Dynamic Competition in Online Retailing: Implications of Network Effects

Zheyu Jiang
Miami Herbert Business School, University of Miami, zxj144@miami.edu

Nan Yang
Miami Herbert Business School, University of Miami, nyang@bus.miami.edu

Renyu Zhang
CUHK Business School, The Chinese University of Hong Kong, philipzhang@cuhk.edu.hk

Problem Definition: Thanks to the ubiquity of social media and the new business model of live-streaming e-commerce, online retailing has recently embraced increasing interactions among customers. Such interactions give rise to strong network effects, i.e., customers are attracted to purchase the products with higher sales, and raise a crucial challenge for competing retailers: current operations decisions (such as promotion, price, and inventory) impact not only the current market, but also the future demands and profits of all the retailers through the network effects. To understand the implications of such network effects, we study a dynamic competition model, in which retailers periodically compete on promotional effort (e.g., advertising) and price, while operating their inventories over a finite planning horizon.

Methodology/Results: We find that, under the Markov perfect equilibrium, the state space of each firm is linearly separable for both the simultaneous (promotion, price, and inventory) competition and the promotion-first competition, which leads to a simple characterization of the equilibrium strategy in both competitions. The equilibrium characterization also enables us to investigate the impact of the network effects, and to compare the equilibria under the different competition modes.

Managerial Implications: The network effects give rise to a natural trade-off between generating current profits and inducing future demands for competing retailers, thus having several important implications upon their operations decisions. The trade-off between current profits and future demands is more intensive at an earlier stage than at later stages, so the equilibrium prices are increasing, whereas the equilibrium promotional efforts are decreasing, over the planning horizon. The retailers need to balance the aforementioned trade-off inter-temporally under the simultaneous competition, whereas they need to balance this trade-off both inter-temporally and intra-temporally under the promotion-first competition. Finally, in the dynamic game between online retailers, the network effects could be a new driving force for the "fat-cat" effect, i.e., the equilibrium promotional efforts are higher under the promotion-first competition than those under the simultaneous competition.

Key words: online retailing; dynamic game; Markov perfect equilibrium; network effects

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1. Introduction

Social media platforms, such as Facebook, YouTube, and WeChat, have witnessed a dramatic growth in scale over the past decade. The number of active users on these platforms already surpassed 4.2 billion in 2021 (Statista 2021a). Such development of online social media drives a rising trend of new online retailing model called *social commerce*, where e-commerce meets online social media. In the U.S., there were about 80 million social commerce buyers who contributed to the $27 billion sales in 2020, which implies a 30% year-over-year increase (see, Statista 2021c,d).

The rise of social commerce also creates successful new business models, such as live-streaming e-commerce and community group buying. Live-streaming e-commerce became popular during the COVID-19 pandemic, and has already grown to a trillion-RMB retail market in China (the live streaming e-commerce market size in China reached about RMB 1237.9 billion in 2021, see, Statista 2021b). Retailers hire social-media influencers/live-streamers (e.g., Austin Li, a star e-commerce live-streamer in China, sold products worth $1.7 billion in Alibaba's Singles Day in 2021, see, Tan 2021) to promote their products on live-streaming platforms (e.g., Tiktok, Facebook, and Amazon). Through live-streaming, the influencers introduce the features of the products, communicate with consumers, and answer their questions. Thanks to such an interactive channel together with the trust between the live-streamers and their followers on social media, liver-streaming e-commerce could substantially boost the consumer conversions for the retailers. In addition to live-streaming e-commerce, grocery retailers also leverage online social networks to boost their sales. This new business model, referred to as community group buying (see, e.g., Li 2020), facilitates consumers to form groups on a social network (e.g., WeChat) and launch group-buying campaigns that offer discounts to the group members if sufficiently many consumers sign up to make a purchase. Community group buying has already greatly impacted the traditional offline grocery retailing market in China, which has long accounted for 90% of the grocery market.

Social commerce involves substantial consumer interactions for an online retailer, which in turn significantly propel the market-diffusion process of the retailer. Indeed, interactions between consumers could improve their surplus of purchasing the product from a retailer under discussion on the platform (see, e.g., Katona et al. 2011). Such phenomenon naturally drives the network effect, referred to as a common effect that a consumer gain additional surplus of purchasing a certain product if the sales volume of the product increases (see, e.g., Economides 1996). Network effects create a dependence between the current decisions and future demands for retailers, because the current decisions affect current sales, which will impact future consumer utilities and, thus, future demands. For example, consumers in live-streaming rooms are easily attracted by the bestsellers therein, and are more likely to revisit the retailer and make a purchase in the near future (see, e.g., Wongkitrungrueng and Assarut 2020). Moreover, different retailers compete with each other in social commerce, so the competition will interact with the strong network effects, which together pose an important challenge for managing their operations strategies (e.g., price, inventory, and promotion policies). Myopically optimizing the current profits may lead to significant losses in future demands and, thus, hurt the profits in the long run. On the other hand, overemphasizing market share growth may significantly compromise current profits, which is another extreme of undesirable outcome. Recognizing the trade-off between current profits and future demands in a competitive market calls the retailers for carefully re-designing their operations strategies under network effects.

It is worth noticing that online retailers have different preferences on the timing of promotions, depending on the specific marketing tool the retailer is adopting. For example, immediately re-targeting consumers who just made a purchase and encouraging them to purchase again with personalized content are fairly effective. See, e.g., Bleier and Eisenbeiss (2015). However, the cart re-targeting that sends reminders to consumers about their carted products are not as effective, and may negatively affect their willingness to purchase. See, e.g., Li et al. (2021). Given that the timing of promotional efforts may have significant impact on their effectiveness, competing retailers must carefully time their price, inventory, and promotion decisions.

The main goal of this paper is to explore the dynamic competition between online retailers in the presence of network effects. Motivated by online retailing practices, we seek to address the following key research question:

> *Taking into account network effects and different competition modes, how should online retailers dynamically compete with each other on price, inventory, and promotion to well balance current profits and future demands?*

To address this question, we develop a periodic-review dynamic competition model, in which online retailers compete with each other in a Markov game over a finite planning horizon. Each retailer makes the promotional effort, retail price, and inventory decisions in each period to maximize its total profit of the entire planning horizon. The promotional effort of the retailer includes all costs of marketing and promotion to its products, such as hiring famous influencers, subsidizing audiences/subscribers in live-streaming, and procuring consumer traffic through platform ads. See, e.g., Li (2020), Li et al. (2021). The random demand of each retailer in each period is determined by its own market size, and the current prices and promotional efforts of all competing retailers in the market. Each retailer chooses its promotional effort, price, and inventory stocking quantity in any period. The promotional effort and price discounts of a retailer boost the current demand of itself and diminish that of its competitors. We emphasize that the salient feature of our model is that the market sizes of the competing retailers are evolving under the network effects throughout the planning horizon. More specifically, to capture the network effects in the social commerce of modern online retailing, we assume that the future market size of each retailer is increasing in its current demand. We also investigate the impact of promotional efforts' timing. We consider two competition modes: (a) the simultaneous competition, under which retailers simultaneously make promotion, price, and inventory decisions in each period; and (b) the promotion-first competition, under which the retailers first choose their promotional efforts, and after observing the promotion decisions of all retailers in the market, make price and inventory decisions in each period.

We use the Markov perfect equilibrium (MPE) paradigm to analyze our dynamic competition model, and employ the linear separability approach (see, e.g., Olsen and Parker 2008) to show that, if the initial inventory level at the beginning of the planning horizon is zero for each retailer, an MPE exists in the dynamic competition model. Under the MPE, the equilibrium profit of each firm in each period is linearly separable in its private information - its own inventory level and market size. We also find mild sufficient conditions to ensure the uniqueness of MPE. Specifically, we study the subgame in each period under the two competition modes, and show that (a) under the simultaneous competition, the subgame can be decomposed into a noncooperative game, in which the firms compete jointly on promotional effort and sales price, and an inventory optimization problem, if the starting inventory level of each retailer at the beginning of the planning horizon is 0; and (b) under a similar assumption regarding the starting inventory level of each firm, the subgame under the promotion-first competition can be converted to a two-stage competition, in which the retailers compete on promotional effort in the first stage and on price in the second stage, when they also optimize their respective inventory stocking quantities. The analysis of the subgame in each period under the two competition modes allows us to identify conditions under which there exists a unique pure strategy Nash equilibrium, thus ensuring the existence and uniqueness of a pure strategy MPE in the Markov game.

1.1. Main Contributions

We make the following contributions in this paper:

Given the prevailing consumer interactions in social commerce, retailers must take into account the network effects in their operations. Despite the abundant literature of dynamic competition on operational decisions, to our best knowledge, we are the first to explicitly model network effects in the dynamic competition between retailers. Our proposed model helps deliver insights on the operations implications of network effects on competing retailers in a dynamic market environment.

We characterize the MPE in the dynamic competition model under the simultaneous competition and the promotion-first competition. The analysis also provides managerial implications for competing retailers facing network effects. We show that retailers can boost promotional efforts, offer price discounts, and, consequently, increase inventory stocking level to balance the trade-off between current profits and future demands that is driven by network effects. The stronger the network effects, the more intensive the aforementioned trade-off, which leads to more promotional efforts, heavier price discounts, and higher inventory levels for all retailers. By adopting these strategies, each retailer leverages the network effects to expand the market size in the future. Furthermore, we show that the trade-off between current profits and future demands becomes less intensive towards the end of the planning horizon regardless of the competition mode. Hence, the equilibrium prices are increasing, whereas the equilibrium promotional efforts are decreasing, over the planning horizon.

By comparing the equilibria of the dynamic competition model under the two competition modes, we derive two critical insights with regard to the timing of promotional efforts. Under the simultaneous competition, the competing retailers need to balance the trade-off between current profits and future demands *inter-temporally*; whereas, under the promotion-first competition, they have to balance this trade-off both *inter-temporally* and *intra-temporally.* Besides that, we identify a new driving force for the "fat-cat" effect (i.e., in each period, the equilibrium promotional efforts may be higher under the promotion-first competition than those under the simultaneous competition): The trade-off between current profits and future demands is more intensive in the promotion-first competition than in the simultaneous competition, thus prompting more promotional efforts under the promotion-first competition.

The rest of this paper is organized as follows. We position this paper in the related literature in Section 2. Section 3 introduces the model setup. We analyze the simultaneous competition model in Section 4, and the promotion-first competition model in Section 5. We compare the equilibrium outcomes in these two competition models in Section 6. Section 7 concludes this paper. All proofs are relegated to the Appendix. We use the **bold face** to represent a vector or a matrix throughout this paper.

2. Literature Review

Our work is related to several streams of research in the literature. First, there are emerging studies on live-streaming and other new business models in online retailing. Wongkitrungrueng and Assarut (2020) find that live-streaming e-commerce can improve customers' trust on the products and sellers, thus increasing customer engagement, loyalty, and purchase. Qi et al. (2020) study the capacity investment problem of a manufacturer which sells products through a live-streaming platform, however has no direct access to demand information and tries to infer it from the commission rate of the platform. Hou et al. (2021) analyze the optimal live-streaming adoption and influencer selection strategy for retailers. Chen et al. (2020) study the position auctions of live-streaming advertising, by endogenizing product information provision in a mechanism design framework. We refer interested readers to Caro et al. (2020) for a comprehensive review on the new trends of retail operations and the directions for future research.

Network effects have drawn significant attention in the economics and operations literature. Several studies embed network effects into a firm's revenue management and sourcing decisions. Dhebar and Oren (1986) characterize the optimal nonlinear pricing strategy for a network product with heterogeneous customers. Xie and Sirbu (1995) examine the equilibrium dynamic pricing strategies of an incumbent and an entrant under network externalities. Bensaid and Lesne (1996) consider the optimal dynamic monopoly pricing under network externalities and show that the equilibrium prices increase as time passes. Bloch and Quérou (2013) study the optimal pricing strategy in a network with a given network structure and characterize the relationship between optimal prices and consumers' centrality. Wang and Wang (2017) endogenize network externalities in consumer choice models and analyze the assortment optimization problem under the choice models. Hu et al. (2020) study the innovation spillover when an innovator outsources its products to a contract manufacturer, which may also be a competitor in the end market. We contribute to this stream of literature by analyzing the impact of network externalities upon the competing firms' operations (i.e., the inventory policies) and marketing (i.e., promotional investments) decisions in a dynamic retail competition.

Our paper is also related to the extensive literature on dynamic pricing and inventory management. This literature diverges into two lines of research: (i) the monopoly model, in which a single firm maximizes its total expected profit over a finite or infinite planning horizon, and (ii) the competition model, in which multiple firms play a non-cooperative game to maximize their respective expected per-period profits over an infinite planning horizon. The literature on the monopoly model of joint pricing and inventory management is very rich. See Chen and Simchi-Levi (2012) for a review. Chen and Simchi-Levi (2004a,b) study the joint pricing and inventory management problem with fixed ordering costs for the finite horizon and infinite horizon. Pang et al. (2012) and Chen et al. (2014) study a difficult problem - the joint pricing and inventory control problem with periodic review and positive lead-time. Feng et al. (2020) study another challenging problem, joint pricing and inventory management under lost sales. Federgruen et al. (2020) propose a novel method to characterize the optimal inventory strategy for a general review inventory control problem which considers bilateral inventory adjustment, associated fixed costs, and capacity limits. Some other features are also considered for the joint pricing and inventory management problem such as customers' bargaining (Feng and Shanthikumar 2018), production substitution (Feng et al. 2019), delayed differentiation (Yang and Zhang 2022a), and online reviews (Yang and Zhang 2022b).

Dynamic competition on inventory and price has also received abundant attention in the literature. Under deterministic demands, Bernstein and Federgruen (2004a) address infinite-horizon models for oligopolies with competing retailers under price-sensitive uncertain demand. Bernstein and Federgruen (2004b) develop a stochastic general equilibrium inventory model, in which retailers compete on both sales price and service level throughout an infinite horizon. Feng et al. (2020) consider a dynamic pricing competition problem in which buyers are allowed to bargain down the prices. Li et al. (2013) study an infinitely repeated contracting problem with imperfect monitoring, in which a manufacturer incentivizes two competing suppliers' private efforts by allocating future business due to suppliers' overall performances. Our work differs from this line of literature in that we study the trade-off between current profits and future demands in the presence of network effects in a dynamic and competitive market. To this end, we adopt the MPE (i.e., the closed-loop equilibrium) in a finite-horizon model as opposed to the commonly used stationary strategy equilibrium (i.e., the open-loop equilibrium) in an infinite-horizon model.

Finally, from the methodological perspective, our work is related to the literature on the analysis of MPE in dynamic competition models. MPE is a prevalent equilibrium concept in the economics literature on dynamic oligopoly models (see, e.g., Maskin and Tirole 1988, Ericson and Pakes 1995, Curtat 1996). In the operations management literature, this equilibrium concept has been widely adopted to study the strategic behaviors in dynamic games. Employing the linear separability approach, Hall and Porteus (2000), Liu et al. (2007) and Olsen and Parker (2008) characterize the MPE in dynamic duopoly models with market size dynamics, and Ahn and Olsen (2007) analyze the structure of the pure strategy MPE in a dynamic inventory competition with subscriptions. Due to limited technical tractability, the analysis of MPE in nonlinear and nonseparable dynamic games is scarce. Martínez-de-Albéniz and Talluri (2011) characterize the MPE price strategy in a finite-horizon dynamic Bertrand competition with fixed capacities. Lu and Lariviere (2012) numerically compute the MPE in an infinite-horizon model, in which a supplier allocates its limited capacity to competing retailers. Olsen and Parker (2014) give conditions under which the stationary infinite-horizon equilibrium is also an MPE in the context of inventory duopolies. Our paper adopts the linear separability approach to characterize the pure strategy MPE in a dynamic joint promotion, price, and inventory competition of online retailers under network effects, and analyze the trade-off between current profits and future demands therein.

3. Model

Consider an online retailing market with N competing retailers, which serve the market with partially substitutable products over a $T-$period planning horizon, labeled backwards as $\{T, T-1, \cdots, 1\}$. In each period t, each firm i selects a promotional effort $\gamma_{i,t} \in [0, \bar{\gamma}_i]$, which represents the effort the firm makes in advertising, consumer subsidy, after-sales service, and other efforts to promote the demand of its product in the current period. With the recent boom of live-streaming online retailing, the promotional effort may take the form of commission fees paid by the retailer to the influencers who help sell its product through live-streaming. In period t, the total promotional investment cost of each firm i is proportional to its realized demand, $D_{i,t}$, and given by $\nu_i(\gamma_{i,t})D_{i,t}$. The per-unit demand cost rate, $\nu_i(\cdot)$, is a non-negative, convexly increasing, and twice continuously differentiable function of the promotional effort $\gamma_{i,t}$, with $\nu_i(0)=0$. Before the demand is realized in period t, each retailer i selects a sales price $p_{i,t} \in [\underline{p}_i, \bar{p}_i]$ and adjusts its inventory level to $x_{i,t}(x_{i,t} \geq I_{i,t})$, where $I_{i,t}$ is the inventory level of retailer i at the beginning of period t. We assume that the excess demand of each firm is fully backlogged. In summary, each retailer i makes three decisions at the beginning of any period t: (i) the promotional effort $\gamma_{i,t}$, (ii) the sales price $p_{i,t}$, and (iii) the inventory level $x_{i,t}$.

The demand of each retailer i in period t depends on the entire vector of promotional efforts $\boldsymbol{\gamma}_t := (\gamma_{1,t}, \gamma_{2,t}, \cdots, \gamma_{N,t})$ and the entire vector of sales prices $\boldsymbol{p}_t := (p_{1,t}, p_{2,t}, \cdots, p_{N,t})$ in period t. We denote the demand of retailer i as $D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$ to capture such dependence. More specifically, we base our analysis on the following multiplicative form of $D_{i,t}(\cdot,\cdot)$:

$$D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) = \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) + \xi_{i,t}, \tag{1}$$

where $\Lambda_{i,t} > 0$ is the market size of retailer i in period t, $d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) > 0$ captures the impact of $\boldsymbol{\gamma}_t$ and $\boldsymbol{p}_t$ on retailer i's demand, and $\xi_{i,t}$ is a continuous random variable with a connected support and a zero mean, i.e., $\mathbb{E}[\xi_{i,t}] = 0$. For each retailer i, the random perturbations $\xi_{i,t}, 1 \leq t \leq T$ are *i.i.d.* throughout the whole planning horizon, and are also independent of the market size vector $\boldsymbol{\Lambda}_t := (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t})$, the price vector $\boldsymbol{p}_t$, and the promotional effort vector $\boldsymbol{\gamma}_t$. Therefore, $d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$ can be viewed as the normalized expected demand of retailer i in period t and $\mathbb{E}[D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)] = \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$. Let $F_i(\cdot)$ be the *c.d.f.* and $\bar{F}_i(\cdot)$ be the *c.c.d.f.* of $\xi_{i,t}$ for all t. The market size $\Lambda_{i,t}$ is observable by retailer i at the beginning of period t through the pre-order sign-ups before the release of its product and/or the subscriptions to the retailer's social media account (on, e.g., Facebook or Tiktok) in period t. Without loss of generality, we assume that the demand of each retailer i in each period t is larger than 0 with probability 1, i.e., $\mathbb{P}[\Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) + \xi_{i,t} \geq 0] = 1$ for any $(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$.

We assume that $d_{i,t}(\cdot,\cdot)$ is twice continuously differentiable on $[0, \bar{\gamma}_1] \times [0, \bar{\gamma}_2] \times \cdots \times [0, \bar{\gamma}_N] \times [\underline{p}_1, \bar{p}_1] \times [\underline{p}_2, \bar{p}_2] \times \cdots \times [\underline{p}_N, \bar{p}_N]$, and satisfies the following monotonicity properties:

$$\frac{\partial d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}} > 0, \ \frac{\partial d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{j,t}} < 0, \ \frac{\partial d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}} < 0, \text{ and } \frac{\partial d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{j,t}} > 0, \text{ for all } j \neq i. \tag{2}$$

In other words, an increase in a retailer's promotional effort increases the current-period demand of itself, and decreases the demands of its competitors. On the other hand, an increase in a retailer's price decreases the demand of itself, and increases the demands of its competitors. Moreover, we assume that $d_i(\cdot,\cdot)$ is log-separable, i.e., $d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=\psi_i(\boldsymbol{\gamma}_t)\rho_i(\boldsymbol{p}_t)$, where $\psi_i(\cdot)$ and $\rho_i(\cdot)$ are positive and twice-continuously differentiable. Inequalities (2) imply that

$$\frac{\partial\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{i,t}}>0,\ \frac{\partial\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{j,t}}<0,\ \frac{\partial\rho_i(\boldsymbol{p}_t)}{\partial p_{i,t}}<0,\ \text{and}\ \frac{\partial\rho_i(\boldsymbol{p}_t)}{\partial p_{j,t}}>0,\ \text{for all } j\neq i.$$

For technical tractability, we assume that $\psi_i(\cdot)$ and $\rho_i(\cdot)$ satisfy the log increasing differences and the diagonal dominance conditions, i.e., for each i and any t,

$$\frac{\partial^2\log\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{i,t}^2}<0,\ \frac{\partial^2\log\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{i,t}\partial\gamma_{j,t}}\geq 0 \text{ if } j\neq i, \text{ and } \left|\frac{\partial^2\log\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{i,t}^2}\right|>\sum_{j\neq i}\frac{\partial^2\log\psi_i(\boldsymbol{\gamma}_t)}{\partial\gamma_{i,t}\partial\gamma_{j,t}}; \tag{3}$$

$$\frac{\partial^2\log\rho_i(\boldsymbol{p}_t)}{\partial p_{i,t}^2}<0,\ \frac{\partial^2\log\rho_i(\boldsymbol{p}_t)}{\partial p_{i,t}\partial p_{j,t}}\geq 0 \text{ if } j\neq i, \text{ and } \left|\frac{\partial^2\log\rho_i(\boldsymbol{p}_t)}{\partial p_{i,t}^2}\right|>\sum_{j\neq i}\frac{\partial^2\log\rho_i(\boldsymbol{p}_t)}{\partial p_{i,t}\partial p_{j,t}}. \tag{4}$$

The log increasing differences and the diagonal dominance assumptions are not restrictive, and can be satisfied by a large set of commonly used demand models in the economics and operations management literature, such as the linear, logit, Cobb-Douglas, and CES demand functions (see, e.g., Milgrom and Roberts 1990, Bernstein and Federgruen 2004a,b).

