there exists a unique Nash equilibrium in the second period, $(\overline{\overline{y}}^N_{A,2}, \overline{\overline{y}}^N_{B,2})$, where $\overline{\overline{y}}^N_{A,2} = [\alpha_{A,2} + \theta(1-\alpha_{A,2})]\widehat{\zeta}^N_{A,2} - \theta(1-\alpha_{A,2})\widehat{\zeta}^N_{B,2}$ and $\overline{\overline{y}}^N_{B,2} = (1-\alpha_{A,2})\widehat{\zeta}^N_{B,2}$. Thus, $y^N_{A,2}$ and $y^N_{B,2}$ follow the forms in Lemma 6.

Then we study the game in the first period. By analogous arguments in the proof of Lemma 3, the best response of retailer A in the first period is as follows: If $y_{B,1} \le (1-\alpha_{A,1})\widehat{\zeta}^N_{A,1}$, where $\widehat{\zeta}^N_{A,1} := F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)}\right)$, $y^*_{A,1}(y_{B,1}) = [\alpha_{A,1} + \theta(1-\alpha_{A,1})]\widehat{\zeta}^N_{A,1} - \theta y_{B,1}$; If $y_{B,1} > (1-\alpha_{A,2})\widehat{\zeta}^N_{A,1}$, $y^*_{A,1}(y_{B,1})$, which is implicitly characterized by the equation

$$p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma$$

$$= [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma(1-\theta)]F\left(\frac{y^*_{A,1}(y_{B,1})}{\alpha_{A,1}}\right) + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma\theta F\left(\frac{y_{B,1} + \theta y^*_{A,1}(y_{B,1})}{1 - \alpha_{A,1} + \theta\alpha_{A,1}}\right),$$

is decreasing in $y_{B,1}$ and goes to $\alpha_{A,1}F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma(1-\theta)}\right)$ as $y_{B,1} \to +\infty$. On the other hand, the best response of firm B is characterized as follows: If $y_{A,1} < \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right)$, $y^*_{B,1}(y_{A,1}) = [1 - \alpha_{A,1} + \theta\alpha_{A,1}]\,F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right) - \theta y_{A,1}$; If $y_{A,1} \geq \alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right)$, $y^*_{B,1}(y_{A,1})$, which is implicity characterized by the equation

$$\begin{aligned} &p - c_B + \rho(p - c_B)(1 - \alpha_{A,2})\gamma \\ &= [p + \rho(p - c_B)(1 - \alpha_{A,2})\gamma(1-\theta)]F\left(\frac{y^*_{B,1}(y_{A,1})}{1 - \alpha_{A,1}}\right) + \rho(p - c_B)(1 - \alpha_{A,2})\gamma\theta F\left(\frac{y_{A,1} + \theta y^*_{B,1}(y_{A,1})}{\alpha_{A,1} + (1 - \alpha_{A,1})\theta}\right), \end{aligned}$$

is decreasing in $y_{A,1}$ and goes to $(1 - \alpha_{A,1})F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma(1-\theta)}\right)$ as $y_{A,1} \to +\infty$.

We show that there exists a unique equilibrium of the two-period game, $(y^N_{A,1}, y^N_{B,1})$, which is in area $\{(y_{A,1}, y_{B,1}) : \frac{1}{\alpha_{A,1}}y_{A,1} \geq \frac{1}{1-\alpha_{A,1}}y_{B,1}\}$. We first show that $\widehat{\zeta}^N_{A,1} = F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}\right) > F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right)$: We have $e_1 + \rho\alpha_{A,2}\gamma e_2 \leq c_B(1 + \rho\alpha_{A,2}\gamma)(1 - \frac{c_A}{c_B}) < p(1 + \rho\alpha_{A,2}\gamma)(1 - \frac{c_A}{c_B})$. Thus, $p - e_1 + \rho\alpha_{A,2}\gamma(p - e_2 - c_A) > p(1 + \rho\alpha_{A,2}\gamma) - \rho\alpha_{A,2}\gamma c_A - (1 + \rho\alpha_{A,2}\gamma)p(1 - \frac{c_A}{c_B}) = \frac{c_A}{c_B}(p + \rho\alpha_{A,2}\gamma(p - c_B)) \geq \frac{c_A}{c_B}(p + \rho(1 - \alpha_{A,2})\gamma(p - c_B))$. Thus, $\frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)} < \frac{c_B}{p+\rho(1-\alpha_{A,2})\gamma(p-c_B)}$. Next, we show that there exists no equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{1}{\alpha_{A,1}}y_{A,1} < \frac{1}{1-\alpha_{A,1}}y_{B,1}\}$, i.e., two firms' best responses do not intersect each other in the area. We assume, to the contrary, there exists an equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{1}{\alpha_{A,1}}y_{A,1} < \frac{1}{1-\alpha_{A,1}}y_{B,1}\}$. Thus, it must satisfy the following system:

$$\begin{cases} p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma = [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma(1-\theta)]F\left(\frac{y_{A,1}}{\alpha_{A,1}}\right) \\ \qquad\qquad + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma\theta F\left(\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}}\right), \\ p - c_B + \rho(p - c_B)(1 - \alpha_{A,2})\gamma = [p + \rho(p - c_B)(1 - \alpha_{A,2})\gamma]F\left(\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}}\right). \end{cases} \tag{32}$$

Since $\frac{1}{\alpha_{A,1}}y_{A,1} < \frac{1}{1-\alpha_{A,1}}y_{B,1}$, we have $\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}} > \frac{y_{A,1}}{\alpha_{A,1}}$ for $\theta \in [0,1]$. Thus, by the first equation in (32), $p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma < [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma]F\left(\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}}\right)$, i.e., $1 - \frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)} < F\left(\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}}\right)$. Furthermore, by the second equation in (32), $1 - \frac{c_B}{p+\rho(1-\alpha_{A,2})\gamma(p-c_B)} = F\left(\frac{y_{B,1}+\theta y_{A,1}}{1-\alpha_{A,1}+\theta\alpha_{A,1}}\right)$. Thus, we have $1 - \frac{c_A}{p_{A,1}+\rho\alpha_{A,2}\gamma(p_{A,2}-c_A)} < 1 - \frac{c_B}{p+\rho(1-\alpha_{A,2})\gamma(p-c_B)}$, leading to a contradiction with the above statement $F^{-1}\left(1 - \frac{c_A}{p_{A,1}+\rho(p_{A,2}-c_A)\alpha_{A,2}\gamma}\right) > F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right)$. Thus, there exists no equilibrium in area $\{(y_{A,1}, y_{B,1}) : \frac{1}{\alpha_{A,1}}y_{A,1} < \frac{1}{1-\alpha_{A,1}}y_{B,1}\}$. Furthermore, by the above arguments, both firms' best responses are decreasing in $y_{B,1}$ in area $\{(y_{A,1}, y_{B,1}) : \frac{1}{\alpha_{A,1}}y_{A,1} \geq \frac{1}{1-\alpha_{A,1}}y_{B,1}\}$, where firm A's best response drops from $[\alpha_{A,1} + \theta(1 - \alpha_{A,1})]\widehat{\zeta}^N_{A,1}$ to $\alpha_{A,1}\widehat{\zeta}^N_A$ on interval $[0, (1 - \alpha_{A,1})\widehat{\zeta}^N_{A,1}]$ and firm B's best response drops from infinity to $\alpha_{A,1}F^{-1}\left(1 - \frac{c_B}{p+\rho(1-\alpha_{A,2})\gamma(p-c_B)}\right)$ on interval $\left((1 - \alpha_{A,1})F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma(1-\theta)}\right), (1 - \alpha_{A,1})F^{-1}\left(1 - \frac{c_B}{p+\rho(p-c_B)(1-\alpha_{A,2})\gamma}\right)\right]$. Therefore, there exists a unique equilibrium, $(y^N_{A,1}, y^N_{B,1})$, which solves the following system:

$$\begin{cases} p_{A,1} - c_A + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma = [p_{A,1} + \rho(p_{A,2} - c_A)\alpha_{A,2}\gamma]F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+\theta(1-\alpha_{A,1})}\right) \\ p - c_B + \rho(p - c_B)(1 - \alpha_{A,2})\gamma = [p + \rho(p - c_B)(1 - \alpha_{A,2})\gamma(1-\theta)]F\left(\frac{y_{B,1}}{1-\alpha_{A,1}}\right) \\ \qquad\qquad + \rho(p - c_B)(1 - \alpha_{A,2})\gamma\theta F\left(\frac{y_{A,1}+\theta y_{B,1}}{\alpha_{A,1}+(1-\alpha_{A,1})\theta}\right). \end{cases} \tag{33}$$

By (33), we have $y_{A,1}^N$ and $y_{B,1}^N$. Furthermore, by substituting $y_{A,t}^N$ and $y_{B,t}^N$, $t=1,2$, we have retailers' profits (30) and (31). *Q.E.D.*

We are going to prove Propositions 7&8. Note that, by assumptions, $e_t \in [0, c_B - c_A]$. Furthermore, retailer A's optimal discount e_t^k should be smaller than $\frac{1}{2\beta}$, $k=S,N$; otherwise, any excessive discount cannot improve retailer A's market split ratio (limited by 1) but reduces the firm's margin.