The key feature of our model is that current promotion, pricing, and inventory decisions impact upon future demands via the network effect. We assume that the market size of each firm evolves in following functional form:

$$\Lambda_{i,t-1}=\alpha_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=k_i^1\Lambda_{i,t}+k_i^2D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=k_i^1\Lambda_{i,t}+k_i^2\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)+k_i^2\xi_{i,t}, \tag{5}$$

where $k_i^1\in(0,1)$ captures the retention of the market size, and $k_i^2\in(0,1]$ captures the network effects for retailer i. Such network effects have recently become very common in online retailing. For example, consumers are likely to be attracted by the transactions in live-streaming e-commerce and revisit the live-streaming room of the retailer again in the future, see Wongkitrungrueng and Assarut (2020). We denote the market dynamics vector by $\boldsymbol{\alpha}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t):=(\alpha_{1,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t),\alpha_{2,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t),\cdots,\alpha_{N,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))$. Following (5), the future market size of each firm depends on its current market size in a Markovian fashion. Thus, the dynamic competition model in this paper falls into the regime of Markov games.

We introduce the following model primitives. Let δ_i be the discount factor of firm i for revenues and costs in future periods, which satisfies $0<\delta_i\leq 1$. w_i, b_i, and h_i are per-unit wholesales price, backlogging cost, and holding cost paid by firm i, respectively. Without loss of generality, we assume that $b_i>(1-\delta_i)w_i$ and $\bar{p}_i>w_i+\nu_i(\bar{\gamma}_i)$ hold for each i and t. The first inequality means that the backlogging penalty is higher than the saving from delaying an order to the next period for each firm in any period, so that no firm will backlog all of its demand. The second inequality means that the margins for the backlogged demands with the highest price and promotional effort are positive for all firms.

We define the safety stock level decision of firm i in period t as $y_{i,t} := x_{i,t} - \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$, and the normalized expected holding and backlogging cost function for firm i in period t:

$$L_i(y_{i,t}) := \mathbb{E}\{h_i(y_{i,t} - \xi_{i,t})^+ + b_i(y_{i,t} - \xi_{i,t})^-\}, \tag{6}$$

where $y_{i,t} \in \mathbb{R}$ for all i. The state of the Markov game is given by:

$$\boldsymbol{I}_t = (I_{1,t}, I_{2,t}, \cdots, I_{N,t}) = \text{the vector for the starting inventories of all firms in period } t,$$

$$\boldsymbol{\Lambda}_t = (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t}) = \text{the vector for the market sizes of all firms in period } t.$$

We use $\mathcal{S} := \mathbb{R}^N \times \mathbb{R}_+^N$ to denote the state space of $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ in the dynamic competition. We define the action space of each firm i in each period t: $\mathcal{A}_{i,t}(I_{i,t}) := [0, \bar{\gamma}_i] \times [\underline{p}_i, \bar{p}_i] \times [I_{i,t}, +\infty)$.

The rest of this paper focuses on the strategic implications of the network effects. We consider the Markov perfect equilibrium (MPE) in our dynamic competition model and study the impact of the network effects on the MPE. An MPE satisfies two conditions: (a) in each period t, each firm i's promotion, price, and inventory strategy depends on the history of the game only through the current period state variables $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, and (b) in each period t, the strategy profile generates a Nash equilibrium in the associated proper subgame. In other words, MPE is a closed-loop equilibrium that satisfies subgame perfection in each period. Because of its simplicity and consistency with rationality, MPE is widely used in dynamic competition models in economics (e.g., Maskin and Tirole 1988) and operations management (e.g., Olsen and Parker 2008) literature.

4. Simultaneous Competition

In this section, we study the simultaneous competition (SC) model where each firm i simultaneously chooses a combined promotion, price, and inventory strategy in any period t. This model applies to the scenarios where the market expanding efforts (e.g., advertising, consumer subsidies, and live-streaming commission fees paid to influencers on a social media, etc.) take effect instantaneously. Hence, in essence, the promotional effort and price decisions are made simultaneously in each period. Our analysis in this section focuses on characterizing the pure strategy MPE and providing insights on the impact of the network effects in the SC model.

4.1. Equilibrium Analysis

To begin our analysis, we show that the simultaneous competition model has a pure strategy MPE. Moreover, we characterize a sufficient condition on the per-unit demand cost rate of promotional effort, $\nu_i(\cdot)$, under which the MPE is unique. Without loss of generality, we assume that, at the end of the planning horizon (i.e., the end of period 1), each firm i salvages all the on-hand inventory at wholesale price w_i. Under an MPE, each firm i should try to maximize its expected payoff in each subgame (i.e., in each period t) conditioned on the realized inventory levels and market sizes in period t, $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$:

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{\tau=1}^{t}\delta_i^{t-\tau}[p_{i,\tau}D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau)-w_{i,\tau}(x_{i,\tau}-I_{i,\tau})-h_{i,\tau}(x_{i,\tau}-D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau))^+\\
&-b_{i,\tau}(x_{i,\tau}-D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau))^- -\nu_{i,\tau}(\gamma_{i,\tau})D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau)]+\delta_i^t w_i I_{i,0}\Big|\boldsymbol{I}_t,\boldsymbol{\Lambda}_t\Big\}, \qquad (7)\\
\text{s.t.}\quad & I_{i,\tau-1}=x_{i,\tau}-D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau)\text{ for each }\tau,\ t\ge\tau\ge1,\\
\text{and}\quad & \Lambda_{i,\tau-1}=k_i^1\Lambda_{i,\tau}+k_i^2D_{i,\tau}(\boldsymbol{\gamma}_\tau,\boldsymbol{p}_\tau)\text{ for each }\tau,\ t\ge\tau\ge1.
\end{aligned}$$

A (pure) Markov strategy profile in the SC model $\boldsymbol{\sigma}^{sc} := \{\boldsymbol{\sigma}_{i,t}^{sc}(\cdot,\cdot) : 1 \le i \le N, T \ge t \ge 1\}$ prescribes each firm i's combined promotion, price, and inventory strategy in each period t, where $\boldsymbol{\sigma}_{i,t}^{sc}(\cdot,\cdot) := (\gamma_{i,t}^{sc}(\cdot,\cdot), p_{i,t}^{sc}(\cdot,\cdot), x_{i,t}^{sc}(\cdot,\cdot))$ is a Borel measurable mapping from $\mathcal{S}$ to $\mathcal{A}_{i,t}(I_{i,t})$. We use $\boldsymbol{\sigma}_t^{sc} := \{\sigma_{i,\tau}^{sc}(\cdot,\cdot) : 1 \le i \le N, t \ge \tau \ge 1\}$ to denote the pure strategy profile in the induced subgame in period t, which prescribes each firm i's (pure) strategy from period t till the end.

Let $V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_t^{sc})$ be the total expected discounted profit of firm i in periods $t, t-1, \cdots, 1, 0$, when starting period t with the state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and the firms play strategy $\boldsymbol{\sigma}_t^{sc}$ in periods $t, t-1, \cdots, 1$. Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\boldsymbol{\sigma}_t^{sc})$ satisfies the following recursive scheme for each firm i in each period t:

$$V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_t^{sc})=J_{i,t}(\boldsymbol{\gamma}_t^{sc}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t),\boldsymbol{p}_t^{sc}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t),\boldsymbol{x}_t^{sc}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t),\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_{t-1}^{sc}),$$

where $\boldsymbol{\gamma}_t^{sc}(\cdot,\cdot) = (\gamma_{1,t}^{sc}(\cdot,\cdot), \gamma_{2,t}^{sc}(\cdot,\cdot), \cdots, \gamma_{N,t}^{sc}(\cdot,\cdot))$ is the period t promotional effort vector prescribed by $\boldsymbol{\sigma}^{sc}$, $p_t^{sc}(\cdot,\cdot) = (p_{1,t}^{sc}(\cdot,\cdot), p_{2,t}^{sc}(\cdot,\cdot), \cdots, p_{N,t}^{sc}(\cdot,\cdot))$ is the period t sales price vector prescribed by $\boldsymbol{\sigma}^{sc}$, $\boldsymbol{x}_t^{sc}(\cdot,\cdot) = (x_{1,t}^{sc}(\cdot,\cdot), x_{2,t}^{sc}(\cdot,\cdot), \cdots, x_{N,t}^{sc}(\cdot,\cdot))$ is the period t post-delivery inventory vector prescribed by $\boldsymbol{\sigma}^{sc}$,

$$\begin{aligned}
J_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,x_{i,t},\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_{t-1}^{sc}) = \mathbb{E}\Big\{&p_{i,t}D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)-w_i(x_{i,t}-I_{i,t})-h_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^+\\
&-b_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^- -\nu_i(\gamma_{i,t})D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\\
&+\delta_iV_{i,t-1}(\boldsymbol{x}_t-\boldsymbol{D}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t),\boldsymbol{\alpha}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)|\boldsymbol{\sigma}_{t-1}^{sc})|\boldsymbol{I}_t,\boldsymbol{\Lambda}_t\Big\}, \qquad (8)
\end{aligned}$$

and $V_{i,0}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = w_i I_{i,0}$. We now formally define the pure strategy MPE in the SC model.

Definition 1. A (pure) Markov strategy $\boldsymbol{\sigma}^{sc*} = \{(\boldsymbol{\gamma}^{sc*}_{i,t}(\cdot,\cdot), \boldsymbol{p}^{sc*}_{i,t}(\cdot,\cdot), \boldsymbol{x}^{sc*}_{i,t}(\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$ is a pure strategy MPE in the SC model if and only if, for each retailer i, each period t, and each state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$,

$$\begin{aligned}&(\gamma^{sc*}_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t), p^{sc*}_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t), x^{sc*}_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t))\\ =&\underset{(\gamma_{i,t},p_{i,t},x_{i,t})\in\mathcal{A}_{i,t}(I_{i,t})}{\arg\max}\left\{J_{i,t}([\gamma_{i,t},\boldsymbol{\gamma}^{sc*}_{-i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)],[p_{i,t},\boldsymbol{p}^{sc*}_{-i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)],[x_{i,t},\boldsymbol{x}^{sc*}_{-i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)],\boldsymbol{I}_t,\boldsymbol{\Lambda}_t\,\middle|\,\boldsymbol{\sigma}^{sc*}_{t-1})\right\}.\end{aligned} \tag{9}$$

By Definition 1, a (pure) Markov strategy profile in the SC model is a pure strategy MPE if it satisfies subgame perfection in each period t. Definition 1 does not guarantee the existence of an MPE, $\boldsymbol{\sigma}^{sc*}$, in the SC model. In Theorem 1, below, we will show a pure strategy MPE exists in the SC model if the initial inventory of each firm is zero. Moreover, under a mild additional assumption on $\nu_{i,t}(\cdot)$, the SC model has a unique pure strategy MPE. By Definition 1, the equilibrium strategy for retailer i in period t, $(\gamma^{sc*}_{i,t}(\cdot,\cdot), p^{sc*}_{i,t}(\cdot,\cdot), x^{sc*}_{i,t}(\cdot,\cdot))$, may depend on the state vector of its competitors $(\boldsymbol{I}_{-i,t}, \boldsymbol{\Lambda}_{-i,t})$. In practice, however, each retailer i's starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$ are generally its private information that is not accessible by its competitors in the market. We will show that the equilibrium strategy profile of each retailer i in each period t is only contingent on its own realized state variables $(I_{i,t}, \Lambda_{i,t})$, but independent of its competitors' private information $(\boldsymbol{I}_{-i,t}, \boldsymbol{\Lambda}_{-i,t})$. The following theorem characterizes the existence and the uniqueness of MPE in the SC model.

Theorem 1. *If $I_{i,T} = 0$ for all i, then the following statements hold for the SC model:*

(a) There exists a pure strategy MPE $\boldsymbol{\sigma}^{sc*} = \{(\gamma^{sc*}_{i,t}(\cdot,\cdot), p^{sc*}_{i,t}(\cdot,\cdot), x^{sc*}_{i,t}(\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$.

(b) For each pure strategy MPE, $\boldsymbol{\sigma}^{sc*}$, there exists two series of vectors $\{\boldsymbol{\beta}^{sc}_t : T \ge t \ge 1\}$ and $\{\boldsymbol{\eta}^{sc}_t : T \ge t \ge 1\}$, where $\boldsymbol{\beta}^{sc}_t = (\beta^{sc}_{1,t}, \beta^{sc}_{2,t}, \cdots, \beta^{sc}_{N,t})$ with $\beta^{sc}_{i,t} > 0$ for each i, and $\boldsymbol{\eta}^{sc}_t = (\eta^{sc}_{1,t}, \eta^{sc}_{2,t}, \cdots, \eta^{sc}_{N,t})$, such that

$$V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}^{sc*}_t) = w_i I_{i,t} + \beta^{sc}_{i,t}\Lambda_{i,t} + \eta^{sc}_{i,t}, \text{ for } 1 \le i \le N \text{ and } 1 \le t \le T. \tag{10}$$

(c) If the following two conditions hold for each i and t:

(C1) $\nu_i'(\cdot) \le 1$ for all $\gamma_{i,t} \in [0, \bar{\gamma}_i]$; and

(C2) $\nu_i''(\gamma_{i,t})[p_{i,t} - w_i - \nu_i(\gamma_{i,t})] + [\nu_i'(\gamma_{i,t})]^2 \ge \nu_i'(\gamma_{i,t})$ for all $p_{i,t} \in [\underline{p}_i, \bar{p}_i]$ and $\gamma_{i,t} \in [0, \bar{\gamma}_i]$;

then we have $\boldsymbol{\sigma}^{sc*}$ is the unique MPE in the SC model. In particular, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, conditions (C1) and (C2) are satisfied.

Theorem 1 demonstrates the existence of a pure strategy MPE in the simultaneous competition model. Moreover, in Theorem 1(b), we show that, for the pure strategy MPE $\boldsymbol{\sigma}^{sc*}$, the corresponding profit function of each retailer i in each period t is linearly separable in its starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$ as long as the starting inventory level of all retailers is 0 (i.e., $I_{i,T}=0$ for all i). We refer to the constant $\beta^{sc}_{i,t}$ as the SC market size coefficient of retailer i in period t. As we will show later, the SC market size coefficients measure the intensity of the trade-off between current profits and future demands for the retailers. The larger the $\beta^{sc}_{i,t}$, the more intensive this trade-off for retailer i in the previous period $t+1$. Theorem 1 also implies that the equilibrium profit of each firm i in each period t only depends on the state variables of itself $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(\boldsymbol{I}_{-i,t}, \boldsymbol{\Lambda}_{-i,t})$.

The linear separability of $V_{i,t}(\cdot,\cdot|\boldsymbol{\sigma}^{sc*}_t)$ (i.e., Theorem 1(b)) enables us to give a sharper characterization of the MPE in the SC model. Recall that $x_{i,t}-\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=y_{i,t}$ and $D_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)+\xi_{i,t}$. Plugging (10) into the objective function of retailer i in period t, we have:

$$\begin{aligned}
J_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,\boldsymbol{x}_t,\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}^{sc*}_{t-1}) &= \mathbb{E}\Big\{p_{i,t}D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)-w_i(x_{i,t}-I_{i,t})-h_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^+\\
&\quad -b_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^- -\nu_i(\gamma_{i,t})D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\\
&\quad +\delta_iV_{i,t-1}(\boldsymbol{x}_t-\boldsymbol{D}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t),\boldsymbol{\alpha}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)|\boldsymbol{\sigma}^{sc*}_{t-1})|\boldsymbol{I}_t,\boldsymbol{\Lambda}_t\Big\}\\
&= w_iI_{i,t}+\Lambda_{i,t}\Big(\delta_i\beta^{sc}_{i,t-1}k^1_i+d_i(\boldsymbol{\gamma}_i,\boldsymbol{p}_t)[p_{i,t}-w_i-\nu_i(\gamma_{i,t})+\delta_i\beta^{sc}_{i,t-1}k^2_i]\Big)\\
&\quad -(1-\delta_i)w_iy_{i,t}-L_i(y_{i,t})+\delta_i\eta^{sc}_{i,t-1},
\end{aligned}\tag{11}$$

where $\beta^{sc}_{i,0}=0, \eta^{sc}_{i,0}=0$ for each i. For each retailer i, the subgame of period t with the starting state $(I_{i,t},\Lambda_{i,t})$ has a constraint on its decisions: $x_{i,t}=y_{i,t}+\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\geq I_{i,t}$. Therefore, the optimal safety-stock level may depend on the starting inventory level $I_{i,t}$ and expected demand $\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)$. As we will show below, as long as the starting inventory of each retailer is 0, the optimal safety-stock $y_{i,t}$ and the equilibrium promotion and pricing decisions $(\gamma^{sc*}_{i,t}, p^{sc*}_{i,t})$ are irrespective of the inventory level in period t. To prove this, we first define some notations.

$$\begin{aligned}
\pi^{sc}_{i,t}(y_{i,t}) &:= -(1-\delta_i)w_iy_{i,t}-L_i(y_{i,t}),\\
\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t) &:= d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)[p_{i,t}-w_i-\nu_i(\gamma_{i,t})+\delta_i\beta^{sc}_{i,t-1}k^2_i],\\
O^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,y_{i,t}|\Lambda_{i,t}) &:= \Lambda_{i,t}\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)+\pi^{sc}_{i,t}(y_{i,t}).
\end{aligned}$$

For retailer i, given the competitors' pricing profile $\boldsymbol{p}_{-i,t}$ and promotion profile $\boldsymbol{\gamma}_{-i,t}$, and starting state $(I_{i,t},\Lambda_{i,t})$, we define

$$\begin{aligned}
&(y^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}), p^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}), \gamma^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}))\\
:=& \operatorname*{arg\,max}_{\gamma_{i,t}\in[0,\overline{\gamma}_i],p_{i,t}\in[\underline{p}_i,\overline{p}_i],y_{i,t}\in\mathbb{R}} O^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,y_{i,t}|\Lambda_{i,t}),\\
=& \operatorname*{arg\,max}_{\gamma_{i,t}\in[0,\overline{\gamma}_i],p_{i,t}\in[\underline{p}_i,\overline{p}_i]} \Lambda_{i,t}\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)+\operatorname*{arg\,max}_{y_{i,t}\in\mathbb{R}}\pi^{sc}_{i,t}(y_{i,t}),
\end{aligned}\tag{12}$$

$$\begin{aligned}
&(y^{sc*}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}), p^{sc*}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}), \gamma^{sc*}_{i,t}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}))\\
:=& \operatorname*{arg\,max}_{\gamma_{i,t}\in[0,\overline{\gamma}_i],p_{i,t}\in[\underline{p}_i,\overline{p}_i],y_{i,t}+\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\geq I_{i,t}} O^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,y_{i,t}|\Lambda_{i,t}).
\end{aligned}\tag{13}$$

Hence, we write the normalized demand, the demand, and the order-up-to level of retailer i under the solution of (12) as

$$\begin{aligned}
d_i^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}) &:= d_i([\gamma_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}),\boldsymbol{\gamma}_{-i,t}],[p_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}),\boldsymbol{p}_{-i,t}]),\\
D_i^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) &:= \Lambda_{i,t} d_i^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})+\xi_{i,t},\\
x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) &:= y_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})+\Lambda_{i,t} d_i^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}).
\end{aligned}$$

Thus, the best response of each retailer i are presented in the following proposition.

PROPOSITION 1. *For each i, given other retailers' pricing decisions $\boldsymbol{p}_{-i,t}$ and promotion decisions $\boldsymbol{\gamma}_{-i,t}$, the following statements hold for the subgame of period t with states $(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)$:*

(a) $y_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t}) = F_i^{-1}(\frac{b_i-(1-\delta_i)w_i}{h_i+b_i})$.

(b) If $x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) \geq I_{i,t}$, *we have that* $y_i^{sc*}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}) = y_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})$, $\gamma_i^{sc*}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}) = \gamma_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})$, *and* $p_i^{sc*}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},I_{i,t},\Lambda_{i,t}) = p_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})$.

(c) $x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) \geq 0$.

(d) If $t \geq 2$, $\mathbb{P}\left[x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) - D_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t},\Lambda_{i,t}) \leq x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t-1},\boldsymbol{p}_{-i,t-1},\Lambda_{i,t-1})\middle|\Lambda_{i,t}\right] = 1$ *for any $\Lambda_{i,t}$, and any $(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})$ and $(\boldsymbol{\gamma}_{-i,t-1},\boldsymbol{p}_{-i,t-1})$.*

Proposition 1 characterizes the best response of retailer i in period t. In particular, given other retailers' pricing and promotion strategy, if the starting inventory level of firm i is below the optimal base-stock level (i.e. $I_{i,t} \leq x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t},\boldsymbol{p}_{-i,t})$), then the optimal safety-stock is given by $F_i^{-1}(\frac{b_i-(1-\delta_i)w_i}{h_i+b_i})$, and the starting inventory level in the next period will be smaller than the corresponding optimal base-stock level with probability 1. We highlight that this sample-path property is a key technical result that serves as the stepping stone to develop the linear separability of the value functions. This technical observation is proved by adopting a similar argument as Yang and Zhang (2022b), which establish a similar separability result in a monopoly dynamic pricing and inventory model in the presence of online reviews. For notational ease, we define $y_{i,t}^{sc*} := F_i^{-1}(\frac{b_i-(1-\delta_i)w_i}{h_i+b_i})$ and $\pi_{i,t}^{sc*} := \pi_{i,t}^{sc}(y_{i,t}^{sc*})$ for the rest of this paper.

Now we define a noncooperative game $\tilde{\mathcal{G}}_t^{sc}$ in which the N retailers compete on price and promotional effort in period t. That is, the payoff function of firm i in period t is $\Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)$, and the action space is defined as

$$\begin{aligned}
\mathcal{A}'_t := \ &\{(\gamma_{i,1},\cdots,\gamma_{i,N},p_{i,1},\cdots,p_{i,N}) \in [0,\overline{\gamma}_1]\times\cdots\times[0,\overline{\gamma}_N]\times[\underline{p}_1,\overline{p}_1]\times\cdots\times[\underline{p}_N,\overline{p}_N]:\\
&p_{i,t}-w_i-\nu_i(\gamma_{i,t})>0, 1\leq i\leq N\},
\end{aligned}$$

i.e., the profit margin of period t should be positive for each retailer i. We characterize the Nash equilibrium of the game $\tilde{\mathcal{G}}_t^{sc}$ in the following proposition.

Proposition 2. *For each period t, the following statements hold:*

(a) The joint promotion and price competition, $\tilde{\mathcal{G}}_t^{sc}$, is a log-supermodular game.

(b) If conditions (C1) and (C2) are satisfied, the following statements hold for the game $\tilde{\mathcal{G}}_t^{sc}$:

(i) $\tilde{\mathcal{G}}_t^{sc}$ has a unique pure strategy Nash equilibrium $(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})$, which is the unique serially undominated strategy of $\tilde{\mathcal{G}}_t^{sc}$.

(ii) The Nash equilibrium of $\tilde{\mathcal{G}}_t^{sc}$ is the unique solution to the following system of equations:

$$\begin{aligned}
&\text{For each } i,\ \frac{\partial_{\gamma_{i,t}}\psi_i(\boldsymbol{\gamma}_t^{sc*})}{\psi_i(\boldsymbol{\gamma}_t^{sc*})} - \frac{\nu_i'(\gamma_{i,t}^{sc*})}{p_{i,t}^{sc*} - w_i - \nu_i(\gamma_{i,t}^{sc*}) + \delta_i \beta_{i,t-1}^{sc} k_i^2} \begin{cases} \le 0, & \text{if } \gamma_{i,t}^{sc*} = 0, \\ = 0, & \text{if } \gamma_{i,t}^{sc*} \in (0, \bar{\gamma}_i), \text{ and,} \\ \ge 0 & \text{if } \gamma_{i,t}^{sc*} = \bar{\gamma}_i; \end{cases} \\
&\text{for each } i,\ \frac{\partial_{p_{i,t}}\rho_i(\boldsymbol{p}_t^{sc*})}{\rho_i(\boldsymbol{p}_t^{sc*})} + \frac{1}{p_{i,t}^{sc*} - w_i - \nu_i(\gamma_{i,t}^{sc*}) + \delta_i \beta_{i,t-1}^{sc} k_i^2} \begin{cases} \le 0, & \text{if } p_{i,t}^{sc*} = \underline{p}_i, \\ = 0, & \text{if } p_{i,t}^{sc*} \in (\underline{p}_i, \bar{p}_i), \\ \ge 0 & \text{if } p_{i,t}^{sc*} = \bar{p}_i. \end{cases}
\end{aligned} \tag{14}$$

(iii) Let $\boldsymbol{\Pi}_t^{sc*} := (\Pi_{1,t}^{sc*}, \Pi_{2,t}^{sc*}, \cdots, \Pi_{N,t}^{sc*})$ be the equilibrium payoff vector of the game $\tilde{\mathcal{G}}_t^{sc}$, where $\Pi_{i,t}^{sc*} = \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})$. We have $\Pi_{i,t}^{sc*} > 0$ for all i. In particular, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, (C1) and (C2) are satisfied.

Proposition 2 shows that the noncooperative game $\tilde{\mathcal{G}}_t^{sc}$ is a log-supermodular game, and has a unique pure strategy Nash equilibrium $(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})$ if the conditions $(C1)$ and $(C2)$ are satisfied. The unique Nash equilibrium, $(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})$, is determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (14). Note that the pure-strategy Nash equilibrium $(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})$ satisfies $\gamma_{i,t}^{sc*} = \gamma_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*})$ and $p_{i,t}^{sc*} = p_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*})$. Therefore, given other retailers' promotion and pricing profile $(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*})$, retailer i chooses promotion $\gamma_{i,t}^{sc*}$, price $p_{i,t}^{sc*}$, and safety-stock level $y_{i,t}^{sc*}$ as long as its starting inventory level $I_{i,t}$ is lower than $x_i^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*}, \Lambda_{i,t})$. Hence, if $I_{i,t} \le x_i^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*}, \Lambda_{i,t})$ for all i, the unique MPE in period t is characterized by $(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*}, \boldsymbol{y}_t^{sc*})$.

Theorem 2. *If $I_{i,T} = 0$ for each i and the conditions (C1) and (C2) are satisfied for each i and t, then the following statements hold for each retailer i and each period t:*

(a) The policy of retailer i in period t under the pure strategy MPE σ^{sc*} is

$$(\gamma_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), p_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), x_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)) = (\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, y_{i,t}^{sc*} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t^{sc*}, \boldsymbol{p}_t^{sc*})). \tag{15}$$

(b) The following recursive relations hold: $\beta_{i,t}^{sc} = \delta_i \beta_{i,t-1}^{sc} k_i^1 + \Pi_{i,t}^{sc*} > 0$, and $\eta_{i,t}^{sc} = \delta_i \eta_{i,t-1}^{sc} + \pi_i^{sc*}$.

(c) The following sample-path property holds:

$$\mathbb{P}\Big[x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*}, \Lambda_{i,t}) - D_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*}, \Lambda_{i,t}) \le x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t-1}^{sc*}, \boldsymbol{p}_{-i,t-1}^{sc*}, \Lambda_{i,t-1}) \Big| \Lambda_{i,t}\Big] = 1,$$

for any $\Lambda_{i,t}$, where $(\boldsymbol{\gamma}_{-i,t}^{sc*}, \boldsymbol{p}_{-i,t}^{sc*})$ and $(\boldsymbol{\gamma}_{-i,t-1}^{sc*}, \boldsymbol{p}_{-i,t-1}^{sc*})$ are the equilibrium promotion and price decisions of retailer i's competitors in periods t and $t-1$, respectively.

Theorem 2(a) demonstrates that, under the MPE σ^{sc*}, each retailer i's joint promotion, price, and inventory policy in each period t only depends on its own state variables $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(\boldsymbol{I}_{-i,t}, \boldsymbol{\Lambda}_{-i,t})$, which are not accessible to retailer i in general. Thus, for each retailer i in each period t, its equilibrium strategy has the desirable feature that it depends on accessible information only. Theorem 2(b) recursively computes the SC market size coefficient vectors $\{\beta_t^{sc} : T \geq t \geq 1\}$.

In some of our analysis below, we will focus on a special case of the SC model, where the market is symmetric, i.e., all competing retailers have identical characteristics. We use the subscript "s" to denote the case of symmetric market. In this case, for all i, j, and t, let $d_s(\cdot) := d_i(\cdot)$, $\rho_s(\cdot) := \rho_i(\cdot)$, $\psi_s(\cdot) := \psi_i(\cdot)$, $\nu_s(\cdot) := \nu_i(\cdot)$, $\xi_{s,t} := \xi_{i,t}$, $k_s^1 := k_i^1$, $k_s^2 := k_i^2$, $w_s := w_i$, $h_s := h_i$, $b_s := b_i$, and $\delta_s := \delta_i$. Thus, we define $\pi_{s,t}^{sc}(y_{i,t}) := -(1-\delta_s)w_s y_{i,t} - L_s(y_{i,t})$, and $\Pi_{s,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) := d_s(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)[p_{i,t} - w_s - \nu_s(\gamma_{i,t}) + \delta_s \beta_{s,t-1}^{sc} k_s^2]$. Then, as Proposition 1 shows, $y_{i,t}^{sc*} = F_s^{-1}(\frac{b_s-(1-\delta_s)w_s}{h_s+b_s})$. By Theorem 1, there exists a unique pure strategy MPE in the symmetric SC model, which we denote as $\boldsymbol{\sigma}_s^{sc*}$. The following proposition is a corollary of Theorems 1-2.

PROPOSITION 3. *If $I_{i,T} = 0$ for all i, the following statements hold for the symmetric SC model:*

(a) For each $t = T, T-1, \cdots, 1$, there exist constants $\beta_{s,t}^{sc} > 0$ and $\eta_{s,t}^{sc}$, such that

$$V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{s,t}^{sc*}) = w_s I_{i,t} + \beta_{s,t}^{sc} \Lambda_{i,t} + \eta_{s,t}^{sc}, \text{ for all } i.$$

(b) In each period t, the game $\mathcal{G}_t^{sc}$ is symmetric. Moreover, $\mathcal{G}_t^{sc}$ has a unique pure strategy Nash equilibrium $(\boldsymbol{\gamma}_{ss,t}^{sc*}, \boldsymbol{p}_{ss,t}^{sc*})$ which is symmetric (i.e., $\boldsymbol{\gamma}_{ss,t}^{sc*} = (\gamma_{s,t}^{sc*}, \gamma_{s,t}^{sc*}, \cdots, \gamma_{s,t}^{sc*})$ for some $\gamma_{s,t}^{sc*}$ and $\boldsymbol{p}_{ss,t}^{sc*} = (p_{s,t}^{sc*}, p_{s,t}^{sc*}, \cdots, p_{s,t}^{sc*})$ for some $p_{s,t}^{sc*}$).

(c) The policy of retailer i in period t under the unique pure strategy MPE $\boldsymbol{\sigma}_s^{sc*}$ is

$$(\gamma_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), p_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), x_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)) = (\gamma_{s,t}^{sc*}, p_{s,t}^{sc*}, y_{s,t}^{sc*} + \Lambda_{i,t} d_s(\boldsymbol{\gamma}_{ss,t}^{sc*}, \boldsymbol{p}_{ss,t}^{sc*})), \text{ for each } (\boldsymbol{I}_t, \boldsymbol{\Lambda}_t).$$

Proposition 3 characterizes the MPE, $\boldsymbol{\sigma}_s^{sc*}$, and the market size coefficients, $\{\beta_{s,t}^{sc} : T \geq t \geq 1\}$, in the symmetric SC model. Proposition 3 shows that, in the symmetric SC model, all competing retailers set the same promotional effort, price, and safety-stock level in each period under MPE, whereas the equilibrium market outcome may vary in different periods.

4.2. Implications of Network Effects

In this subsection, we study the operations implications of network effects in the SC model. We focus on how network effects drive the trade-off between current profits and future demands in a dynamic and competitive market. Therefore, following the results in Theorem 1 and Proposition 3, we assume that $I_{i,T} = 0$ and conditions (C1) and (C2) are satisfied for all i and t in this section. To begin with, we characterize the impact of the market size coefficient vectors $\{\boldsymbol{\beta}_t^{sc} : T \geq t \geq 1\}$ upon the equilibrium market outcome. The following theorem serves as the building block of our subsequent analysis of the trade-off between current profits and future demands in the SC model.

Theorem 3. *For each period t, the following statements hold for the symmetric SC model:*

(a) $\gamma_{s,t}^{sc*}$ is continuously increasing in $\beta_{s,t-1}^{sc}$ and k_s^2, whereas $p_{s,t}^{sc*}$ is continuously decreasing in $\beta_{s,t-1}^{sc}$ and k_s^2.

(b) If $\psi_{s,t}(\cdot)$ and $\rho_{s,t}(\cdot)$ satisfy the following monotonicity condition

$$\sum_{i=1}^{N} \frac{\partial \psi_{s,t}(\boldsymbol{\gamma}_t)}{\partial \gamma_{i,t}} > 0, \text{ for all } \boldsymbol{\gamma}_t, \text{ and } \sum_{i=1}^{N} \frac{\partial \rho_{s,t}(\boldsymbol{p}_t)}{\partial p_{i,t}} < 0, \text{ for all } \boldsymbol{p}_t, \tag{16}$$

$\Pi_{s,t}^{sc*} := \Pi_{i,t}^{sc}\left(\boldsymbol{\gamma}_{ss,t}^{sc*}, \boldsymbol{p}_{ss,t}^{sc*}\right)$ is continuously increasing in $\beta_{s,t-1}^{sc}$ and k_s^2.

(c) If the monotonicity condition (16) holds, $\beta_{s,t}^{sc}$ is continuously increasing in $\beta_{s,t-1}^{sc}$, k_s^1 and k_s^2.

Theorem 3 shows that, in the symmetric SC model, the market size coefficients $\{\beta_{s,t}^{sc} : T \geq t \geq 1\}$ quantify the intensity of the trade-off between current profits and future demands. More specifically, if $\beta_{s,t-1}^{sc}$ is larger, each retailer i faces a stronger trade-off in period t. Therefore, to balance this enhanced trade-off, each retailer would decrease the equilibrium price and increase the equilibrium promotional effort, as shown in Theorem 3(a). Moreover, Theorem 3(c) characterizes the relationship between this trade-off in different periods, demonstrating that if the trade-off is more intensive in the next period, it is also stronger in the current period under a mild condition (16). The monotonicity condition (16) implies that a uniform increase of all N retailers' promotional efforts leads to an increase in the demand of each retailer, and a uniform price increase by all N retailers gives rise to a decrease in the demand of each firm. This condition is commonly used in the literature (see, e.g., Bernstein and Federgruen 2004b, Allon and Federgruen 2007), and often referred to as the "dominant diagonal" condition for linear demand models.

Now we directly study the impact of network effects. We compare two symmetric SC models where all parameters are same except for the strength of network effect, i.e., the value of k_s^2. Assume that $\hat{k}_s^2 \geq k_s^2$. Then we have the following theorem as a corollary of Proposition 3 and Theorem 3.

Theorem 4. *Consider the symmetric SC model. For each period t, the following statements hold:*

(a) $\hat{\gamma}_{s,t}^{sc*} \geq \gamma_{s,t}^{sc*}$ and, thus, $\hat{\gamma}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \geq \gamma_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and all i.

(b) $\hat{p}_{s,t}^{sc*} \leq p_{s,t}^{sc*}$ and, thus, $\hat{p}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \leq p_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and all i.

(c) If the monotonicity condition (16) holds, we have $\hat{x}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \geq x_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and all i.

Theorem 4 highlights the impact of market size dynamics upon the equilibrium market outcome in the symmetrical SC model. Specifically, Theorem 4(a) shows that each retailer should increase its promotional effort in each period under a stronger network effect, in order to induce higher future demands. Analogously, Theorem 4(b) shows that the enhanced the network effect gives rise to a lower equilibrium price of each retailer in each period. Under the monotonicity condition (16), Theorem 4(a,b) imply that the equilibrium expected demand of each retailer in each period is higher under a stronger network effect. As a consequence, to match supply with the current demand and to induce high future demands, each retailer should increase its base stock level in each period under the network effect, as shown in Theorem 4(c).

Theorem 4 identifies effective strategies for firms to balance the trade-off between current profits and future demands under network effects. To balance this trade-off, the retailers can employ two strategies to exploit network effects: (a) offering price discounts and (b) improving promotional efforts. Offering price discounts and improving promotional efforts do not only increase the current profits but give rise to higher current demands and, thus, induce higher future demands via network effects. In a nutshell, the uniform idea of all these strategies is that, to balance the trade-off between current profits and future demands under network effects, the competing retailers should induce higher future demands at the cost of reduced current margins.

Next, we analyze the trade-off between current profits and future demands from an inter-temporal perspective. Under network effects, how should the competing retailers adjust their promotion and price strategies throughout the sales season to balance this trade-off? To address this question, we characterize the evolution of the equilibrium market outcome in the symmetric SC model.

THEOREM 5. *Consider the symmetric SC model. If the monotonicity condition* (16) *holds for each period t, the following statements hold:*

(a) $\beta^{sc}_{s,t} \geq \beta^{sc}_{s,t-1}$, $\gamma^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t-1}$, $p^{sc*}_{s,t} \leq p^{sc*}_{s,t-1}$.

(b) $\gamma^{sc*}_{i,t}(\boldsymbol{I},\boldsymbol{\Lambda}) \geq \gamma^{sc*}_{i,t-1}(\boldsymbol{I},\boldsymbol{\Lambda})$, $p^{sc*}_{i,t}(\boldsymbol{I},\boldsymbol{\Lambda}) \leq p^{sc*}_{i,t-1}(\boldsymbol{I},\boldsymbol{\Lambda})$, *and* $x^{sc*}_{i,t}(\boldsymbol{I},\boldsymbol{\Lambda}) \geq x^{sc*}_{i,t-1}(\boldsymbol{I},\boldsymbol{\Lambda})$ *for each i and each* $(\boldsymbol{I},\boldsymbol{\Lambda}) \in \mathcal{S}$.

Theorem 5 sheds light on how to balance the above trade-off from an inter-temporal perspective. More specifically, we show that, if the market is symmetric and stationary, this trade-off is more intensive (i.e., $\beta^{sc}_{s,t}$ is larger) at the early stage of the sales season. Moreover, the equilibrium price is increasing, whereas the equilibrium promotional effort is decreasing, over the planning horizon. The network effects have stronger impact upon future demands (and, hence, future profits) when the remaining planning horizon is longer. Therefore, to adaptively balance the trade-off throughout the sales season, all the retailers increase their prices and decrease their promotional efforts towards the end of the sales season. In a dynamic competition, our analysis justifies the widely used introductory price and promotion strategy with which retailers offer discounts and launch promotional campaigns at the beginning of a sales season to attract more early purchases (see, e.g., Cabral et al. 1999, Parker and Van Alstyne 2005, Eisenmann et al. 2006).

To summarize, under network effects, the competing retailers have to trade off between generating current profits and inducing future demands. To effectively balance this trade-off, the retailers should (a) increase promotional efforts and (b) offer price discounts. Moreover, this trade-off is more intensive (a) with stronger network effects, or (b) at the early stage of the sales season.

5. Promotion-First Competition

Promotion timing may be different for different promotion tools. For example, highly personalized re-targeting is more effective early in a consumer's purchase decision procedure. See, e.g., Bleier and Eisenbeiss (2015). To capture this context, we consider the promotion-first competition (PF) model, i.e., in each period t, each retailer i first selects its promotional effort and then, after observing the current-period promotional efforts of all retailers, chooses a combined price and inventory strategy. This model applies broadly to the scenario where promotion is more effective to expand the market size at an early stage than at later stages, so it should be settled in advance of price and inventory decision.

Employing the linear separability approach based on a sample-path argument, we will show that, in the PF model, the retailers engage in a two-stage competition in each period, where they compete on promotional effort in the first stage, and on price in the second. Similar to the SC model, retailers optimize their inventory individually in the second stage competition. We will also demonstrate that the trade-off between current profits and future demands has more involved managerial implications in the PF model than in the SC model. In the SC model, the competing retailers balance this trade-off inter-temporally, whereas they balance it both inter-temporally and intra-temporally in the PF model.

For tractability, we make the following additional assumption throughout this section:

$$\rho_i(\boldsymbol{p}_t) = \phi_i - \theta_{ii} p_{i,t} + \sum_{j \neq i} \theta_{ij} p_{j,t}, \text{ for each } i \text{ and } t, \tag{17}$$

where $\phi_i, \theta_{ii} > 0$ and $\theta_{ij} \geq 0$ for each i and j. Moreover, we assume that the diagonal dominance conditions hold for each $\rho_i(\cdot)$, i.e., for each i and t, $\theta_{ii} > \sum_{j \neq i} \theta_{ij}$ and $\theta_{ii} > \sum_{j \neq i} \theta_{ji}$. In addition, we make the same assumption as Allon and Federgruen (2007) as follows:

Assumption 1. *For each i, the minimum [maximum] allowable price $\underline{p}_i$ [$\bar{p}_i$] is sufficiently low [high] so that it will have no impact on the equilibrium market behavior.*

We will give a sufficient condition for Assumption 1 in the discussion after Proposition 6.

5.1. Equilibrium Analysis

In this subsection, we use the linear separability approach to characterize the pure strategy MPE in the PF model. In this model, a (pure) Markov strategy profile of retailer i in period t is given by $\boldsymbol{\sigma}_{i,t}^{pf} = (\gamma_{i,t}^{pf}(\cdot,\cdot), p_{i,t}^{pf}(\cdot,\cdot,\cdot), x_{i,t}^{pf}(\cdot,\cdot,\cdot))$, where $\gamma_{i,t}^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ prescribes the promotional effort given the state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, and $(p_{i,t}^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t), x_{i,t}^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t))$ prescribes the price and the post-delivery inventory level, given the state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and the current period promotional effort vector $\boldsymbol{\gamma}_t$. Let $\boldsymbol{\gamma}_t^{pf}(\cdot,\cdot) := (\gamma_{1,t}^{pf}(\cdot,\cdot), \gamma_{2,t}^{pf}(\cdot,\cdot), \cdots, \gamma_{N,t}^{pf}(\cdot,\cdot))$, $\boldsymbol{p}_t^{pf}(\cdot,\cdot,\cdot) := (p_{1,t}^{pf}(\cdot,\cdot,\cdot), p_{2,t}^{pf}(\cdot,\cdot,\cdot), \cdots, p_{N,t}^{pf}(\cdot,\cdot,\cdot))$, and $\boldsymbol{x}_t^{pf}(\cdot,\cdot,\cdot) := (x_{1,t}^{pf}(\cdot,\cdot,\cdot), x_{2,t}^{pf}(\cdot,\cdot,\cdot), \cdots, x_{N,t}^{pf}(\cdot,\cdot,\cdot))$. We use $\boldsymbol{\sigma}_t^{pf}$ to denote the (pure) strategy profile of all retailers in the subgame of period t, which prescribes their (pure) strategies from period t to the end of the planning horizon.