Proof of Proposition 7: (a). By Lemma 5, we have the closed forms of $y_{A,1}^S$ and $y_{B,1}^S$. Thus, we have $\frac{\partial y_{A,1}^S}{\partial e_1} = \frac{1}{(-\gamma c_B+\gamma p+p)(-\gamma c_A-e_1-\gamma e_2+\gamma p+p)^2}\phi_{A,1}^S(\gamma)$, where $\phi_{A,1}^S(\gamma)$ has the same sign as $\frac{\partial y_{A,1}^S}{\partial e_1}$ with $\frac{\partial^3 \phi_{A,1}^S(\gamma)}{\partial \gamma^3} = 6\beta(p-c_B)(\mu+\sigma)(c_A+e_2-p)^2 > 0$. Moreover, we have $\frac{\partial y_{B,1}^S}{\partial e_1} = \frac{\beta c_B(\gamma(\mu+\sigma)+2\sigma)-\beta(\gamma+1)p(\mu+\sigma)}{-\gamma c_B+\gamma p+p} < 0$.

By Lemma 6, we have the closed forms of $y_{A,1}^N$ and $y_{B,1}^N$. Thus, we have $\frac{\partial y_{A,1}^N}{\partial e_1} = \phi_{A,1}^N(\gamma)/[(-\gamma c_A - 2\beta\gamma c_A e_2 - 2e_1 - 2\beta\gamma e_2^2 - \gamma e_2 + 2\beta\gamma e_2 p + \gamma p + 2p)^2(\gamma c_B\theta - \gamma c_B - 2\beta\gamma c_B e_2\theta + 2\beta\gamma c_B e_2 + 2\beta\gamma e_2\theta p - 2\beta\gamma e_2 p - \gamma\theta p + \gamma p + 2p)]$, where $\phi_{A,1}^N(\gamma)$ has the same sign as $\frac{\partial y_{A,1}^N}{\partial e_1}$ and $\frac{\partial^3\phi_{A,1}^N(\gamma)}{\partial\gamma^3} = -6\beta(\theta-1)(c_B - p)(2\beta e_2 - 1)(2\beta e_2+1)^2(\mu+\sigma)(c_A+e_2-p)^2 \geq 0$ since $\beta e_2 \leq \frac{1}{2}$. Moreover, $\frac{\partial y_{B,1}^N}{\partial e_1} = \phi_{B,1}^N(\gamma)/[(\gamma c_A + 2\beta\gamma c_A e_2 + 2e_1 + 2\beta\gamma e_2^2 + \gamma e_2 - 2\beta\gamma e_2 p - \gamma p - 2p)^2(\gamma c_B\theta - \gamma c_B - 2\beta\gamma c_B e_2\theta + 2\beta\gamma c_B e_2 + 2\beta\gamma e_2\theta p - 2\beta\gamma e_2 p - \gamma\theta p + \gamma p + 2p)]$, where $\phi_{B,1}^N(\gamma)$ has the same sign as $\frac{\partial y_{B,1}^N}{\partial e_1}$ and $\frac{\partial^3\phi_{B,1}^N(\gamma)}{\partial\gamma^3} = 6\beta(\theta-1)(c_B-p)(2\beta e_2 - 1)(2\beta e_2+1)^2(\mu+\sigma)(c_A+e_2-p)^2 \leq 0$ since $\beta e_2 \leq \frac{1}{2}$.

Therefore, there exits a γ_1^D such that, if $\gamma > \gamma_1^D$, $y_{A,1}^k$ increases in e_1 and $y_{B,1}^k$ decreases in e_1, $k=S,N$.