Let $V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_t^{pf})$ be the total expected discounted profit of retailer i in periods $t, t-1, \cdots, 1, 0$, when starting period t with the state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and the retailers play strategy $\boldsymbol{\sigma}_t^{pf}$ in periods $t, t-1, \cdots, 1$. Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\boldsymbol{\sigma}_t^{pf})$ satisfies the following recursive scheme for each retailer i and each period t:

$$V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_t^{pf}) = J_{i,t}(\boldsymbol{\gamma}_t^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), \boldsymbol{p}_t^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)), \boldsymbol{x}_t^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t^{pf}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)), \boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{t-1}^{pf}),$$

where

$$\begin{aligned} J_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t, x_{i,t}, \boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{t-1}^{pf}) = \mathbb{E}\{ & p_{i,t} D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) - w_i(x_{i,t} - I_{i,t}) - h_i(x_{i,t} - D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))^+ \\ & - b_i(x_{i,t} - D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))^- - \nu_{i,t}(\gamma_i) D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) \\ & + \delta_i V_{i,t-1}(\boldsymbol{x}_t - \boldsymbol{D}_t(\boldsymbol{\gamma}_t, \boldsymbol{p}_t), \boldsymbol{\alpha}_t(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) | \boldsymbol{\sigma}_{t-1}^{pf}) | \boldsymbol{I}_t, \boldsymbol{\Lambda}_t \}, \end{aligned} \tag{18}$$

and $V_{i,0}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = w_{i,0} I_{i,0}$. We now define the pure strategy MPE in the PF model.

Definition 2. A (pure) Markov strategy $\boldsymbol{\sigma}^{pf*} = \{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$ is a pure strategy MPE in the PF model if and only if, for each firm i, period t, and state variable $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \in \mathcal{S}$,

$$\begin{aligned} & (p_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t), x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)) \\ & = \underset{p_{i,t} \in [\underline{p}_i, \bar{p}_i], x_{i,t} \ge I_{i,t}}{\arg\max} J_{i,t}\left(\boldsymbol{\gamma}_t, [p_{i,t}, \boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)], [x_{i,t}, \boldsymbol{x}_{-i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)], \boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{t-1}^{pf*}\right), \end{aligned} \tag{19}$$

and

$$\begin{aligned} \gamma_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \underset{\gamma_{i,t} \in [0, \bar{\gamma}_i]}{\arg\max} J_{i,t} & \left([\gamma_{i,t}, \boldsymbol{\gamma}_{-i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)], \boldsymbol{p}_t^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, [\gamma_{i,t}, \boldsymbol{\gamma}_{-i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)]), \right. \\ & \left. \boldsymbol{x}_t^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, [\gamma_{i,t}, \boldsymbol{\gamma}_{-i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)]), \boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{t-1}^{pf*}\right). \end{aligned} \tag{20}$$

Definition 2 suggests that a pure strategy MPE in the PF model is a (pure) Markov strategy profile that satisfies subgame perfection in each stage of the competition in each period t. The following theorem shows that there exists a pure strategy MPE in the PF model.

THEOREM 6. *If $I_{i,T}=0$ for all i, the following statements hold for the PF model:*

(a) There exists a pure strategy MPE $\boldsymbol{\sigma}^{pf*}$.

(b) For each pure strategy MPE $\boldsymbol{\sigma}^{pf*}$, there exists two series of vectors $\{\boldsymbol{\beta}_t^{pf}: T\geq t\geq 1\}$ and $\{\boldsymbol{\eta}_t^{pf}: T\geq t\geq 1\}$, where $\boldsymbol{\beta}_t^{pf}=(\beta_{1,t}^{pf},\beta_{2,t}^{pf},\cdots,\beta_{N,t}^{pf})$ with $\beta_{i,t}^{pf}>0$ for each i and t, and $\boldsymbol{\eta}_t^{pf}=(\eta_{1,t}^{pf},\eta_{2,t}^{pf},\cdots,\eta_{N,t}^{pf})$, such that

$$V_{i,t}(I_{i,t},\Lambda_{i,t}|\boldsymbol{\sigma}_t^{pf*})=w_{i,t}I_{i,t}+\beta_{i,t}^{pf}\Lambda_{i,t}+\eta_{i,t}^{pf},\ \text{for each } i,\ t,\ \text{and } (\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)\in\mathcal{S}. \tag{21}$$

(c) If $\nu_{i,t}(\gamma_{i,t})=\gamma_{i,t}$ for each i and t, $\boldsymbol{\sigma}^{pf*}$ is the unique MPE in the PF model.

Theorem 6 demonstrates the existence of a pure strategy MPE in the PF model. As in the SC model, we show that, for each pure strategy MPE $\boldsymbol{\sigma}^{pf*}$, the associated profit function of each firm i in each period t is linearly separable in its own starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. We refer to the constant $\beta_{i,t}^{pf}$ as the PF market size coefficient of firm i in period t, which measures the intensity of the trade-off between current profits and future demands in the PF model.

The linear separability of $V_{i,t}(\cdot,\cdot|\boldsymbol{\sigma}_t^{pf*})$ enables us to have a sharper characterization of MPE in the PF model. As in the SC model, we can rewrite the objective function of retailer i in period t as follows.

$$\begin{aligned}
J_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t,\boldsymbol{x}_t,\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_{t-1}^{pf*}) &= \mathbb{E}\{p_{i,t}D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)-w_i(x_{i,t}-I_{i,t})-h_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^+\\
&\quad -b_i(x_{i,t}-D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t))^- -\nu_i(\gamma_{i,t})D_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\\
&\quad +\delta_i V_{i,t-1}(\boldsymbol{x}_t-D_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t),\boldsymbol{\alpha}_t(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)|\boldsymbol{\sigma}_{t-1}^{pf*})|\boldsymbol{I}_t,\boldsymbol{\Lambda}_t\}\\
&= w_iI_{i,t}+\Lambda_{i,t}\{\delta_i\beta_{i,t-1}^{pf}k_i^1+d_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)[p_{i,t}-w_i-\nu_i(\gamma_{i,t})+\delta_i\beta_{i,t-1}^{pf}k_i^2]\}\\
&\quad -(1-\delta_i)w_iy_{i,t}-L_i(y_{i,t})+\delta_i\eta_{i,t-1}^{pf},
\end{aligned} \tag{22}$$

where $\beta_{i,0}^{pf}=0$ and $\eta_{i,0}^{pf}=0$ for each i. We observe from (22) that, all retailers participate in a two-stage competition: in the first stage, the retailers compete on promotional effort; in the second stage, they compete on price and optimize their inventory levels individually. For each period t, by backward induction, we start with analyzing the optimal safety-stock level and pricing strategy in equilibrium, for a given promotion strategy $\boldsymbol{\gamma}_t$. For a given promotion vector $\boldsymbol{\gamma}_t$, we define the following functions:

$$\begin{aligned}
\pi_{i,t}^{pf}(y_{i,t}) &:= -(1-\delta_i)w_iy_{i,t}-L_i(y_{i,t}),\\
\Pi_{i,t}^{pf}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t) &:= d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)[p_{i,t}-w_i-\nu_i(\gamma_{i,t})+\delta_i\beta_{i,t-1}^{pf}k_i^2],\\
O_{i,t}^{pf}(\boldsymbol{p}_t,y_{i,t}|\boldsymbol{\gamma}_t,\Lambda_{i,t}) &:= \Lambda_{i,t}\Pi_{i,t}^{pf}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t)+\pi_{i,t}^{pf}(y_{i,t}).
\end{aligned}$$

Therefore, given $\boldsymbol{I}_t$, $\boldsymbol{\Lambda}_t$, and promotion strategy $\boldsymbol{\gamma}_t$, we define the best response of retailer i when other retailers pick pricing decisions $\boldsymbol{p}_{-i,t}$ as,

$$
\begin{aligned}
&(y_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t), p_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t)) := \underset{p_{i,t}\in[\underline{p}_i,\overline{p}_i], y_{i,t}\in\mathbb{R}}{\arg\max} O_{i,t}^{pf}(\boldsymbol{p}_t, y_{i,t}|\boldsymbol{\gamma}_t,\Lambda_{i,t}),\\
=\ &\underset{p_{i,t}\in[\underline{p}_i,\overline{p}_i]}{\arg\max}\Lambda_{i,t}\Pi_{i,t}^{pf}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t) + \underset{y_{i,t}\in\mathbb{R}}{\arg\max}\pi_{i,t}^{pf}(y_{i,t}), \qquad (23)\\
&(y_{i,t}^{pf*}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t, I_{i,t},\Lambda_{i,t}), p_{i,t}^{pf*}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t, I_{i,t},\Lambda_{i,t}))\\
:=\ &\underset{p_{i,t}\in[\underline{p}_i,\overline{p}_i], y_{i,t}+\Lambda_{i,t}d_i(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)\geq I_{i,t}}{\arg\max} O_{i,t}^{sc}(\boldsymbol{p}_t, y_{i,t}|\boldsymbol{\gamma}_t,\Lambda_{i,t}). \qquad (24)
\end{aligned}
$$

Hence, when the inventory level constraints are not binding, the retailers can optimize their inventory decisions individually. We can write demand functions and inventory levels associated with the solution to (23) as:

$$
\begin{aligned}
d_i^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t) &:= d_i(\boldsymbol{\gamma}_t,[p_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t),\boldsymbol{p}_{-i,t}]),\\
D_i^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) &:= \Lambda_{i,t}d_i^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t)+\xi_{i,t},\\
x_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) &:= y_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t)+\Lambda_{i,t}d_i^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t).
\end{aligned}
$$

The following proposition presents a similar result of Proposition 1 for the PF model.

PROPOSITION 4. *For each i, given any promotion profile $\boldsymbol{\gamma}_t$, the following statements hold for the subgame of period t with states $(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)$ when other firms set price $\boldsymbol{p}_{-i,t}$:*

(a) $y_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t) = F_i^{-1}(\frac{b_i-(1-\delta_i)w_i}{h_i+b_i})$.

(b) If $x_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) \geq I_{i,t}$, $y_i^{pf*}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,I_{i,t},\Lambda_{i,t}) = y_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t)$, $p_i^{sc*}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,I_{i,t},\Lambda_{i,t}) = p_{i,t}^{sc}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t)$.

(c) $x_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) \geq 0$.

(d) If $t\geq 2$, $\mathbb{P}\Big[x_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) - D_{i,t}^{pf}(\boldsymbol{p}_{-i,t},\boldsymbol{\gamma}_t,\Lambda_{i,t}) \leq x_{i,t}^{pf}(\boldsymbol{p}_{-i,t-1},\boldsymbol{\gamma}_{t-1},\Lambda_{i,t-1})\Big|\Lambda_{i,t}\Big] = 1$ *for any* $\Lambda_{i,t-1}$, *and any* $(\boldsymbol{\gamma}_t,\boldsymbol{p}_{-i,t})$ *and* $(\boldsymbol{\gamma}_{t-1},\boldsymbol{p}_{-i,t-1})$.

Proposition 4 characterizes the best response of firm i in the second stage of the game. Most importantly, part (d) shows that the critical sample-path property holds for the PF model: Under equilibrium, the starting inventory of any retailer in any period is below the equilibrium base-stock level of the next period. This property ensures that linear separability also holds for the PF model. Define $y_{i,t}^{pf*} := F_i^{-1}(\frac{b_i-(1-\delta_i)w_i}{h_i+b_i})$, which is the solution to the first-order condition $\partial_{y_{i,t}}\pi_{i,t}^{pf}(y_{i,t}) = 0$, and $\pi_{i,t}^{pf*} := \pi_{i,t}^{pf}(y_{i,t}^{pf*})$.

We now analyze the competition of each period in more detail. Consider a two-stage game in which N retailers compete on promotion profile $\boldsymbol{\gamma}_t$ in the first stage and then on price in the second. Given a promotion profile $\boldsymbol{\gamma}_t$, the second-stage game $\mathcal{G}_t^{pf,2}(\boldsymbol{\gamma}_t)$ has a payoff function

$O^{pf}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\boldsymbol{\gamma}_t, \Lambda_{i,t})$ for all i, and action space is the set of $(\boldsymbol{p}_t, \boldsymbol{y}_t)$ such that $p_{i,t} \in [\underline{p}_i, \overline{p}_i]$ and $y_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) \geq I_{i,t}$ for all i.

Similar to the SC model, we first define a noncooperative game $\tilde{\mathcal{G}}^{pf,2}_t(\boldsymbol{\gamma}_t)$ for any $\boldsymbol{\gamma}_t$, in which the payoff function is $\Pi^{pf}_{i,t}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t)$, and the action space is $[\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \cdots \times [\underline{p}_N, \overline{p}_N]$. We define A as an $N \times N$ matrix with entries defined by $A_{ii} := 2\theta_{ii}$ and $A_{ij} := -\theta_{ij}$ where $i \neq j$. By Lemma 2(a) in the Appendix, A is invertible. Let $\boldsymbol{f}(\boldsymbol{\gamma}_t)$ be an $N-$dimensional vector with $f_i(\boldsymbol{\gamma}_t) := \phi_i + \theta_{ii}(w_i + \nu_i(\gamma_{i,t}) - \delta_i \beta^{pf}_{i,t-1} k_i^2)$. We characterize the Nash equilibrium of the game $\tilde{\mathcal{G}}^{pf,2}_t(\boldsymbol{\gamma}_t)$ in the following proposition.

Proposition 5. *For each period t and any given $\boldsymbol{\gamma}_t$, the following statements hold:*

(a) The price competition $\tilde{\mathcal{G}}^{pf,2}_t(\boldsymbol{\gamma}_t)$ has a unique pure strategy Nash equilibrium $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)$.

(b) $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t) = \boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}_t)$. Moreover, $p^{pf*}_{i,t}(\boldsymbol{\gamma}_t)$ is continuously increasing in $\gamma_{j,t}$ for each i and j.

(c) Let $\boldsymbol{\Pi}^{pf*,2}_t(\boldsymbol{\gamma}_t) := (\Pi^{pf*,2}_{1,t}(\boldsymbol{\gamma}_t), \Pi^{pf*,2}_{2,t}(\boldsymbol{\gamma}_t), \cdots, \Pi^{pf*,2}_{N,t}(\boldsymbol{\gamma}_t))$ be the equilibrium payoff vector of the game $\tilde{\mathcal{G}}^{pf,2}_t(\boldsymbol{\gamma}_t)$ in period t, where $\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) = \Pi^{pf}_{i,t}(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)|\boldsymbol{\gamma}_t)$. We have $\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) = \theta_{ii}\psi_i(\boldsymbol{\gamma}_t)(p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i \beta^{pf}_{i,t-1} k_i^2)^2 > 0$ for all i.

Let $\boldsymbol{p}^{pf*}_{-i,t}(\boldsymbol{\gamma}_t)$ be the equilibrium pricing policy except for retailer i given $\boldsymbol{\gamma}_t$. Given the results in Proposition 4 and Proposition 5, we are able to characterize the equilibrium of the second-stage price competition with inventory optimization.

Proposition 6. *For a promotion decision $\boldsymbol{\gamma}_t$, if $x^{pf}_{i,t}(\boldsymbol{p}^{pf*}_{-i,t}(\boldsymbol{\gamma}_t), \boldsymbol{\gamma}_t, \Lambda_{i,t}) \geq I_{i,t}$ for all i, then the second-stage game $\mathcal{G}^{pf,2}_t(\boldsymbol{\gamma}_t)$ in period t has a unique pure strategy Nash equilibrium $(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t), \boldsymbol{y}^{pf*}_t)$. Furthermore, the equilibrium payoff vector is $(O^{pf*,2}_{1,t}(\boldsymbol{\gamma}_t), O^{pf*,2}_{2,t}(\boldsymbol{\gamma}_t), \cdots, O^{pf*,2}_{N,t}(\boldsymbol{\gamma}_t))$, where $O^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) = O^{pf}_{i,t}(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t), y^{pf*}_{i,t}|\boldsymbol{\gamma}_t, \Lambda_{i,t}) = \pi^{pf*}_{i,t} + \Lambda_{i,t}\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t)$.*

Proposition 6 shows that, for any given promotional effort vector $\boldsymbol{\gamma}_t$, the second-stage price competition $\mathcal{G}^{pf,2}_t(\boldsymbol{\gamma}_t)$ has a unique pure strategy Nash equilibrium $(\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}_t), \boldsymbol{y}^{pf*}_t)$ if the starting inventory of retailers in period t is smaller than a threshold. By Proposition 5(b), we have $p^{pf*}_{i,t}(\boldsymbol{0}) \leq p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) \leq p^{pf*}_{i,t}(\bar{\boldsymbol{\gamma}}_t)$ for each i and $\boldsymbol{\gamma}_t$, where $\boldsymbol{0}$ is an N-dimensional vector with each entry equal to 0 and $\bar{\boldsymbol{\gamma}}_t := (\bar{\gamma}_{1,t}, \bar{\gamma}_{2,t}, \cdots, \bar{\gamma}_{N,t})$. Thus, a sufficient condition for Assumption 1 is that $\underline{p}_{i,t} \leq p^{pf*}_{i,t}(\boldsymbol{0})$ and $\bar{p}_{i,t} \geq p^{pf*}_{i,t}(\bar{\boldsymbol{\gamma}}_t)$ for all i and t.

Now we study the first-stage promotion competition. In each period t, we define a game $\tilde{\mathcal{G}}^{pf,1}_t$ with the payoff function of retailer i as $O^{pf,1}_{i,t}(\boldsymbol{\gamma}_t) := \pi^{pf*}_{i,t} + \Lambda_{i,t}\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t)$ and an action space $[0, \overline{\gamma}_1] \times [0, \overline{\gamma}_2] \times \cdots \times [0, \overline{\gamma}_N]$. Notice that, for the game $\tilde{\mathcal{G}}^{pf,1}_t$, we implicitly assume that $x^{pf}_{i,t}(\boldsymbol{p}^{pf*,2}_{-i,t}(\boldsymbol{\gamma}_t), \boldsymbol{\gamma}_t, \Lambda_{i,t}) \geq I_{i,t}$ holds for all i, which we will verify later to close the loop.

Proposition 7. *The following statements hold for the game $\tilde{\mathcal{G}}^{pf,1}_t$:*

(a) The first-stage promotion competition $\tilde{\mathcal{G}}^{pf,1}_t$ is a log-supermodular game.

(b) If $\nu_i(\gamma_{i,t}) = \gamma_{i,t}$ for each i, then:

(i) There exists a unique pure strategy Nash equilibrium in the game $\tilde{\mathcal{G}}_t^{pf,1}$, which is the unique serially undominated strategy of $\tilde{\mathcal{G}}_t^{pf,1}$.

(ii) The unique Nash equilibrium of $\tilde{\mathcal{G}}_t^{pf,1}$, $\boldsymbol{\gamma}_t^{pf*}$, is the solution to the following system of equations:

$$\frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\boldsymbol{\gamma}_t^{pf*})}{\psi_{i,t}(\boldsymbol{\gamma}_t^{pf*})} - \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})}{p_{i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}) - w_i - \gamma_{i,t}^{pf*} + \delta_i \beta_{i,t-1}^{pf} k_i^2} \begin{cases} \leq 0, & \text{if } \gamma_{i,t}^{pf*} = 0, \\ = 0, & \text{if } \gamma_{i,t}^{pf*} \in (0, \bar{\gamma}_i), \\ \geq 0 & \text{if } \gamma_{i,t}^{pf*} = \bar{\gamma}_i, \end{cases} \qquad \text{for each } i. \tag{25}$$

As shown in Proposition 7, in the PF model, the first-stage promotion competition in period t has a unique pure strategy Nash equilibrium. Moreover, the unique Nash equilibrium promotional effort vector $\boldsymbol{\gamma}_t^{pf*}$ can be determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (25). We also denote $\Pi_{i,t}^{pf*,1} := \Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*})$ for the rest of this paper. Proposition 7 immediately implies that, as long as $x_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}), \boldsymbol{\gamma}_t^{pf*}, \Lambda_{i,t}) \geq I_{i,t}$ for all i, retailer i will play $\gamma_{i,t}^{pf*}$ under the MPE for all i. Furthermore, the equilibrium payoff of retailer i is given by $O_{i,t}^{pf*,1} := O_{i,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}) = \pi_{i,t}^{pf*} + \Lambda_{i,t}\Pi_{i,t}^{pf*,1}$.

The following theorem summarizes Theorem 6 and Propositions 4 - 7, and characterizes the MPE in the PF model.

THEOREM 7. *If $I_{i,T} = 0$ and $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i, the following statements hold for each retailer i and each period t:*

(a) The policy of retailer i in period t under the unique pure strategy MPE $\boldsymbol{\sigma}^{pf*}$ is

$$(\gamma_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), p_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t), x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)) = (\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\boldsymbol{\gamma}_t), y_{i,t}^{pf*} + \Lambda_{i,t}\rho_{i,t}(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))\psi_{i,t}(\boldsymbol{\gamma}_t)). \tag{26}$$

In particular, for any $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, the associated (pure strategy) equilibrium price and inventory decisions of retailer i are $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*})$ and $y_{i,t}^{pf*} + \Lambda_{i,t}\rho_{i,t}(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t^{pf*}))\psi_{i,t}(\boldsymbol{\gamma}_t^{pf*})$, respectively.

(b) The following recursive relations hold: $\beta_{i,t}^{pf} = \delta_i \beta_{i,t-1}^{pf} k_i^1 + \Pi_{i,t}^{pf*,1} > 0$ and $\eta_{i,t}^{pf} = \delta_i \eta_{i,t-1}^{pf} + \pi_{i,t}^{pf*}$.

(c) The following sample-path property holds:

$$\mathbb{P}\Big[x_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}), \boldsymbol{\gamma}_t^{pf*}, \Lambda_{i,t}) - D_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}), \boldsymbol{\gamma}_t^{pf*}, \Lambda_{i,t}) \leq x_{i,t-1}^{pf}(\boldsymbol{p}_{-i,t-1}^{pf*}(\boldsymbol{\gamma}_{t-1}^{pf*}), \boldsymbol{\gamma}_{t-1}^{pf*}, \Lambda_{i,t-1}) \Big| \Lambda_{i,t}\Big] = 1,$$

for any $\Lambda_{i,t}$.

Theorem 7(a) characterizes the unique pure strategy MPE in the PF model. It also demonstrates that, under the unique pure strategy MPE $\boldsymbol{\sigma}^{pf*}$, retailer i's promotion, price, and inventory decisions in each period t depend on its private information (i.e., $(I_{i,t}, \Lambda_{i,t})$) only, but not on that of its competitors (i.e., $(\boldsymbol{I}_{-i,t}, \boldsymbol{\Lambda}_{-i,t})$). Hence, the unique pure strategy MPE in the PF model also has the desirable feature that the strategy of each firm is contingent on its accessible information only. Theorem 7(b) recursively determines the PF coefficient vectors, $\{\boldsymbol{\beta}_t^{pf} : T \geq t \geq 1\}$

and $\{\boldsymbol{\eta}_t^{pf} : T \geq t \geq 1\}$, associated with the unique pure strategy MPE $\boldsymbol{\sigma}^{pf*}$. Theorem 7(c) verifies the desired sample path that the starting inventory in the next period is small enough to ensure the existence and uniqueness of MPE $\boldsymbol{\sigma}_{t-1}^{pf*}$, when firms adopt the pure strategy MPE, $(\gamma_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t), p_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t), x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t))$.

As in the SC model, we will perform some of our analysis below with the symmetric PF model, where all retailers have identical characteristics. We use the subscript "s" to denote the case of symmetric market in the PF model. In this case, $\rho_s(\boldsymbol{p}_t) = \phi_s - \theta_{sa} p_{i,t} + \sum_{j \neq i} \theta_{sb} p_{j,t}$ for some nonnegative constants ϕ_s, θ_{sa}, and θ_{sb}, where $\theta_{sa} > (N-1)\theta_{sb}$. We use $\boldsymbol{\sigma}_s^{pf*}$ to denote the unique pure strategy MPE in the symmetric PF model. The following proposition characterizes $\boldsymbol{\sigma}_s^{pf*}$ in the PF model.

Proposition 8. *If $I_{i,T} = 0$ and $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i, the following statements hold for the symmetric PF model:*

(a) For each $t = T, T-1, \cdots, 1$, there exists two constants $\beta_{s,t}^{pf} > 0$ and $\eta_{s,t}^{pf}$, such that

$$V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}_{s,t}^{pf*}) = w_s I_{i,t} + \beta_{s,t}^{pf} \Lambda_{i,t} + \eta_{s,t}^{pf}, \text{ for all } i.$$

(b) In each period t, the second-stage price competition $\mathcal{G}_{s,t}^{pf,2}(\boldsymbol{\gamma}_t)$ is symmetric if $\gamma_{i,t} = \gamma_{j,t}$ for all $1 \leq i, j \leq N$. In this case, $\mathcal{G}_{s,t}^{pf,2}(\boldsymbol{\gamma}_t)$ has a unique pure strategy Nash equilibrium $\boldsymbol{p}_{ss,t}^{pf*}(\boldsymbol{\gamma}_t)$, which is symmetric (i.e., $\boldsymbol{p}_{ss,t}^{pf*}(\boldsymbol{\gamma}_t) = (p_{s,t}^{pf*}(\boldsymbol{\gamma}_t), p_{s,t}^{pf*}(\boldsymbol{\gamma}_t), \cdots, p_{s,t}^{pf*}(\boldsymbol{\gamma}_t))$ for some $p_{s,t}^{pf*}(\boldsymbol{\gamma}_t) \in [\underline{p}_s, \bar{p}_s]$).

(c) In each period t, the first-stage promotion competition $\mathcal{G}_{s,t}^{pf,1}$ is symmetric. Moreover, $\mathcal{G}_{s,t}^{pf,1}$ has a unique pure strategy Nash equilibrium $\boldsymbol{\gamma}_{ss,t}^{pf*}$, which is symmetric (i.e., $\boldsymbol{\gamma}_{ss,t}^{pf*} = (\gamma_{s,t}^{pf*}, \gamma_{s,t}^{pf*}, \cdots, \gamma_s^{pf*})$ for some $\gamma_{s,t}^{pf*} \in [0, \bar{\gamma}_s]$).

(d) Under the unique pure strategy MPE $\boldsymbol{\sigma}_s^{pf*}$, the policy of firm i in period t is

$$(\gamma_{i,t}^{pf*}(\boldsymbol{\gamma}_t, I_{i,t}, \Lambda_{i,t}), p_{i,t}^{pf*}(\boldsymbol{\gamma}_t, I_{i,t}, \Lambda_{i,t}), x_{i,t}^{pf*}(\boldsymbol{\gamma}_t, I_{i,t}, \Lambda_{i,t})) = (\gamma_{s,t}^{sc*}, p_{i,t}^{pf*}(\boldsymbol{\gamma}_t), y_{s,t}^{pf*} + \Lambda_{i,t}\rho_{s,t}(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))\psi_{s,t}(\boldsymbol{\gamma}_t)),$$

for all $(I_{i,t}, \Lambda_{i,t})$ and $\boldsymbol{\gamma}_t$. In particular, for each firm i and any $(I_{i,t}, \Lambda_{i,t})$, the equilibrium price is $p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*})$, and the equilibrium post-delivery inventory level is $y_{s,t}^{pf*} + \Lambda_{i,t}\rho_{s,t}(\boldsymbol{p}_{ss,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}))\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{pf*})$.

Proposition 8 shows that, in the symmetric PF model, all competing retailers make the same promotional effort, charge the same price, and maintain the same safety stock in each period. The PF market size coefficient is also identical for all retailers in each period.

5.2. Implications of Network Effects

In this subsection, we study the operations implications of network effects in the PF model. As in the SC model, we assume that the initial inventory level of each retailer is zero, that is, $I_{i,T} = 0$ for all i. To ensure the uniqueness of MPE, we also assume that $\nu_i(\gamma_{i,t}) = \gamma_{i,t}$ for all i. We first characterize the impact of the PF market size coefficient vectors, $\{\boldsymbol{\beta}_t^{pf} : T \geq t \geq 1\}$.

THEOREM 8. *For each period t, the following statements hold:*

(a) For each i, j, and $\boldsymbol{\gamma}_t$, $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t)$ is continuously decreasing in $\beta_{j,t-1}^{pf}$ and k_j^2.

(b) If the PF model is symmetric, $\gamma_{s,t}^{pf*}$ is continuously increasing in $\beta_{s,t-1}^{pf}$ and k_s^2.

(c) If the PF model is symmetric and the monotonicity condition (16) holds, $\Pi_{s,t}^{pf*,1}$ is continuously increasing in $\beta_{s,t-1}^{pf}$ and k_s^2.

(d) If the PF model is symmetric and the monotonicity condition (16) holds, $\beta_{s,t}^{pf}$ is continuously increasing in k_s^1, $\beta_{s,t-1}^{pf}$ and k_s^2.

Theorem 8 demonstrates that the market size coefficients $\{\beta_{i,t}^{pf} : 1 \leq i \leq N, T \geq t \geq 1\}$ quantify the intensity of the trade-off between current profits and future demands in the PF model. More specifically, a larger $\beta_{i,t-1}^{pf}$ implies more intensive trade-off for firm i in period t.

As in the SC model, we first compare two PF models with different network effect intensities. Assume that $\hat{k}_s^2 \geq k_s^2$ for each firm i. The following theorem characterizes the impact of network effect intensity in the PF model.

THEOREM 9. *Consider the symmetric PF model. For each period t, the following statements hold:*

(a) $\hat{p}_{s,t}^{pf*}(\boldsymbol{\gamma}_t) \leq p_{s,t}^{pf*}(\boldsymbol{\gamma}_t)$ for all $\boldsymbol{\gamma}_t$, and thus, $\hat{p}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t) \leq p_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, $\boldsymbol{\gamma}_t$ and all i.

(b) $\hat{x}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t) \geq x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, $\boldsymbol{\gamma}_t$ and all i.

(c) $\hat{\gamma}_{s,t}^{pf*} \geq \gamma_{s,t}^{pf*}$, and thus, $\hat{\gamma}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \geq \gamma_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for all $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ and all i.

Theorem 9(a) reveals the impact of the trade-off between current profits and future demands upon the competing firms' price strategy in the PF model. Specifically, given any outcome of the first-stage promotion competition $\boldsymbol{\gamma}_t$, in the second-stage price competition, each firm should charge a lower retail price under a more intensive network effect, so as to exploit the network effect and induce higher future demands. Theorem 9(b) shows that, in each period t, the equilibrium post-delivery inventory levels contingent on any realized promotional effort vector $\boldsymbol{\gamma}_t$ are also higher in the PF model under the stronger network effect. Theorem 9(c) sheds light on how network effects influence the equilibrium promotion strategies. In the symmetric PF model, the equilibrium promotional effort of each firm i in each period t is higher when the network effect is stronger.

Note that, in the PF model, the equilibrium price and inventory outcomes under the network effect $\hat{k}_s^2$, $\hat{p}_{s,t}^{pf*}(\hat{\boldsymbol{\gamma}}_{ss,t}^{pf*})$ and $\hat{x}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \hat{\boldsymbol{\gamma}}_{ss,t}^{pf*})$, may be either higher or lower than those under a lower network effect k_s^2, $\boldsymbol{p}_{ss,t}^{pf*}(\gamma_{s,t}^{pf*})$ and $x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_{ss,t}^{pf*})$. This phenomenon contrasts with the equilibrium market outcomes in the SC model, where the equilibrium price (resp. post-delivery inventory level) of each retailer in each period is lower (resp. higher) under a stronger network effect (i.e., Theorem

4(b-c)). This discrepancy is driven by the fact that, in the PF model, each retailer observes the promotion decisions of its competitors before making its pricing decision. Hence, under a stronger network effect, the competing retailers may either charge lower prices to induce more future demands or increase the prices to exploit the better market condition from the increased promotional efforts (recall that $\hat{\gamma}^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t}$). In general, either effect may dominate, so we do not have a general monotonicity relationship between either the equilibrium price outcomes (i.e., $\hat{p}^{pf*}_{s,t}(\hat{\boldsymbol{\gamma}}^{pf*}_{ss,t})$ and $\boldsymbol{p}^{pf*}_{ss,t}(\gamma^{pf*}_{s,t})$) or the equilibrium inventory outcomes (i.e., $\hat{x}^{pf*}_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \hat{\boldsymbol{\gamma}}^{pf*}_{ss,t})$ and $x^{pf*}_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}^{pf*}_{ss,t})$). Therefore, the trade-off between current profits and future demands in the PF model is more involved than that in the SC model. The competing retailers only need to balance this trade-off inter-temporally in the SC model, whereas they have to balance it both inter-temporally and intra-temporally in the PF model.

The following theorem is a counterpart of Theorem 5, and characterizes the evolution of the equilibrium market outcome in the stationary symmetric PF model.

THEOREM 10. *Consider the symmetric PF model. If the monotonicity condition* (16) *holds for each period t, the following statements hold:*

(a) $\beta^{pf}_{s,t} \geq \beta^{pf}_{s,t-1}$, $p^{pf*}_{s,t}(\boldsymbol{\gamma}) \leq p^{pf*}_{s,t-1}(\boldsymbol{\gamma})$ *for each* $\boldsymbol{\gamma}$, *and* $\gamma^{pf*}_{s,t} \geq \gamma^{pf*}_{s,t-1}$.

(b) $p^{pf*}_{i,t}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) \leq p^{pf*}_{i,t-1}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$, $x^{pf*}_{i,t}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) \geq x^{pf*}_{i,t-1}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$, *and* $\gamma^{pf*}_{i,t}(\boldsymbol{I}, \boldsymbol{\Lambda}) \geq \gamma^{pf*}_{i,t-1}(\boldsymbol{I}, \boldsymbol{\Lambda})$ *for each* i, $\boldsymbol{\gamma}$, $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$.

Analogous to Theorem 5, Theorem 10 justifies the widely used introductory price and promotion strategy. More specifically, this result shows that if the market is stationary and symmetric in the PF model, the competing retailers should decrease the promotional efforts (i.e., $\gamma^{pf*}_{s,t}$), and increase the prices contingent on any realized promotional efforts (i.e., $p^{pf*}_{s,t}(\boldsymbol{\gamma}_t)$), over the planning horizon. Hence, Theorem 10 suggests that, in the PF model, the trade-off between current profits and future demands is more intensive at the early stage of the sales season than at later stages.

To conclude this section, we remark that, because of the aforementioned intra-temporal trade-off under the promotion-first competition, Theorems 9 - 10 cannot give the monotone relationships on the equilibrium outcomes of each retailer i's price (i.e., $p^{pf*}_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}^{pf*}_{ss,t})$) and post-deliver inventory level (i.e., $x^{pf*}_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}^{pf*}_{ss,t})$).

6. Comparison of the Two Competition Models

As demonstrated above, the trade-off between current profits and future demands is more involved in the PF model than that in the SC model. In this section, we compare the unique MPE in the SC model and that in the PF model, and discuss how this trade-off influences the equilibrium market outcomes under different competitions.

THEOREM 11. *Consider the symmetric SC and PF models. Assume that, for each retailer i and each period t, (i) $I_{i,T}=0$, (ii) the demand function $\rho_{i,t}(\cdot)$ is linear and given by (17), (iii) $\nu_{i,t}(\gamma_{i,t})=\gamma_{i,t}$, (iv) the monotonicity condition (16) holds, and (v) Assumption 1 holds. The following statements hold:*

(a) If $\beta^{pf}_{s,t-1}\geq\beta^{sc}_{s,t-1}$, $\gamma^{pf*}_{s,t}\geq\gamma^{sc*}_{s,t}$.

(b) There exists an $\epsilon\in[0,\frac{1}{N-1}]$, such that, if $\theta_{sb}\leq\epsilon\theta_{sa}$, we have

(i) $\beta^{pf}_{s,t}\geq\beta^{sc}_{s,t}$ and, thus, $V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}^{pf*}_t)\geq V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}^{sc*}_t)$ for each firm i and all $(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)\in\mathcal{S}$;

(ii) $\gamma^{pf*}_{s,t}\geq\gamma^{sc*}_{s,t}$.

Theorem 11 shows that, if the product differentiation is sufficiently high (as captured by the condition that $\theta_{sb}\leq\epsilon\theta_{sa}$), the PF competition leads to stronger trade-off between current profits and future demands (i.e., $\beta^{pf}_{s,t}\geq\beta^{sc}_{s,t}$). As a consequence, the competing retailers should set higher promotional efforts in the PF model. Compared with the simultaneous competition, the promotion-first competition enables the retailer to responsively adjust their sales prices in accordance to the market condition and their competitors' promotion strategies. If the product differentiation is sufficiently high, such pricing flexibility gives rise to higher expected profits of all retailers and more intensive trade-off in the PF model.

Theorem 11 also reveals the "fat-cat" effect in our dynamic competition model: When the price decisions are made after observing the promotional efforts in each period, the firms tend to "over-invest" in promotional efforts. As shown in the literature (e.g., Fudenberg and Tirole 1984, Allon and Federgruen 2007), one driving force for this phenomenon is that, under the PF competition, the firms can charge higher prices in the subsequent price competition with increased promotional efforts in each period. Theorem 11 identifies a new driving force for the "fat-cat" effect: The firms under the PF competition make more promotional efforts to balance the more intensive trade-off between current profits and future demands. Therefore, our analysis delivers a new insight to the literature that network effects may give rise to the "fat-cat" effect in dynamic competition.

7. Conclusion

This paper studies the dynamic joint promotion, price, and inventory competition between online retailers. A salient feature of our model is the social interactions between consumers, which drive network effects rendering the current decisions of retailers to influence their future demands. Our model highlights an important trade-off in a dynamic and competitive market: the one between generating current profits and inducing future demands. We characterize the impact of this trade-off upon the equilibrium market outcome under network effects, and identify the effective strategies to balance this trade-off in dynamic competition.

We employ the linear separability approach based on a sample-path property of inventory dynamics to characterize the pure strategy MPE both in the SC model and in the PF model. An important feature of the MPE in both models is that the equilibrium strategy of each retailer in each period only depends on the private inventory and market size information of itself, but not on that of its competitors. Moreover, the trade-off between current profits and future demands is more intensive if the network effect are stronger, and its intensity decreases over the planning horizon. The trade-off is more involved in the PF model than in the SC model. This is because the competing retailers need to balance this trade-off both inter-temporally and intra-temporally in the PF model, whereas they only need to balance it inter-temporally in the SC model. More specifically, in the SC model, to effectively balance the trade-off, the firms should (a) increase promotional efforts, and (b) offer price discounts. In the PF model, the firms should increase promotional efforts under stronger network effects. Given the same promotional effort in the first-stage competition, the firms need to decrease their sales prices under stronger network effects. However, with an increased promotional effort in the first-stage competition, the equilibrium prices in the second-stage competition may either decrease or increase. Analogously, the equilibrium post-delivery inventory levels may either decrease or increase in the PF model under stronger network effects. Finally, we identify the "fat-cat" effect in our dynamic competition model: If the product differentiation is sufficiently high, under the MPE, the retailers make more promotional efforts in the PF model than in the SC model. The driving force of this phenomenon is that the trade-off between current profits and future demands is more intensive under the promotion-first competition than under the simultaneous competition.

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Online Appendices to "Dynamic Competition in Online Retailing: The Implications of Network Effects"

We use ∂ to denote the derivative operator of a single variable function, and ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. The following lemma is indebted from Lemma 4 of Yang and Zhang (2022a) and is used throughout our analysis. For completeness, we also give its proof.

Lemma 1. *Let $G_i(z, \boldsymbol{Z})$ be a continuously differentiable function in $(z, \boldsymbol{Z})$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $\boldsymbol{Z} \in \mathbb{R}^{n_i}$ for $i = 1, 2$. For $i = 1, 2$, let $(z_i, \boldsymbol{Z}_i) := \arg\max_{(z,\boldsymbol{Z})} G_i(z, \boldsymbol{Z})$ be the optimizers of $G_i(\cdot, \cdot)$. If $z_1 < z_2$, we have: $\partial_z G_1(z_1, \boldsymbol{Z}_1) \le \partial_z G_2(z_2, \boldsymbol{Z}_2)$.*

Proof: $z_1 < z_2$, so $\underline{z} \le z_1 < z_2 \le \bar{z}$. Hence, $\partial_z G_1(z_1, \boldsymbol{Z}_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \le 0 & \text{if } z_1 = \underline{z}; \end{cases}$ and $\partial_z G_2(z_2, \boldsymbol{Z}_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \ge 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z G_1(z_1, \boldsymbol{Z}_1) \le 0 \le \partial_z G_2(z_2, \boldsymbol{Z}_2)$. *Q.E.D.*

Proof of Theorems 1-2 and Propositions 1-2: We show Theorem 1, Proposition 1, Proposition 2, and Theorem 2 together by strong backward induction. More specifically, we show that, if $V_{i,\tau}(I_{i,\tau}, \Lambda_{i,\tau}|\sigma_\tau^{sc*}) = w_i I_{i,\tau} + \beta_{i,\tau}^{sc}\Lambda_{i,\tau} + \eta_{i,\tau}^{sc}$ for all i and $\tau \le t-1$, (1) Proposition 1(a-c) hold for period t, Proposition 1(d) holds for period t if $t \ge 2$, (2) Proposition 2 holds for period t, (3) there exists a Markov strategy profile $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \le i \le N\}$ which forms a Nash equilibrium in the subgame of period t, (4) under conditions (C1) and (C2) in Theorem 1(c), the Nash equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \le i \le N\}$, is unique, (5) there exists a positive vector $\boldsymbol{\beta}_t^{sc}$ and a vector $\boldsymbol{\eta}_t^{sc}$, such that $V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_t^{sc*}) = w_i I_{i,t} + \beta_{i,t}^{sc}\Lambda_{i,t} + \eta_{i,t}^{sc}$ for all i, and (6) starting inventory in the next period, $I_{i,t-1}$, is smaller than the optimal base-stock level $x_{i,t-1}^{sc*}(\cdot,\cdot)$ with probability one. Because $V_{i,0}(I_0, \Lambda_0) = w_i I_{i,0}$ for all i, the initial condition is satisfied.

Since $V_{i,t-1}(\boldsymbol{I}_{t-1}, \boldsymbol{\Lambda}_{t-1}|\boldsymbol{\sigma}_{t-1}^{sc*}) = w_i I_{i,t-1} + \beta_{i,t-1}^{sc}\Lambda_{i,t-1} + \eta_{i,t-1}^{sc}$ for all i, the problem defined in (12) can be decomposed into two problems which solve $\max\limits_{y_{i,t}} \pi_{i,t}^{sc}(y_{i,t})$ and $\max\limits_{(\gamma_{i,t}, p_{i,t})} \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$, respectively. Because $\pi_{i,t}^{sc}(y_{i,t})$ is concave and continuously differentiable, its optimizer satisfies the first order condition, that is, $-(h_i + b_i)F(y_{i,t}^{sc*}) + b_i - (1-\delta_i)w_i = 0$. Hence, Proposition 1(a) holds for period t. To prove Proposition 1(b), we need to prove that $O_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t, y_{i,t}|\Lambda_{i,t})$ is jointly log-concave in $(\gamma_{i,t}, p_{i,t}, y_{i,t})$ given $\boldsymbol{\gamma}_{-i,t}$, $\boldsymbol{p}_{-i,t}$ and $\Lambda_{i,t}$. Since $\overline{p}_i - w_i - \nu_i(\overline{\gamma}_i) > 0$, we can restrict the feasible action set of $(\gamma_{i,t}, p_{i,t})$ to $\mathcal{A}'_{i,t} := \{(\gamma_{i,t}, p_{i,t}) \in [0, \overline{\gamma_i}] \times [\underline{p}_i, \overline{p}_i] : p_{i,t} - w_i - \nu_i(\gamma_{i,t}) > 0\}$, which is a nonempty and complete sublattice of $\mathbb{R}^2$. Thus, $\Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) > 0$ and $\log \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) = \log \psi_i(\boldsymbol{\gamma}_t) + \log \rho_i(\boldsymbol{p}_t) + \log[p_{i,t} - w_i - \nu_i(\gamma_{i,t}) + \delta_i \beta_{i,t-1}^{sc} k_i^2]$ is well defined on $\mathcal{A}'_{i,t}$. By (3) and (4), $\log \psi_i(\boldsymbol{\gamma}_t)$ and $\log \rho_i(\boldsymbol{p}_t)$ are strictly concave in $\gamma_{i,t}$ and $p_{i,t}$ respectively given $\boldsymbol{\gamma}_{-i,t}$ and $\boldsymbol{p}_{-i,t}$. Because $\nu(\gamma_{i,t})$ is convexly increasing, $p_{i,t} - w_i - \nu_i(\gamma_{i,t}) + \delta_i \beta_{i,t-1}^{sc} k_i^2$ is concave in $(\gamma_{i,t}, p_{i,t})$. Since $log(\cdot)$ is concave and increasing, then $\log[p_{i,t} - w_i - \nu_i(\gamma_{i,t}) + \delta_i \beta_{i,t-1}^{sc} k_i^2]$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$, so is $\log \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$. Therefore, (12) and (13) are both well defined. If $x_{i,t}^{sc}(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t}, \Lambda_{i,t}) \ge I_{i,t}$, the inventory constraint is not binding. Thus, Proposition 1(b) holds for period t.