(b). We have $\frac{\partial y_{A,1}^S}{\partial e_2} = \frac{\gamma c_A\sigma(2\beta e_1\theta - 2\beta e_1 - \theta - 1)}{(-\gamma c_A - e_1 - \gamma e_2 + \gamma p + p)^2} \leq 0$. Moreover, by Lemma 5, $y_{B,1}^S$ is independent of e_2.

(c). We have $\frac{\partial y_{A,1}^N}{\partial e_2} = [2\gamma\sigma\psi_{A,1}^N(\theta,\gamma)]/[(-\gamma c_A - 2\beta\gamma c_A e_2 - 2e_1 - 2\beta\gamma e_2^2 - \gamma e_2 + 2\beta\gamma e_2 p + \gamma p + 2p)^2(\gamma c_B\theta - \gamma c_B - 2\beta\gamma c_B e_2\theta + 2\beta\gamma c_B e_2 + 2\beta\gamma e_2\theta p - 2\beta\gamma e_2 p - \gamma\theta p + \gamma p + 2p)^2]$, where $\psi_{A,1}^N(\theta,\gamma)$ has the same sign as $\frac{\partial y_{A,1}^N}{\partial e_2}$. If $\theta = 1$, $\psi_{A,1}^N(1,\gamma)$ is a linear function in γ with $\frac{\partial\psi_{A,1}^N(1,\gamma)}{\partial\gamma} = 2c_A p(p-c_B)(2\beta e_1 - 1)\left(4\beta c_A - 4\beta^2 e_2^2 + 4\beta e_2 - 4\beta p + 1\right)$. Note that, if $\beta \leq \frac{1}{4(p-c_A)}$, $\frac{\partial\psi_{A,1}^N(1,\gamma)}{\partial\gamma} < 0$ for all feasible e_1 and e_2, since $e_t \leq \min\{c_B - c_A, \frac{1}{2\beta}\}$, $t=1,2$.

On the other hand, $\frac{\partial y_{B,1}^N}{\partial e_2} = [2\gamma\sigma(p-c_B)(1-2\beta e_1)\psi_{B,1}^N(\theta,\gamma)]/[(\gamma c_A + 2\beta\gamma c_A e_2 + 2e_1 + 2\beta\gamma e_2^2 + \gamma e_2 - 2\beta\gamma e_2 p - \gamma p - 2p)^2(\gamma c_B\theta - \gamma c_B - 2\beta\gamma c_B e_2\theta + 2\beta\gamma c_B e_2 + 2\beta\gamma e_2\theta p - 2\beta\gamma e_2 p - \gamma\theta p + \gamma p + 2p)^2]$, where $\psi_{B,1}^N(\theta,\gamma)$ has the same sign as $\frac{\partial y_{B,1}^N}{\partial e_2}$. If $\theta = 1$, $\psi_{B,1}^N(1,\gamma)$ is a linear function in γ with $\frac{\partial\psi_{B,1}^N(1,\gamma)}{\partial\gamma} = 2c_A p\left(4\beta c_A - 4\beta^2 e_2^2 + 4\beta e_2 - 4\beta p + 1\right)$. Note that, if $\beta \leq \frac{1}{4(p-c_A)}$, $\frac{\partial\psi_{B,1}^N(1,\gamma)}{\partial\gamma} > 0$ for all feasible e_1 and e_2, since $e_t \leq \min\{c_B - c_A, \frac{1}{2\beta}\}$, $t=1,2$.

Therefore, by the above statements, when $\theta = 1$, there exists a $\gamma_2^D > 0$ such that $y_{A,1}^N$ decreases in e_2 and $y_{B,1}^N$ increases in e_2 if $\beta < \beta^D := \frac{1}{4(p-c_A)}$ and $\gamma > \gamma_2^D$. Thus, by the continuity of $y_{A,1}^N$ and $y_{B,1}^N$ in θ, there exists a $\theta^D \in [0,1]$ such that the statement in part (c) holds. *Q.E.D.*

Proof of Proposition 8: Because $e_t = \frac{1}{\beta}(\alpha_{A,t} - \frac{1}{2})$, in this proof, we use $(\alpha_{A,1}, \alpha_{A,2})$ as decisions of retailer A at Stage 0. Thus, retailer A's profit in Scenario k can be written as $\Pi_A^k(\alpha_{A,1}, \alpha_{A,2})$, $k=S,N$. By assumptions, $\alpha_{A,t} \leq \bar{\alpha}_A := \min\{1, \frac{1}{2} + \beta(c_B - c_A)\}$, $t=1,2$. To show part (a)-(b), it is sufficient to show that there exists a $\gamma^D > 0$ such that $\alpha_{A,1}^S \geq \alpha_{A,2}^S = \frac{1}{2}$ and $\alpha_{A,1}^N \leq \alpha_{A,2}^N$ if $\gamma \geq \gamma^D$.