Next, we prove Proposition 1(c) for period t. Given policy profile $\boldsymbol{\gamma}_{-i,t}$ and $\boldsymbol{p}_{-i,t}$, for any $\gamma_{i,t} \in [0,\bar{\gamma}_i]$, $p_{i,t} \in [\underline{p}_i, \overline{p}_i]$, $x_{-i,t}$, and $I_{i,t} \geq 0$, $\Lambda_{i,t} \geq 0$,

$$\partial_{x_{i,t}} J_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t, [0, x_{-i,t}], I_{i,t}, \Lambda_{i,t} | \sigma^{sc}_{t-1}) = b_i - (1-\delta_i)w_i - (h_i + b_i)F_i\left(-\Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)\right) = b_i - (1-\delta_i)w_i > 0,$$

where the second equality holds because $F_i\left(-\Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)\right) = \mathbb{P}(\xi_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) \leq 0) = 0$ for any $\gamma_{i,t} \in [0,\bar{\gamma}_i]$, $p_{i,t} \in [\underline{p}_i, \overline{p}_i]$, $I_{i,t} \geq 0$ $\Lambda_{i,t} \geq 0$, and the last equality holds because $D_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t}) \geq 0$ with probability 1. If $t \geq 2$, because $V_{i,t-2}(I_{i,t-2}, \Lambda_{i,t-2} | \boldsymbol{\sigma}^{sc*}_{t-2}) = w_i I_{i,t-2} + \beta^{sc}_{i,t-2}\Lambda_{i,t-2} + \eta^{sc}_{i,t-2}$ for all i, all argument above remains valid for period $t-1$. Thus, by Proposition 1(a), $y^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1}) = y^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t}) = F_i^{-1}(\frac{b_i - (1-\delta_i)w_i}{h_i + b_i})$ for any policy $(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1})$ in period $t-1$. We have

$$\begin{aligned} & x^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t}, \Lambda_{i,t}) - D^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t}, \Lambda_{i,t}) \leq x^{sc}_{i,t}(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1}, \Lambda_{i,t-1}) \\ \iff\ & y^{sc}_i(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t}) - \xi_{i,t} \leq y^{sc}_i(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1}) + \Lambda_{i,t-1} d^{sc}_i(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1}) \\ \iff\ & 0 \leq \Lambda_{i,t-1} d^{sc}_i(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1}) + \xi_{i,t} \end{aligned}$$

which occurs with probability 1 for any policy $(\boldsymbol{\gamma}_{-i,t-1}, \boldsymbol{p}_{-i,t-1})$ and any state $\Lambda_{i,t-1}$. Therefore, Proposition 1(d) holds for period t.

We now show that Proposition 2 holds for period t. First, the action space $\mathcal{A}'_t$ is nonempty and complete sublattice of $\mathbb{R}^{2n}$. Because $\rho_{i,t}(\cdot)$ and $\psi_{i,t}(\cdot)$ satisfy (3) and (4), for each i and $j \neq i$, we have

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{i,t}} = \frac{\partial^2 \log(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)}{\partial \gamma_{i,t} \partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} \geq 0,$$

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{j,t}} = 0, \quad \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} = \frac{\partial^2 \log(\psi_{i,t}(\boldsymbol{\gamma}_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} \geq 0,$$

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial \gamma_{j,t}} = 0, \text{ and } \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial p_{j,t}} = \frac{\partial^2 \log(\rho_{i,t}(\boldsymbol{p}_t))}{\partial p_{i,t} \partial p_{j,t}} \geq 0.$$

Hence, $\tilde{\mathcal{G}}^{sc}_t$ is a log-supermodular game and, thus, has pure strategy Nash equilibria which are the smallest and largest undominated strategies (see Theorem 5 in Milgrom and Roberts 1990). Proposition 2(a) follows.

Next, we show that if conditions (C1) and (C2) in Theorem 1(c) hold, the Nash equilibrium of $\mathcal{G}^{sc,1}_t$ is unique. First, we show that under conditions (C1) and (C2) in Theorem 1(c),

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}^2} < 0, \ \left| \frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}^2} \right| > \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial p_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial \gamma_{j,t}}, \tag{27}$$

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}^2} < 0, \text{ and } \left| \frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}^2} \right| > \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{j,t}}. \tag{28}$$

Note that, by (4), $\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}^2} = \frac{\partial^2 \log \rho_{i,t}(\boldsymbol{p}_t)}{\partial p_{i,t}^2} - \frac{1}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} < 0$, and $\left|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}^2}\right| = \left|\frac{\partial^2 \log \rho_{i,t}(\boldsymbol{p}_t)}{\partial p_{i,t}^2}\right| + \frac{1}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2}$. Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial \gamma_{j,t}} = 0$ for $j \neq i$, and $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial \gamma_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2}$, we have

$$\begin{aligned} \left| \frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial p_{i,t}^2} \right| &= \left| \frac{\partial^2 \log \rho_{i,t}(\boldsymbol{p}_t)}{\partial p_{i,t}^2} \right| + \frac{1}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} \\ &> \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial p_{j,t}} + \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} \\ &= \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial p_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial p_{i,t} \partial \gamma_{j,t}}, \end{aligned}$$

where the inequality follows from (4) and condition (C1). Hence, (27) holds for all i and all $(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$.

Since $\nu''_{i,t}(\cdot) \geq 0$ and (3), we have

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}^2} = \frac{\partial^2 \log \psi_{i,t}(\boldsymbol{\gamma}_t)}{\partial \gamma_{i,t}^2} - \frac{\nu''_{i,t}(\boldsymbol{\gamma}_t)(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2) + (\nu'_{i,t}(\boldsymbol{\gamma}_t))^2}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} < 0,$$

and

$$\Big|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}^2}\Big| = \Big|\frac{\partial^2 \log \psi_{i,t}(\boldsymbol{\gamma}_t)}{\partial \gamma_{i,t}^2}\Big| + \frac{\nu''_{i,t}(\boldsymbol{\gamma}_t)(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2) + (\nu'_{i,t}(\boldsymbol{\gamma}_t))^2}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2}.$$

Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{j,t}} = 0$ for $j \neq i$, and $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2}$, we have

$$\begin{aligned}
\Big|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)}{\partial \gamma_{i,t}^2}\Big| &= \Big|\frac{\partial^2 \log \psi_{i,t}(\boldsymbol{\gamma}_t)}{\partial \gamma_{i,t}^2}\Big| + \frac{\nu''_{i,t}(\boldsymbol{\gamma}_t)(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2) + (\nu'_{i,t}(\gamma_{i,t}))^2}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} \\
&> \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i \beta^{sc}_{i,t-1} k_i^2)^2} \\
&= \sum_{j \neq i} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))}{\partial \gamma_{i,t} \partial p_{j,t}},
\end{aligned}$$

where the inequality follows from (4), $\delta_i \beta^{sc}_{i,t-1} k_i^2 > 0$, and condition (C2). Hence, (28) holds for all i and all $(\boldsymbol{\gamma}_t, \boldsymbol{p}_t)$.

We now show that if (27) and (28) hold, $\tilde{\mathcal{G}}^{sc}_t$ has a unique Nash equilibrium. Recall that the set of Nash equilibria in $\tilde{\mathcal{G}}^{sc}_t$ forms a complete lattice (see Theorem 2 in Zhou 1994). If, to the contrary, there exist two distinct equilibria $(\boldsymbol{\gamma}^*_t, \boldsymbol{p}^*_t)$ and $(\hat{\boldsymbol{\gamma}}^*_t, \hat{\boldsymbol{p}}^*_t)$, where $\hat{p}^*_{i,t} \geq p^*_{i,t}$ for all i and $\hat{\gamma}^*_{j,t} \geq \gamma^*_{j,t}$ for all j, with the inequality being strict for some i or j. If, for some i, $\hat{p}^*_{i,t} > p^*_{i,t}$, $\hat{p}^*_{i,t} - p^*_{i,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l, and $\hat{p}^*_{i,t} - p^*_{i,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, without loss of generality, we assume that $i = 1$. Lemma 1 suggests that

$$\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\boldsymbol{\gamma}}^*_t, \hat{\boldsymbol{p}}^*_t)) \geq \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^*_t, \boldsymbol{p}^*_t)). \tag{29}$$

On the other hand, by Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\boldsymbol{\gamma}}^*_t, \hat{\boldsymbol{p}}^*_t)) - \partial_{p_{1,t}} \log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^*_t, \boldsymbol{p}^*_t)) \\
= &\int_{s=0}^{1} \Big[\sum_{j=1}^{N} (\hat{p}^*_{j,t} - p^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\boldsymbol{\gamma}^*_t + s\hat{\boldsymbol{\gamma}}^*_t, (1-s)\boldsymbol{p}^*_t + s\hat{\boldsymbol{p}}^*_t))}{\partial p_{1,t} \partial p_{j,t}} \\
&+ \sum_{j=1}^{N} (\hat{\gamma}^*_{j,t} - \gamma^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\boldsymbol{\gamma}^*_t + s\hat{\boldsymbol{\gamma}}^*_t, (1-s)\boldsymbol{p}^*_t + s\hat{\boldsymbol{p}}^*_t))}{\partial p_{1,t} \partial \gamma_{j,t}}\Big] \mathrm{d}s \\
\leq &\int_{s=0}^{1} \Big[\sum_{j=1}^{N} (\hat{p}^*_{1,t} - p^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\boldsymbol{\gamma}^*_t + s\hat{\boldsymbol{\gamma}}^*_t, (1-s)\boldsymbol{p}^*_t + s\hat{\boldsymbol{p}}^*_t))}{\partial p_{1,t} \partial p_{j,t}} \\
&+ \sum_{j=1}^{N} (\hat{p}^*_{1,t} - p^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\boldsymbol{\gamma}^*_t + s\hat{\boldsymbol{\gamma}}^*_t, (1-s)\boldsymbol{p}^*_t + s\hat{\boldsymbol{p}}^*_t))}{\partial p_{1,t} \partial \gamma_{j,t}}\Big] \mathrm{d}s < 0,
\end{aligned}$$

where the first inequality follows from $\hat{p}^*_{1,t} - p^*_{1,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l and $\hat{p}^*_{1,t} - p^*_{1,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, and the second from $\hat{p}^*_{1,t} - p^*_{1,t} > 0$ and (27). This contradicts (29).

If, for some j, $\hat{\gamma}^*_{j,t} > \gamma^*_{j,t}$, $\hat{\gamma}^*_{j,t} - \gamma^*_{j,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l, and $\hat{\gamma}^*_{j,t} - \gamma^*_{j,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, without loss of generality, we assume that $j = 1$. Following a similar argument as above, we can derive a contradiction regarding $\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\boldsymbol{\gamma}}^*_t, \hat{\boldsymbol{p}}^*_t))$ and $\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^*_t, \boldsymbol{p}^*_t))$. Therefore, if conditions (C1) and (C2) in Theorem 1(c) hold, there exists a unique Nash equilibrium in $\tilde{\mathcal{G}}^{sc}_t$. If $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, we have $\nu'_{i,t}(\gamma_{i,t}) = 1$ and $\nu''_{i,t}(\gamma_{i,t}) = 0$ for all $\gamma_{i,t}$. So $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ implies conditions (C1) and (C2) in Theorem 1(c).

Note that we have already shown $\log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$ for any given $(\boldsymbol{\gamma}_{-i,t}, \boldsymbol{p}_{-i,t})$. Therefore, the first-order conditions with respect to $\gamma_{i,t}$ and $p_{i,t}$ are necessary and sufficient for $(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t)$ to be the unique Nash equilibrium in $\mathcal{G}_t$. Thus, the Nash equilibrium of $\tilde{\mathcal{G}}^{sc}_t$ is a solution to the system of equations (14). Since $\tilde{\mathcal{G}}^{sc}_t$ has a unique equilibrium, (14) has a unique solution, which coincides with the unique pure strategy Nash equilibrium of $\tilde{\mathcal{G}}^{sc}_t$. As shown above, for all i, $\Pi^{sc}_{i,t}(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t) > d_i(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t)(p_{i,t} - w_i - \nu_{i,t}(\gamma_{i,t})) \geq 0$, where the first inequality holds because $\delta_i \beta^{sc}_{i,t-1} k^2_i > 0$. Hence, $\Pi^{sc*}_{i,t} = \Pi^{sc}_{i,t}(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t) > 0$ for all i. Hence, Proposition 2 holds for period t.

Next, we show that $\{(\gamma^{sc*}_{i,t}, p^{sc*}_{i,t}, y^{sc*}_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t) : 1 \leq i \leq N\}$ is the unique equilibrium in the subgame of period t if $I_{i,t} \leq x^{sc}_{i,t}(\boldsymbol{\gamma}^{sc*}_{-i,t}, \boldsymbol{p}^{sc*}_{-i,t}, \Lambda_{i,t})$ for all i. We have shown that $(\boldsymbol{y}^{sc*}_t, \boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t)$ is the unique solution to the system of first order conditions, $\partial_{y_{i,t}} \pi^{sc}_{i,t}(y_{i,t}) = 0, 1 \leq i \leq N$, and Equation (14). Because $\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t) > 0$, $\partial_{\gamma_{i,t}} \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))$ and $\partial_{p_{i,t}} \log(\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t))$ have the same sign of $\partial_{\gamma_{i,t}} O^{sc}_{i,t}(y_{i,t}, \boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t})$ and $\partial_{p_{i,t}} O^{sc}_{i,t}(y_{i,t}, \boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t})$. Then $(\boldsymbol{y}^{sc*}_t, \boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t)$ is also the unique solution to the system: for each i, $\partial_{y_{i,t}} O^{sc}_{i,t}(y_{i,t}, \boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t}) = 0$,

$$
\begin{aligned}
&\partial_{\gamma_{i,t}} O^{sc}_{i,t}(y_{i,t}, \boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t}) \begin{cases} \leq 0, & \text{if } \gamma^{sc*}_{i,t} = 0, \\ = 0, & \text{if } \gamma^{sc*}_{i,t} \in (0, \bar{\gamma}_i), \\ \geq 0 & \text{if } \gamma^{sc*}_{i,t} = \bar{\gamma}_i; \end{cases} \text{ and,} \\
&\partial_{p_{i,t}} O^{sc}_{i,t}(y_{i,t}, \boldsymbol{\gamma}_t, \boldsymbol{p}_t, \Lambda_{i,t}) \begin{cases} \leq 0, & \text{if } p^{sc*}_{i,t} = \underline{p}_i, \\ = 0, & \text{if } p^{sc*}_{i,t} \in (\underline{p}_i, \bar{p}_i), \\ \geq 0 & \text{if } p^{sc*}_{i,t} = \bar{p}_i. \end{cases}
\end{aligned} \tag{30}
$$

which also satisfies the constraint $y^{sc*}_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t) \geq I_{i,t}$ for all i by the inductive assumption. Because Equation (30) uniquely defines the Nash Equilibrium in $\tilde{\mathcal{G}}^{sc}_t$, $(\boldsymbol{y}^{sc*}_t, \boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t)$ characterizes the pure strategy unique Nash equilibrium in the subgame of period t. That is, given that $V_{i,t-1}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}^{sc*}_{t-1})$ is linearly separable, there is a unique pure strategy subgame perfect equilibrium policy $(\gamma^{sc*}_{i,t}, p^{sc*}_{i,t}, y^{sc*}_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t))$. This proves point (4) of the induction step.

Next, we show that there exists a vector $\boldsymbol{\eta}^{sc}_t = (\eta^{sc}_{1,t}, \eta^{sc}_{2,t}, \cdots, \eta^{sc}_{N,t})$, and a positive vector $\boldsymbol{\beta}^{sc}_t = (\beta^{sc}_{1,t}, \beta^{sc}_{2,t}, \cdots, \beta^{sc}_{N,t})$ such that $V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}^{sc*}_t) = w_i I_{i,t} + \beta^{sc}_{i,t} \Lambda_{i,t} + \eta^{sc}_{i,t}$. By (11), if $I_{i,t} \leq x_{i,t}(\boldsymbol{\gamma}^{sc*}_{-i,t}, \boldsymbol{p}^{sc*}_{-i,t}, \Lambda_{i,t})$, we have that

$$
\begin{aligned}
V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}^{sc*}_t) &= J_{i,t}(\gamma^{sc*}_{i,t}, p^{sc*}_{i,t}, y^{sc*}_{i,t} + \Lambda_{i,t} d_i(\boldsymbol{\gamma}^{sc*}_t, \boldsymbol{p}^{sc*}_t), \boldsymbol{I}_t, \boldsymbol{\Lambda}_t | \boldsymbol{\sigma}^{sc*}_{t-1}) \\
&= w_i I_{i,t} + (\delta_i \beta^{sc}_{i,t-1} k^1_i + \Pi^{sc*}_{i,t}) \Lambda_{i,t} - (1 - \delta_i) w_i y^{sc*}_{i,t} - L_i(y^{sc*}_{i,t}) + \delta_i \eta^{sc}_{i,t-1}
\end{aligned}
$$

Since $\beta^{sc}_{i,t-1} \geq 0$, $\Pi^{sc*}_{i,t} > 0$, and $\pi^{sc*}_{i,t} = -(1 - \delta_i) w_i y^{sc*}_{i,t} - L_i(y^{sc*}_{i,t})$, $\beta^{sc}_{i,t} = \delta_i \beta^{sc}_{i,t-1} \mu_{i,t} + \Pi^{sc*}_{i,t} > 0$ and $\eta^{sc}_{i,t} = \delta_i \eta^{sc}_{i,t-1} + \pi^{sc*}_{i,t}$. This proves point (5) of the induction step.

Finally, by Proposition 1(d), if $I_{i,t} \leq x^{sc*}_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, the probability that the starting inventory level in the next period is smaller than the optimal base-stock level is 1. This completes the induction and, thus, the proof of Theorem 1, Proposition 1, Proposition 2, and Theorem 2. *Q.E.D.*

Proof of Proposition 3: By Theorems 1-2, and Propositions 1-2, it suffices to show that, if there exist constants $\beta^{sc}_{s,t-1}\geq 0$ and $\eta^{sc}_{s,t-1}$, such that $V_{i,t-1}(\boldsymbol{I}_{t-1},\boldsymbol{\Lambda}_{t-1}|\boldsymbol{\sigma}^{sc*}_{t-1})=w_{s,t}I_{i,t-1}+\beta^{sc}_{s,t-1}\Lambda_{i,t-1}+\eta^{sc}_{i,t-1}$ for all i, we have: (a) $y^{sc*}_{i,t}=y^{sc*}_{j,t}$ for all $i\neq j$; (b) the unique Nash equilibrium in $\mathcal{G}_t$ is symmetric, i.e., $(\gamma^{sc*}_{i,t},p^{sc*}_{i,t})=(\gamma^{sc*}_{j,t},p^{sc*}_{j,t})$ for all $i\neq j$, and (c) there exist constants $\beta^{sc}_{s,t}>0$ and $\eta^{sc}_{s,t}$, such that $V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\sigma^{sc*}_{s,t})=w_{s,t}I_{i,t}+\beta^{sc}_{s,t}\Lambda_{i,t}+\eta^{sc}_{s,t}$ for all i. Since $V_{i,0}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t)=w_{s,0}I_{i,0}$ for all i, the initial condition is satisfied with $\beta^{sc}_{s,0}=0$.

Since $V_{i,t-1}(\boldsymbol{I}_{t-1},\boldsymbol{\Lambda}_{t-1}|\boldsymbol{\sigma}^{sc*}_{t-1})=w_{s,t}I_{i,t-1}+\beta^{sc}_{s,t}\Lambda_{i,t-1}+\eta^{sc}_{s,t-1}$ for all i, by (11), we have $\pi^{sc}_{i,t}(\boldsymbol{y}_t)=-(1-\delta_s)w_sy_{i,t}-L_s(y_{i,t})$. Then $y^{sc*}_{i,t}=F^{-1}(\frac{b_s-(1-\delta_s)w_s}{h_s+b_s})$ for all i since $y^{sc*}_{i,t}$ satisfies the first-order condition. Thus $\pi^{sc*}_{i,t}=\pi^{sc}_{i,t}(y^{sc*}_{s,t})=\pi^{sc}_{j,t}(y^{sc*}_{s,t})=\pi^{sc*}_{j,t}$ for all $i\neq j$. We denote $y^{sc*}_{s,t}=y^{sc*}_{i,t}$ for each i, and $\pi^{sc*}_{s,t}=\pi^{sc*}_{i,t}$ for each i. Observe that, the objective functions of $\mathcal{G}_t$, $\{\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)=\rho_s(\boldsymbol{p}_t)\psi_s(\boldsymbol{\gamma}_t)[p_{i,t}-w_s-\nu_s(\gamma_{i,t})+\delta_s\beta^{sc}_{s,t-1}k^2_s]:1\leq i\leq N\}$ are symmetric. Thus, by the uniqueness of the Nash equilibrium in $\mathcal{G}_t$, it is symmetric.

Hence, $\Pi^{sc*}_{i,t}=\Pi^{sc}_{i,t}(\boldsymbol{\gamma}^{sc*}_{ss,t},\boldsymbol{p}^{sc*}_{ss,t})=\Pi^{sc}_{j,t}(\boldsymbol{\gamma}^{sc*}_{ss,t},\boldsymbol{p}^{sc*}_{ss,t})=\Pi^{sc*}_{j,t}>0$. Thus, we denote the payoff of each firm i as $\Pi^{sc*}_{s,t}$. By Theorem 2(a-b), for any $i\neq j$, $\beta^{sc}_{i,t}=\delta_s\beta^{sc}_{s,t-1}k^1_s+\Pi^{sc*}_{i,t}=\delta_s\beta^{sc}_{s,t-1}\mu_{s,t}+\Pi^{sc*}_{j,t}=\beta^{sc}_{j,t}>0$, and $\eta^{sc}_{i,t}=\delta_s\eta^{sc}_{i,t-1}+\pi^{sc*}_i=\delta_s\eta^{sc}_{i,t-1}+\pi^{sc*}_j=\eta^{sc}_{j,t}$. Thus, we denote the SC market size coefficient of each firm i as $\beta^{sc}_{s,t}$. This completes the induction and, thus, the proof of Proposition 3. *Q.E.D.*

Proof of Theorem 3: Part (a). Let $\zeta^{sc}_{i,t-1}=\delta_s\beta^{sc}_{s,t-1}k^2_s$. Then the objective function of each firm i in $\mathcal{G}_{s,t}$ is denoted as $\Pi^{sc}_{i,t}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t|\zeta^{sc}_{s,t-1})=d_s(\boldsymbol{\gamma}_t,\boldsymbol{p}_t)[p_{i,t}-w_s-v_s(\gamma_{i,t})+\zeta^{sc}_{s,t-1}]$ to capture the dependence of the objective functions on $\zeta^{sc}_{s,t-1}$. The unique symmetric Nash equilibrium in $\mathcal{G}_{s,t}$ is denoted as $(\boldsymbol{\gamma}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}),\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}))$, where $\boldsymbol{\gamma}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1})=(\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}),\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}),\cdots,\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))$ and $\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1})=(p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}),p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}),\cdots,p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))$. Because $\beta^{sc}_{s,t-1}>0$, $\delta_s>0$ and $k^2_s>0$, it suffices to show that, if $\bar{\zeta}^{sc}_{s,t-1}>\zeta^{sc}_{s,t-1}$, $\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})\geq\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$, and $p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})\leq p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$.

We first show that $p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})\leq p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$ for all $\bar{\zeta}^{sc}_{s,t-1}>\zeta^{sc}_{s,t-1}$. Assume, to the contrary, that $p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})>p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$. Lemma 1 implies that $\partial_{p_{1,t}}\log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1}),\boldsymbol{p}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1})|\bar{\zeta}^{sc}_{s,t-1}))\geq\partial_{p_{1,t}}\log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}),\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1})|\zeta^{sc}_{s,t-1}))$. By (4) and Newton-Leibniz formula, we have

$$\begin{aligned}&\partial_{p_{1,t}}\log\rho_{s,t}(\boldsymbol{p}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1}))-\partial_{p_{1,t}}\log\rho_{s,t}(\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}))\\=&\int_{s=0}^{1}\Big[\sum_{j=1}^{N}(p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})-p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))\frac{\partial^2\log\rho_{s,t}((1-s)\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1})+s\boldsymbol{p}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1}))}{\partial p_{1,t}\partial p_{j,t}}\Big]\,\mathrm{d}s<0.\end{aligned}$$

Hence, the above two arguments suggest that

$$p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})-w_s-\nu_s(\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1}))+\bar{\zeta}^{sc}_{s,t-1}<p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})-w_s-\nu_s(\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))+\zeta^{sc}_{s,t-1}. \tag{31}$$

Since $p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})>p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$, $\bar{\zeta}^{sc}_{s,t-1}>\zeta^{sc}_{s,t-1}$, and $\nu_s(\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1}))>\nu_s(\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))$. Thus, $\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})>\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})$. Lemma 1 yields that $\partial_{\gamma_{1,t}}\log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1}),\boldsymbol{p}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1})|\bar{\zeta}^{sc}_{s,t-1}))\geq\partial_{\gamma_{1,t}}\log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}),\boldsymbol{p}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1})|\zeta^{sc}_{s,t-1}))$, i.e.,

$$\begin{aligned}&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\bar{\zeta}^{sc}_{s,t-1}))-\frac{\nu'_s(\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1}))}{p^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1})-w_s-\nu_s(\gamma^{sc*}_{s,t}(\bar{\zeta}^{sc}_{s,t-1}))+\bar{\zeta}^{sc}_{s,t-1}}\\\geq&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}(\zeta^{sc}_{s,t-1}))-\frac{\nu'_s(\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))}{p^{sc*}_{s,t}(\zeta^{sc}_{s,t-1})-w_s-\nu_s(\gamma^{sc*}_{s,t}(\zeta^{sc}_{s,t-1}))+\zeta^{sc}_{s,t-1}}\end{aligned} \tag{32}$$

Since $\nu_s(\cdot)$ is convexly increasing, $\nu_s'(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \geq \nu_s'(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))$, inequality (31) implies that

$$-\frac{\nu_s'(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))}{p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})-w_s-\nu_s(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))+\bar{\zeta}_{s,t-1}^{sc}} < -\frac{\nu_s'(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))}{p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})-w_s-\nu_s(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))+\zeta_{s,t-1}^{sc}}.$$

Hence, (32) suggests that $\partial_{\gamma_{1,t}} \log\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) > \partial_{\gamma_{1,t}} \log\psi_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))$. However, by (3) and Newton-Leibniz formula, we have

$$\begin{aligned}&\partial_{\gamma_{1,t}} \log\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) - \partial_{\gamma_{1,t}} \log\psi_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))\\ =& \int_{s=0}^{1}\Big[\sum_{j=1}^{N}(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})-\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))\frac{\partial^2\log\psi_{s,t}((1-s)\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})+s\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}\Big]\,\mathrm{d}s<0,\end{aligned}$$

which leads to a contradiction. Therefore, for all $\bar{\zeta}_{s,t-1}^{sc} > \zeta_{s,t-1}^{sc}$, we have $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \leq p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$.

We now show that $\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \geq \gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ for all $\bar{\zeta}_{s,t-1}^{sc} > \zeta_{s,t-1}^{sc}$. The proof follows symmetric arguments of proving $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \leq p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ for all $\bar{\zeta}_{s,t-1}^{sc} > \zeta_{s,t-1}^{sc}$. We omit the proof here to avoid repeated arguments. The continuity of $\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ and $p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ in $\zeta_{s,t-1}^{sc}$ follows directly from that $\Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t,\boldsymbol{p}_t|\zeta_{s,t-1}^{sc})$ is twice continuously differentiable and the implicit function theorem. This completes the proof of part (a).

Part (b). Because $\delta_s>0$, $\beta_{s,t-1}^{sc}>0$, and $k_s^2>0$, it suffices to show that $\Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ is continuously increasing in $\zeta_{s,t-1}^{sc}$, where $\Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}) := \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}),\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))$.

Assume that $\bar{\zeta}_{s,t-1}^{sc} > \zeta_{s,t-1}^{sc}$. Since part (c) implies that $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \leq p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ and $\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) \geq \gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$, the monotonicity condition (16) implies that $\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) \geq \rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))$ and $\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) \geq \psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))$. We will prove that $\Pi_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) > \Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ in three cases. First, if $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) = p_{s,t}^{sc*}(\zeta^{sc})$ and $\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) = \gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$, by $\bar{\zeta}_{s,t-1}^{sc} > \zeta_{s,t-1}^{sc}$, we have $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})-w_s-\nu_s(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))+\bar{\zeta}_{s,t-1}^{sc} > p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})-w_s-\nu_s(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))+\zeta_{s,t-1}^{sc}$. Thus,

$$\begin{aligned}\Pi_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) &= \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}),\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})|\bar{\zeta}_{s,t-1}^{sc})\\ &= (p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})-\delta_s w_{s,t-1}-\nu_s(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))+\bar{\zeta}_{s,t-1}^{sc})\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))\\ &> (p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})-\delta_s w_{s,t-1}-\nu_s(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))+\zeta_{s,t-1}^{sc})\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))\\ &= \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}),\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})|\zeta_{s,t-1}^{sc}) = \Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}).\end{aligned}$$

Second, if $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) < p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ and $\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) = \gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$, Lemma 1 yields that

$$\partial_{p_{1,t}}\log(\Pi_{1,t}^{sc}(\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}),\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})|\bar{\zeta}_{s,t-1}^{sc})) \leq \partial_{p_{1,t}}\log(\Pi_{1,t}^{sc}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}),\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})|\zeta_{s,t-1}^{sc})).$$

By (4) and Newton-Leibniz formula, we have

$$\begin{aligned}&\partial_{p_{1,t}}\log\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})) - \partial_{p_{1,t}}\log\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))\\ =&\int_{s=0}^{1}\left[\sum_{j=1}^{N}(p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})-p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))\frac{\partial^2\log\rho_{s,t}((1-s)\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})+s\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))}{\partial p_{1,t}\partial p_{j,t}}\right]\mathrm{d}s<0.\end{aligned}$$

Hence, the above two arguments imply that

$$p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})-\delta_s w_{s,t-1}-\nu_s(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))+\bar{\zeta}_{s,t-1}^{sc} > p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})-\delta_s w_{s,t-1}-\nu_s(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}))+\zeta_{s,t-1}^{sc}.$$

Therefore,

$$\begin{aligned}\Pi_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) &= \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}), \boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})|\bar{\zeta}_{s,t-1}^{sc}) \\ &= (p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) - \delta_s w_{s,t-1} - \nu_s(\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) + \bar{\zeta}_{s,t-1}^{sc})\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}))\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc})) \\ &> (p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}) - \delta_s w_{s,t-1} - \nu_s(\gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})) + \zeta_{s,t-1}^{sc})\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}))\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})) \\ &= \Pi_{i,t}^{sc}(\boldsymbol{\gamma}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc}), \boldsymbol{p}_{ss,t}^{sc*}(\zeta_{s,t-1}^{sc})|\zeta_{s,t-1}^{sc}) = \Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc}).\end{aligned}$$

The last case is that $p_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) = p_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ and $\gamma_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) > \gamma_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$. The proof of $\Pi_{s,t}^{sc*}(\bar{\zeta}_{s,t-1}^{sc}) > \Pi_{s,t}^{sc*}(\zeta_{s,t-1}^{sc})$ follows symmetric arguments in the second case. Hence, by Theorem 2(a), the continuity of $\Pi_{s,t}^{sc*}$ in $\zeta_{s,t-1}^{sc}$ follows directly from the continuous differentiability of $\Pi_{i,t}^{sc}(\boldsymbol{\gamma}_t, \boldsymbol{p}_t|\zeta_{s,t-1}^{sc})$ in $(\boldsymbol{\gamma}_t, \boldsymbol{p}_t, \zeta_{s,t-1}^{sc})$ and the continuity of $(\boldsymbol{\gamma}_{ss,t}^{sc*}, \boldsymbol{p}_{ss,t}^{sc*})$ in $\zeta_{s,t-1}^{sc}$. This completes the proof of part (b).

Part (c). By Theorem 2, $\beta_{s,t}^{sc} = \delta_s \beta_{s,t-1}^{sc} k_s^1 + \Pi_{s,t}^{sc*}$, then $\beta_{s,t}^{sc}$ is continuously increasing in k_s^1. By part (b), $\Pi_{s,t}^{sc*}$ is continuously increasing in $\beta_{s,t-1}^{sc}$ and k_s^2, then $\beta_{s,t}^{sc}$ is also continuously increasing in $\beta_{s,t-1}^{sc}$ and k_s^2. *Q.E.D.*

Proof of Theorem 4: Part (a), (b). By Theorem 3(a), $\hat{\gamma}_{s,t}^{sc*} \geq \gamma_{s,t}^{sc*}$ and $\hat{p}_{s,t}^{sc*} \leq p_{s,t}^{sc*}$ for all t because $\hat{k}_s^2 \geq k_s^2$. Thus, by Theorem 2(b), $\hat{\gamma}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \hat{\gamma}_{s,t}^{sc*} \geq \gamma_{s,t}^{sc*} = \gamma_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$, and $\hat{p}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \hat{p}_{s,t}^{sc*} \leq p_{s,t}^{sc*} = p_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for all t and $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \in \mathcal{S}$. This proves part (a) and (b).

Part (c). By Proposition 3(d), $\hat{x}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \hat{y}_{s,t}^{sc*} + \Lambda_{i,t}\rho_{s,t}(\hat{p}_{ss,t}^{sc*})\psi_{s,t}(\hat{\gamma}_{ss,t}^{sc*})$ and $x_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = y_{s,t}^{sc*} + \Lambda_{i,t}\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*})\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*})$. Proposition 1 implies that $\hat{y}_{s,t}^{sc*} = F^{-1}(\frac{b_s-(1-\delta_s)w_s}{h_s+b_s}) = y_{s,t}^{sc*}$. By parts (a) and (b), $\hat{p}_{s,t}^{sc*} \leq p_{s,t}^{sc*}$ and $\hat{\gamma}_{s,t}^{sc*} \geq \gamma_{s,t}^{sc*}$, the monotonicity condition (16) yields that $\rho_{s,t}(\hat{p}_{ss,t}^{sc*}) \geq \rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*})$, and $\psi_{s,t}(\hat{\gamma}_{ss,t}^{sc*}) \geq \psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*})$. Therefore, for each $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \in \mathcal{S}$,

$$\hat{x}_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \hat{y}_{s,t}^{sc*} + \Lambda_{i,t}\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*})\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}) \geq y_{s,t}^{sc*} + \Lambda_{i,t}\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*})\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}) = x_{i,t}^{sc*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t).$$

This completes the proof of part (c). *Q.E.D.*

Proof of Theorem 5: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$, (1) $\gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*}$, (2) $\gamma_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) \geq \gamma_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, (3) $p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*}$, (4) $p_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) \leq p_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, (5) $x_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) \geq x_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, and (6) $\beta_{s,t}^{sc} \geq \beta_{s,t-1}^{sc}$. Since, by Proposition 3(a), $\beta_{s,1}^{sc} \geq \beta_{s,0}^{sc} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 3(a), $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$ suggests that $\gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*}$ and $p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*}$. Hence, $\gamma_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) = \gamma_{s,t}^{sc*} \geq \gamma_{s,t-1}^{sc*} = \gamma_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ and $p_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) = p_{s,t}^{sc*} \leq p_{s,t-1}^{sc*} = p_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$. Because the monotonicity condition (16) holds, we have $\rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*}) \geq \rho_{s,t-1}(\boldsymbol{p}_{ss,t-1}^{sc*})$, and $\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*}) \geq \psi_{s,t-1}(\boldsymbol{\gamma}_{ss,t-1}^{sc*})$. Therefore, for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$,

$$x_{i,t}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}) = y_{s,t}^{sc*} + \rho_{s,t}(\boldsymbol{p}_{ss,t}^{sc*})\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{sc*})\Lambda_i \geq y_{s,t-1}^{sc*} + \rho_{s,t-1}(\boldsymbol{p}_{ss,t-1}^{sc*})\psi_{s,t-1}(\boldsymbol{\gamma}_{ss,t-1}^{sc*})\Lambda_i = x_{i,t-1}^{sc*}(\boldsymbol{I}, \boldsymbol{\Lambda}).$$

Finally, $\beta_{s,t}^{sc} \geq \beta_{s,t-1}^{sc}$ follows immediately from Theorem 3(f) and $\beta_{s,t-1}^{sc} \geq \beta_{s,t-2}^{sc}$. This completes the induction and, thus, the proof of Theorem 5. *Q.E.D.*

Before presenting the proofs of the results in the PF model, we give the following lemma that is used throughout the rest of our proofs.

LEMMA 2. *Let $\boldsymbol{A}$ be an $N \times N$ matrix with entries defined by $A_{ii} = 2\theta_{ii}$ and $A_{ij} = -\theta_{ij}$ where $i \neq j$. The following statements hold:*

(a) $\boldsymbol{A}$ is invertible. Moreover, $(\boldsymbol{A}^{-1})_{ij} \geq 0$ for all $1 \leq i, j \leq N$.

(b) $\frac{1}{2} \leq \theta_{ii}(\boldsymbol{A}^{-1})_{ii} < 1$.

(c) $\frac{1}{2} \leq \sum\limits_{j=1}^{N} \theta_{jj}(\boldsymbol{A}^{-1})_{ij} < 1$.

Proof: Part (a) follows from Lemma 2(a) in Bernstein and Federgruen (2004c) and **Part (b)** follows from Lemma 2(c) in Bernstein and Federgruen (2004c).

Part (c). Let $\mathcal{I}$ be the $N \times N$ identity matrix. Let $\boldsymbol{B}$ be the $N \times N$ matrix with $(\boldsymbol{B})_{ij} = 0$, if $i = j$, and $(\boldsymbol{B})_{ij} = \frac{\theta_{ij}}{\theta_{ii}}$ if $i \neq j$. Let $\boldsymbol{C}$ be the $N \times N$ diagonal matrix with $(\boldsymbol{C})_{ij} = 2\theta_{ii}$ if $i = j$, and $(\boldsymbol{C})_{ij} = 0$ if $i \neq j$. Because $\theta_{ii} > \sum\limits_{j \neq i} \theta_{ij}$, $\boldsymbol{B}$ is a substochastic matrix.

Observe that, $A = \boldsymbol{C}(\mathcal{I} - \frac{1}{2}\boldsymbol{B})$ and, hence, $\boldsymbol{A}^{-1} = (\mathcal{I} - \frac{1}{2}\boldsymbol{B})^{-1}\boldsymbol{C}^{-1}$. Let $\boldsymbol{\theta}$ be the $N \times N-$dimensional diagonal matrix with diagonal entries $(\theta_{11}, \theta_{22} \cdots, \theta_{NN})$. Thus, $\sum\limits_{j=1}^{N} \theta_{jj}(\boldsymbol{A}^{-1})_{ij} = (\boldsymbol{A}^{-1}\boldsymbol{\theta})_{ii}$. Moreover,

$$\boldsymbol{A}^{-1}\boldsymbol{\theta} = \left(\mathcal{I} - \frac{1}{2}\boldsymbol{B}\right)^{-1} \boldsymbol{C}^{-1}\boldsymbol{\theta} = \left(\mathcal{I} - \frac{1}{2}\boldsymbol{B}\right)^{-1} (\boldsymbol{C}^{-1}\boldsymbol{\theta}) = \frac{1}{2}\left(\mathcal{I} - \frac{1}{2}\boldsymbol{B}\right)^{-1},$$

where the last equality follows from $\boldsymbol{C}^{-1}\boldsymbol{\theta} = \frac{1}{2}\mathcal{I}$. Therefore,

$$\sum_{j=1}^{N} \theta_{jj}(\boldsymbol{A}^{-1})_{ij} = \frac{1}{2}\sum_{j=1}^{N}\left[\left(\mathcal{I} - \frac{1}{2}\boldsymbol{B}\right)^{-1}\right]_{ij} = \frac{1}{2}\sum_{j=1}^{N}\left[\mathcal{I} + \sum_{l=1}^{+\infty}\left(\frac{1}{2}\right)^{l}(\boldsymbol{B})^{l}\right]_{ij},$$

where the second equality follows from the fact that $\mathcal{I} - \frac{1}{2}\boldsymbol{B}$ is a diagonal dominant matrix. Thus, for all i, $\sum_{j=1}^{N} \theta_{jj}(\boldsymbol{A}^{-1})_{ij} \geq \frac{1}{2}\sum_{j=1}^{N} \mathcal{I}_{ij} = \frac{1}{2}$. On the other hand, for all i,

$$\frac{1}{2}\sum_{j=1}^{N}\left[\mathcal{I} + \sum_{l=1}^{+\infty}\left(\frac{1}{2}\right)^{l}(\boldsymbol{B})^{l}\right]_{ij} = \frac{1}{2}\sum_{j=1}^{N}\left[\sum_{l=0}^{+\infty}\left(\frac{1}{2}\right)^{l}(\boldsymbol{B})^{l}\right]_{ij} = \frac{1}{2}\sum_{l=0}^{+\infty}\left[\left(\frac{1}{2}\right)^{l}\sum_{j=1}^{N}(\boldsymbol{B})^{l}_{ij}\right] < \frac{1}{2}\sum_{l=0}^{+\infty}\left(\frac{1}{2}\right)^{l} = 1,$$

where the inequality follows from that $\boldsymbol{B}$ is a sub-stochastic matrix. This completes the proof of part (c). *Q.E.D.*

Proof of Theorems 6-7 and Propositions 4-7: We show Theorem 6, Propositions 4-7, and Theorem 7 together by strong backward induction. More specifically, we show that, if $V_{i,\tau}(I_{i,\tau}, \Lambda_{i,\tau}|\boldsymbol{\sigma}_{\tau}^{pf*}) = w_{i,\tau}I_{i,\tau} + \beta_{i,\tau}^{pf}\Lambda_{i,\tau} + \eta_{i,\tau}^{pf}$ for all i and $1 \leq \tau \leq t-1$, (1) Proposition 4(a-c) holds for period t, and Proposition 4(d) holds for period t if $t \geq 2$, (2) Proposition 5 holds for period t, (3) Proposition 6 holds for period t, (4) Proposition 7 holds for period t, (5) there exists a Markov strategy profile $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, which forms an equilibrium in the subgame of period t, (6) if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i and $\gamma_{i,t}$, the equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, is unique, (7) there exists a positive vector $\boldsymbol{\beta}_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$ and a vector $\boldsymbol{\eta}_t^{pf} = (\eta_{1,t}^{pf}, \eta_{2,t}^{pf}, \cdots, \eta_{N,t}^{pf})$, such that $V_{i,t}(I_{i,t}, \Lambda_{i,t}|\boldsymbol{\sigma}_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t} + \eta_{i,t}^{pf}$ for all i, and (8) the starting inventory level in the next period,

$t-1$, is smaller $x^{pf*}_{i,t-1}(\cdot,\cdot,\cdot)$ with probability 1 for $t \geq 2$, and the initial inventory level is always smaller than $x^{pf*}_{i,T}(\cdot,\cdot,\cdot)$. Because $V_{i,0}(I_0,\Lambda_0) = w_{i,0} I_{i,0}$ for all i, the initial condition is satisfied.