Scenario S: By Lemma 5, we have the closed form of $\Pi_A^S(\alpha_{A,1},\alpha_{A,2})$. Then $\frac{\partial \Pi_A^S(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,2}} = \psi_D^S(\gamma)/[2\beta p^2(-\gamma c_B+\gamma p+p)^2(2\beta p-2\alpha_{A,2}+1)^2(-\gamma+2\beta\gamma c_A-2\beta\gamma p-2\beta p+2\alpha_{A,1}+2\gamma\alpha_{A,2}-1)^2]$, where $\psi_D^S(\gamma)$ has the same sign as $\frac{\partial \Pi_A^S(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,2}}$ with $\frac{\partial^5 \psi_D^S(\gamma)}{\partial \gamma^5} = -240\mu p^2\alpha_{A,1}(c_B-p)^2(2\beta p-2\alpha_{A,2}+1)^2(-2\beta c_A+2\beta p-2\alpha_{A,2}+1)^2<0$ for all feasible $(\alpha_{A,1},\alpha_{A,2})$. Thus, there exists a $\gamma_D^S>0$ such that $\psi_D^S(\gamma)<0$ if $\gamma\geq\gamma_D^S$. Hence, if $\gamma\geq\gamma_D^S$, $\Pi_A^S(\alpha_{A,1},\alpha_{A,2})$ is decreasing in $\alpha_{A,2}$ for all feasible $(\alpha_{A,1},\alpha_{A,2})$. Therefore, $\alpha_{A,1}^S\geq\alpha_{A,2}^S=\frac{1}{2}$.

Scenario N: By Lemma 6, we have the closed form of $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$. We prove Scenario N in two steps. First, we have $\frac{\partial \Pi_A^N(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,1}} = -\psi_1^N(\gamma)/[2\beta(-\gamma c_B\theta+\gamma c_B+\gamma c_B\theta\alpha_{A,2}-\gamma c_B\alpha_{A,2}+\gamma\theta p-\gamma p-\gamma\theta p\alpha_{A,2}+\gamma p\alpha_{A,2}-p)^2\left(2\beta\gamma c_A\alpha_{A,2}-2\beta p-2\beta\gamma p\alpha_{A,2}+2\alpha_{A,1}+2\gamma\alpha_{A,2}^2-\gamma\alpha_{A,2}-1\right)^3]$, where $\psi_1^N(\gamma)$ has the same sign as $\frac{\partial \Pi_A^N(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,1}}$ with $\frac{\partial^5 \psi_1^N(\gamma)}{\partial \gamma^5} = 120(\theta-1)^2(\alpha_{A,2}-1)^2\alpha_{A,2}^3(c_B-p)^2(-2\beta c_A+2\beta p-2\alpha_{A,2}+1)^3[\mu(-2\beta c_A+2\beta p-4\alpha_{A,1}+1)-2\beta c_A\sigma]$. Note that $\frac{\partial^5 \psi_1^N(\gamma)}{\partial \gamma^5}<0$ if and only if $\mu(-2\beta c_A+2\beta p-4\alpha_{A,1}+1)-2\beta c_A\sigma<0$, i.e., $\alpha_{A,1}>\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu}$. Since $\alpha_{A,1}\in[\frac{1}{2},\bar{\alpha}_A]$, $\alpha_{A,1}>\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu}$ holds for all feasible $\alpha_{A,1}$ if and only if $\frac{1}{2}\geq\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu}$, which holds if and only if condition (i) holds: $\left(0<p\leq\frac{1}{2\beta}\wedge 0<c_A<p\right)$ or $\left(p>\frac{1}{2\beta}\wedge\frac{2\beta\mu p-\mu}{2\beta\mu+2\beta\sigma}\leq c_A<p\right)$. Thus, if condition (i) holds, there exists a $\gamma_{D,1}^N>0$ such that $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$ is decreasing in $\alpha_{A,1}$ for all feasible $\alpha_{A,1}$ when $\gamma\geq\gamma_{D,1}^N$. Thus, $\alpha_{A,1}^N=\frac{1}{2}\leq\alpha_{A,2}^N$. Therefore, we have proved part (b) under condition (i). Now we study $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$ if condition (i) does not hold, i.e., $p>\frac{1}{2\beta}$ and $0<c_A<\frac{2\beta\mu p-\mu}{2\beta\mu+2\beta\sigma}$.

Second, we consider the case where condition (i) does not hold. In this case, we have $\frac{\partial \Pi_A^N(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,2}} = -\psi_2^N(\gamma)/[2\beta p^2(2\beta p-2\alpha_{A,2}+1)^2(\gamma c_B\theta-\gamma c_B-\gamma c_B\theta\alpha_{A,2}+\gamma c_B\alpha_{A,2}-\gamma\theta p+\gamma p+\gamma\theta p\alpha_{A,2}-\gamma p\alpha_{A,2}+p)^3\left(2\beta\gamma c_A\alpha_{A,2}-2\beta p-2\beta\gamma p\alpha_{A,2}+2\alpha_{A,1}+2\gamma\alpha_{A,2}^2-\gamma\alpha_{A,2}-1\right)^3]$, where $\psi_2^N(\gamma)$ has the same sign as $\frac{\partial \Pi_A^N(\alpha_{A,1},\alpha_{A,2})}{\partial \alpha_{A,2}}$ with $\frac{\partial^7 \psi_2^N(\gamma)}{\partial \gamma^7} = -5040(\theta-1)^3\mu p^2(\alpha_{A,2}-1)^3\alpha_{A,2}^3(c_B-p)^3(2\beta p-2\alpha_{A,2}+1)^2(-2\beta c_A+2\beta p-4\alpha_{A,2}+1)(-2\beta c_A+2\beta p-2\alpha_{A,2}+1)^3$. Thus, $\frac{\partial^7 \psi_2^N(\gamma)}{\partial \gamma^7}>0$ if and only if $-2\beta c_A+2\beta p-4\alpha_{A,2}+1>0$, which holds if and only if $\alpha_{A,2}<\frac{1}{4}(-2\beta c_A+2\beta p+1)$. Note that $\frac{1}{4}(-2\beta c_A+2\beta p+1)>\frac{1}{2}$ always holds. Thus, there exists a $\gamma_{D,2}^N\geq\gamma_{D,1}^N>0$ such that the following two sub-cases hold if $\gamma\geq\gamma_{D,2}^N$: First, if condition (i) does not hold and condition (ii) holds, i.e., $0<c_A<\frac{2\mu p}{2\mu+3\sigma}$, $-\frac{\mu}{2c_A\mu+2c_A\sigma-2\mu p}<\beta\leq-\frac{3}{2c_A-2p}$ and $c_B\geq\frac{2\beta c_A+2\beta p-1}{4\beta}$, we have $\frac{1}{4}(-2\beta c_A+2\beta p+1)\in[\frac{1}{2},\bar{\alpha}_A]$. In this case, $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$ increases in $\alpha_{A,2}$ on $[\frac{1}{2},\frac{1}{4}(-2\beta c_A+2\beta p+1)]$ and decreases on $[\frac{1}{4}(-2\beta c_A+2\beta p+1),\bar{\alpha}_A]$; by the arguments above, $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$ increases in $\alpha_{A,1}$ on $[\frac{1}{2},\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu}]$ and decreases on $[\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu},\bar{\alpha}_A]$. Thus, the optimal solution is $(\alpha_{A,1}^N,\alpha_{A,2}^N)=(\frac{-2\beta c_A\mu-2\beta c_A\sigma+\mu+2\beta\mu p}{4\mu},\frac{1}{4}(-2\beta c_A+2\beta p+1))$, which satisfies $\alpha_{A,1}^N\leq\alpha_{A,2}^N$. Second, both conditions (i)-(ii) do not hold, then $\frac{1}{4}(-2\beta c_A+2\beta p+1)>\bar{\alpha}_A$. In this case, $\Pi_A^N(\alpha_{A,1},\alpha_{A,2})$ increases in $\alpha_{A,2}$ on $[\frac{1}{2},\bar{\alpha}_A]$. Thus, $\alpha_{A,2}^N=\bar{\alpha}_A\geq\alpha_{A,1}^N$.

Let $\gamma_D:=\max\{\gamma_D^S,\gamma_{D,2}^N\}$, we have proved part (a)-(b). *Q.E.D.*