First, we start with proving Proposition 4. The proof follows a same logic in proof of Proposition 1. Since $V_{i,t-1}(\boldsymbol{I}_{t-1}, \boldsymbol{\Lambda}_{t-1}|\boldsymbol{\sigma}^{pf*}_{t-1}) = w_i I_{i,t-1} + \beta^{pf}_{i,t-1}\Lambda_{i,t-1} + \eta^{pf}_{i,t-1}$ for all i, the problem defined in (23) can be decomposed into two problems which optimize over $\max_{y_{i,t}} \pi^{pf}_{i,t}(y_{i,t})$ and $\max_{p_{i,t}} \Pi^{pf}_{i,t}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t)$ respectively. Thus, Proposition 4(a) follows the fact that $\pi^{pf}_{i,t}(y_{i,t})$ is concave and continuously differentiable. To prove Proposition 4(b), we need to prove that $O^{pf}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\gamma_t, \Lambda_{i,t})$ is jointly concave in $(p_{i,t}, y_{i,t})$ given $\boldsymbol{\gamma}_t$, $\boldsymbol{p}_{-i,t}$ and $\Lambda_{i,t}$. By the form of $\rho_i(\boldsymbol{p}_t)$ defined in (17), $\partial^2_{p_{i,t}} \Pi^{pf}_{i,t}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t) = -2\theta_{ii}\psi_i(\boldsymbol{\gamma}_t) < 0$, $\Pi^{pf}_{i,t}(\cdot, \boldsymbol{p}_{-i,t}|\boldsymbol{\gamma}_t)$ is strictly concave in $p_{i,t}$ for any given $\boldsymbol{p}_{-i,t}$. Thus, we derive the jointly concavity of $O^{pf}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\gamma_t, \Lambda_{i,t})$. Therefore, Proposition 4(b) follows the concavity of function $O^{pf}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\gamma_t, \Lambda_{i,t})$ in $(y_{i,t}, p_{i,t})$. Proposition 4 parts (c-d) follow from the same arguments as the proof of Proposition 1. We conclude the proof of Proposition 4 for period t.

We now show Proposition 5 holds in period t. By Theorem 1.2 in Fudenberg and Tirole (1991), $\tilde{\mathcal{G}}^{pf,2}_t(\boldsymbol{\gamma}_t)$ has a pure strategy Nash equilibrium $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)$ because the payoff function of firm i is concave in $p_{i,t}$. Since, for each i and t, $\underline{p}_{i,t}$ is sufficiently low whereas $\bar{p}_{i,t}$ is sufficiently high so that they will not affect the equilibrium behaviors of all firms, $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)$ can be characterized by first-order conditions $\partial_{p_{i,t}}\Pi^{pf}_{i,t}(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)|\boldsymbol{\gamma}_t) = 0$ for each i, i.e.,

$$\begin{aligned} & -\theta_{ii}[p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2]\psi_i(\boldsymbol{\gamma}_t) + \rho_{i,t}(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t))\psi_i(\boldsymbol{\gamma}_t) = 0 \\ \Longleftrightarrow & -2\theta_{ii}p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) + \sum_{j\neq i}\theta_{ij}p^{pf*}_{j,t}(\boldsymbol{\gamma}_t) + f_{i,t}(\boldsymbol{\gamma}_t) = 0, \text{ for all } i. \end{aligned} \tag{33}$$

In terms of the matrix language, we have $\boldsymbol{A}\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t) = \boldsymbol{f}(\boldsymbol{\gamma}_t)$. By Lemma 2(a), $\boldsymbol{A}$ is invertible and, thus, $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)$ is uniquely determined by $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t) = \boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}_t)$. To show that $p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) = \sum_j (\boldsymbol{A}^{-1})_{ij} f_j(\boldsymbol{\gamma}_t)$ is continuously increasing in $\gamma_{j,t}$, we observe that $\frac{\partial p^{pf*}_{i,t}(\boldsymbol{\gamma}_t)}{\partial \gamma_{j,t}} = (\boldsymbol{A}^{-1})_{ij}\theta_{jj}\nu'_j(\gamma_{j,t})$. Since, by Lemma 2(a), $(\boldsymbol{A}^{-1})_{ij} \geq 0$ for all i and j, we have $\partial_{\gamma_{j,t}} p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) \geq 0$ and, thus, $p^{pf*}_{i,t}(\boldsymbol{\gamma}_t)$ is continuously increasing in $\gamma_{j,t}$ for each j.

Now, we compute $\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t)$.

$$\begin{aligned} \Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) &= \psi_i(\boldsymbol{\gamma}_t)\rho_i(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t))\left[p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2\right] \\ &= \psi_i(\boldsymbol{\gamma}_t)[\phi_i - \theta_{ii}p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) + \sum_{j\neq i}\theta_{ij}p^{pf*}_{j,t}(\boldsymbol{\gamma}_t)]\left[p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2\right] \\ &= \psi_i(\boldsymbol{\gamma}_t)\left[\theta_{ii}p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - f_i(\boldsymbol{\gamma}_t) + \phi_i\right]\left[p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2\right] \\ &= \psi_i(\boldsymbol{\gamma}_t)\theta_{ii}\left[p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2\right]^2, \end{aligned}$$

where the third equality follows from (33) and the last from $f_i(\boldsymbol{\gamma}_t) = \phi_i + \theta_{ii}(w_i + \nu_i(\gamma_{i,t}) - \delta_i\beta^{pf}_{i,t-1}k_i^2)$. The above computation also implies that $\rho_i(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t)) = \theta_{ii}(p^{pf*}_{i,t}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta^{pf}_{i,t-1}k_i^2)$. We now show that $\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) > 0$. Note that $\Pi^{pf*,2}_{i,t}(\boldsymbol{\gamma}_t) = \frac{\psi_i(\boldsymbol{\gamma}_t)}{\theta_{ii}}[\rho_{i,t}(\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t))]^2 > 0$, where the inequality follows from the assumption that $\rho_{i,t}(\cdot) > 0$ and $\psi_i(\cdot) > 0$ for all $\boldsymbol{p}_t$ and $\boldsymbol{\gamma}_t$ respectively. This completes the proof of Proposition 5 for period t.

Next, we show Proposition 6 for period t. By Proposition 4(a) and Proposition 5, $(\boldsymbol{y}^{pf*}_t, \boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}_t))$ is the unique solution to the system

$$\text{For each } i, \begin{cases} \partial_{y_{i,t}} O^{pf}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\gamma_t, \Lambda_{i,t}) = \partial_{y_{i,t}}\pi^{pf}_{i,t}(y_{i,t}) = 0, \\ \partial_{p_{i,t}} O^{sc}_{i,t}(\boldsymbol{p}_t, y_{i,t}|\gamma_t, \Lambda_{i,t}) = \Lambda_{i,t}\partial_{p_{i,t}}\Pi^{pf}_{i,t}(\boldsymbol{p}_t|\boldsymbol{\gamma}_t) = 0. \end{cases}$$

By the assumption of Proposition 6, $(\boldsymbol{y}_t^{pf*}, \boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))$ is also feasible, which proves Proposition 6 for period t.

Next, we show Proposition 7 for period t. Because the payoff function of firm i in game $\tilde{\mathcal{G}}_t^{pf,1}$ is $\pi_{i,t}^{pf*} + \Lambda_{i,t}\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)$, it is equivalent for firm i to optimize the second term $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)$. Thus, in the proof of Proposition 7, we will use $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)$ as the payoff function of firm i for all i. Since $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t) > 0$ for all $\boldsymbol{\gamma}_t$, $\log(\Pi_{i,t}^{pf,1}(\cdot))$ is well defined. Therefore, $\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)) = \log(\theta_{ii}) + 2\log(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \nu_i(\gamma_{i,t}) + \delta_i\beta_{i,t-1}^{pf}k_i^2) + \log(\psi_i(\boldsymbol{\gamma}_t))$. Since $p_{j,t}^{pf*}(\boldsymbol{\gamma}_t) = \sum_{l=1}^N (\boldsymbol{A}^{-1})_{jl} f_l(\boldsymbol{\gamma}_t) = \sum_{l=1}^N [(\boldsymbol{A}^{-1})_{jl}(\phi_l + \theta_{ll}(w_l + \nu_l(\gamma_{l,t}) - \delta_i\beta_{i,t-1}^{pf}k_i^2))]$ for all j, by direct computation,

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} = \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})\theta_{jj}(\boldsymbol{A}^{-1})_{ij}\nu_i'(\gamma_{i,t})\nu_j'(\gamma_{j,t})}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \nu_{i,t}(\gamma_{i,t}) + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2} + \frac{\partial^2 \log(\psi_i(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } j \neq i. \tag{34}$$

By Lemma 2(a,b), $1 - \theta_{ii}(\boldsymbol{A}^{-1})_{ii} > 0$ and $(\boldsymbol{A}^{-1})_{ij} \geq 0$. Thus, the first term of (34) is non-negative. Because $\psi_{i,t}(\cdot)$ satisfies (3),

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq \frac{\partial^2 \log(\psi_i(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq 0, \text{ for all } j \neq i.$$

and, thus, $\tilde{\mathcal{G}}_t^{pf,1}$ is a log-supermodular game. The feasible action set of player i, $[0, \bar{\gamma}_{i,t}]$, is a compact subset of $\mathbb{R}$. Therefore, by Theorem 2 in Zhou (1994), the pure strategy Nash equilibria of $\tilde{\mathcal{G}}_t^{pf,1}$ is a nonempty complete sublattice of $\mathbb{R}^N$

We now show that if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, the Nash equilibrium of $\tilde{\mathcal{G}}_t^{pf,1}$ is unique. We first show that

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2} < 0, \text{ and } \left|\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2}\right| > \sum_{j\neq i} \frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } i \text{ and } \boldsymbol{\gamma}_t. \tag{35}$$

Since $\nu_{l,t}(\gamma_{l,t}) = \gamma_{l,t}$ for all l (i.e., $\nu_{l,t}'(\cdot) \equiv 1$ for all l), direct computation yields that

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2} = \frac{\partial^2 \log(\psi_i(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2} - \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})^2}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \gamma_{i,t} + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2}.$$

Inequality (3) implies that $\partial^2_{\gamma_{i,t}}\log(\psi_i(\boldsymbol{\gamma}_t)) < 0$ and, thus, $\partial^2_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)) < 0$. Moreover,

$$\left|\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2}\right| = \left|\frac{\partial^2 \log(\psi_{i,t}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2}\right| + \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})^2}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \gamma_{i,t} + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2}$$

and

$$\sum_{j\neq i} \frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} = \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} + \sum_{j\neq i} \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})\theta_{jj}(\boldsymbol{A}^{-1})_{ij}}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \gamma_{i,t} + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2}.$$

Inequality (3) implies that $\left|\frac{\partial^2 \log(\psi_{i,t}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}^2}\right| > \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\boldsymbol{\gamma}_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}$. Lemma 2(b) implies that $1 - \theta_{ii}(\boldsymbol{A}^{-1})_{ii} > 0$. Moreover, Lemma 2(c) suggests that $1 - (\boldsymbol{A}^{-1})_{ii}\theta_{ii} > \sum_{j\neq i}(\boldsymbol{A}^{-1})_{ij}\theta_{jj}$ and, hence,

$$\frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})^2}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \gamma_{i,t} + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2} > \sum_{j\neq i} \frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})\theta_{jj}(\boldsymbol{A}^{-1})_{ij}}{(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_i - \gamma_{i,t} + \delta_i\beta_{i,t-1}^{pf}k_i^2)^2}.$$

Therefore, inequality (35) holds for all $\boldsymbol{\gamma}_t$.

Because $\tilde{\mathcal{G}}_t^{pf,1}$ is a log-supermodular game, by Theorem 5 in Milgrom and Roberts (1990), if there are two distinct pure strategy Nash equilibria $\hat{\boldsymbol{\gamma}}_t^{pf*} \neq \boldsymbol{\gamma}_t^{pf*}$, we must have $\hat{\gamma}_{i,t}^{pf*} \geq \gamma_{i,t}^{pf*}$ for each i, with the inequality being strict for some i. Without loss of generality, we assume that $\hat{\gamma}_{1,t}^{pf*} > \gamma_{1,t}^{pf*}$ and $\hat{\gamma}_{1,t}^{pf*} - \gamma_{1,t}^{pf*} \geq \hat{\gamma}_{i,t}^{pf*} - \gamma_{i,t}^{pf*}$ for each i. Lemma 1 yields that

$$\frac{\partial \log(\Pi_{1,t}^{pf,1}(\hat{\boldsymbol{\gamma}}_t^{pf*}))}{\partial\gamma_{1,t}} \geq \frac{\partial \log(\Pi_{1,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}))}{\partial\gamma_{1,t}} \tag{36}$$

Since $\partial_{\gamma_{1,t}}\partial_{\gamma_{i,t}}\log(\Pi_{1,t}^{pf,1}(\boldsymbol{\gamma}_t))$ is Lebesgue integrable for all $i\neq 1$ and $\boldsymbol{\gamma}_t$, Newton-Leibniz formula implies that

$$\begin{aligned}\frac{\partial\log(\Pi_{1,t}^{pf,1}(\hat{\boldsymbol{\gamma}}_t^{pf*}))}{\partial\gamma_{1,t}}-\frac{\partial\log(\Pi_{1,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}))}{\partial\gamma_{1,t}}&=\int_{s=0}^{1}\sum_{j=1}^{N}(\hat{\gamma}_{j,t}^{pf*}-\gamma_{j,t}^{pf*})\frac{\partial^2\log(\Pi_{1,t}^{pf,1}((1-s)\boldsymbol{\gamma}_t^{pf*}+s\hat{\boldsymbol{\gamma}}_t^{pf*}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}\,\mathrm{d}s\\&\le\int_{s=0}^{1}\sum_{j=1}^{N}(\hat{\gamma}_{1,t}^{pf*}-\gamma_{1,t}^{pf*})\frac{\partial^2\log(\Pi_{1,t}^{pf,1}((1-s)\boldsymbol{\gamma}_t^{pf*}+s\hat{\boldsymbol{\gamma}}_t^{pf*}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}\,\mathrm{d}s<0,\end{aligned}$$

where the first inequality follows from $\hat{\gamma}_{1,t}^{pf*}-\gamma_{1,t}^{pf*}\ge\hat{\gamma}_{i,t}^{pf*}-\gamma_{i,t}^{pf*}$ for all i, and the second from (35), and $\hat{\gamma}_{1,t}^{pf*}-\gamma_{1,t}^{pf*}>0$. This contradicts (36). Thus, $\mathcal{G}_t^{pf,1}$ has a unique pure strategy Nash equilibrium $\boldsymbol{\gamma}_t^{pf*}$.

We now show that the unique pure strategy Nash equilibrium $\boldsymbol{\gamma}_t^{pf*}$ can be characterized by the system of first-order conditions (25). First, (35) implies that $\log(\Pi_{i,t}^{pf,1}(\cdot,\boldsymbol{\gamma}_{-i,t}))$ is strictly concave in $\gamma_{i,t}$ for any i and any fixed $\boldsymbol{\gamma}_{-i,t}$. Hence, $\boldsymbol{\gamma}_t^{pf*}$ must satisfy the system of first-order conditions, i.e., for each i, $\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}))\le 0$ if $\gamma_{i,t}^{pf*}=0$; $\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}))=0$ if $\gamma_{i,t}^{pf*}\in(0,\bar{\gamma}_{i,t})$; and $\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t^{pf*}))\ge 0$ if $\gamma_{i,t}^{pf*}=\bar{\gamma}_{i,t}$. Differentiate $\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))$, and we have

$$\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))=\frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\boldsymbol{\gamma}_t)}{\psi_{i,t}(\boldsymbol{\gamma}_t)}-\frac{2(1-\theta_{ii}(\boldsymbol{A}^{-1})_{ii})}{p_{i,t}^{pf*}(\boldsymbol{\gamma}_t)-w_i-\gamma_{i,t}+\delta_i\beta_{i,t-1}^{pf}k_i^2}.$$

So $\boldsymbol{\gamma}_t^{pf*}$ satisfies the system of first-order conditions (25). This completes the proof of Proposition 7 for period t.

By the concavity of the payoff function $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)$, the Nash equilibrium is characterized by the system of first-order conditions, (25), which has a unique solution $\boldsymbol{\gamma}_t^{pf*}$ if $\nu_i(\gamma_{i,t})=\gamma_{i,t}$ for all i. Because $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)>0$ is positive for any $\boldsymbol{\gamma}_t$, $\partial_{\gamma_{i,t}}\log(\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t))$ has the same sign of $\partial_{\gamma_{i,t}}\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)$. Thus, $\boldsymbol{\gamma}_t^{pf*}$ is also the unique solution to the system:

$$\partial_{\gamma_{i,t}}O_{i,t}^{pf,1}(\boldsymbol{\gamma}_t)\begin{cases}\le 0, & \text{if } \gamma_{i,t}^{pf*}=0,\\ =0, & \text{if } \gamma_{i,t}^{pf*}\in(0,\bar{\gamma}_i),\ \text{ for each } i.\\ \ge 0 & \text{if } \gamma_{i,t}^{pf*}=\bar{\gamma}_i,\end{cases}\tag{37}$$

The solution to (37) uniquely captures the Nash equilibrium in the first-stage game in period t. Because $x_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}),\boldsymbol{\gamma}_t^{pf*},\Lambda_{i,t})\ge I_{i,t}$ for all i, $\boldsymbol{\gamma}_t^{pf*}$ is a feasible decision. Therefore, $\{(\gamma_{i,t}^{pf*},p_{i,t}^{pf*}(\boldsymbol{\gamma}_t),y_{i,t}^{pf*}+\Lambda_{i,t}\rho_{i,t}(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))\psi_{i,t}(\boldsymbol{\gamma}_t)):1\le i\le N\}$ is the unique equilibrium in the subgame of period t.

Next, we show that there exists a positive vector $\boldsymbol{\beta}_t^{pf}=(\beta_{1,t}^{pf},\beta_{2,t}^{pf},\cdots,\beta_{N,t}^{pf})$ and $\boldsymbol{\eta}_t^{pf}=(\eta_{1,t}^{pf},\eta_{2,t}^{pf},\cdots,\eta_{N,t}^{pf})$, such that $V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_t^{pf*})=w_iI_{i,t}+\beta_{i,t}^{pf}\Lambda_{i,t}+\eta_{i,t}^{pf}$. By (22), we have that

$$\begin{aligned}V_{i,t}(\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_t^{pf*})=&J_{i,t}(\gamma_{i,t}^{pf*},p_{i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}),\Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t^{pf*}))\psi_{i,t}(\boldsymbol{\gamma}_t^{pf*}),\boldsymbol{I}_t,\boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_{t-1}^{pf*})\\=&w_{i,t}I_{i,t}+(\delta_i\beta_{i,t-1}^{pf}k_i^1+\Pi_{i,t}^{pf*,1})\Lambda_{i,t}+\pi_{i,t}^{pf*}+\delta_i\eta_{i,t-1}^{pf}.\end{aligned}$$

Since $\beta_{i,t-1}^{pf}>0$, $\beta_{i,t}^{pf}=\delta_i\beta_{i,t-1}^{pf}k_i^1+\Pi_{i,t}^{pf*,1}>0$, and $\eta_{i,t}^{pf}=\pi_{i,t}^{pf*}+\delta_i\eta_{i,t-1}^{pf}$.

Last, for $t\ge 2$, we can find $\boldsymbol{p}_{t-1}^{pf*}(\boldsymbol{\gamma}_{t-1}^{pf*})$, $\boldsymbol{\gamma}_{t-1}^{pf*}$ and $\boldsymbol{y}_{t-1}^{pf*}$ by inductive hypothesis. By Proposition 4(d)

$$\mathbb{P}\Big[x_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}),\boldsymbol{\gamma}_t^{pf*},\Lambda_{i,t})-D_{i,t}^{pf}(\boldsymbol{p}_{-i,t}^{pf*}(\boldsymbol{\gamma}_t^{pf*}),\boldsymbol{\gamma}_t^{pf*},\Lambda_{i,t})\le x_{i,t-1}^{pf}(\boldsymbol{p}_{-i,t-1}^{pf*}(\boldsymbol{\gamma}_{t-1}^{pf*}),\boldsymbol{\gamma}_{t-1}^{pf*},\Lambda_{i,t-1})|\Lambda_{i,t}\Big]=1.$$

This completes the induction and, thus, the proof of Theorem 6, Proposition 4, Proposition 5, Proposition 6, Proposition 7, and Theorem 7. *Q.E.D.*

Proof of Proposition 8: By Theorems 6-7, and Propositions 4-**??**, it suffices to show that, if there exists constants $\beta_{s,t-1}^{pf} \geq 0$ and $\eta_{s,t-1}^{pf}$, such that $V_{i,t-1}(\boldsymbol{I}_{t-1}, \boldsymbol{\Lambda}_{t-1}|\boldsymbol{\sigma}_{t-1}^{pf*}) = w_s I_{i,t-1} + \beta_{s,t-1}^{pf}\Lambda_{i,t-1} + \eta_{s,t-1}^{pf}$ for all i, we have: (1) the optimal safety stock level is symmetric, i.e., $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ for all i, j; (2) the unique Nash equilibrium in $\mathcal{G}_t^{pf,2}(\boldsymbol{\gamma}_t)$ is symmetric if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, (c), the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$, $\boldsymbol{\gamma}_t^{pf*}$ is symmetric, and (d) there exists constants $\beta_{s,t}^{pf} > 0$ and $\eta_{s,t}^{pf}$, such that $V_{i,t}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t|\boldsymbol{\sigma}_{s,t}^{pf*}) = w_{s,t}I_{i,t} + \beta_{s,t}^{pf}\Lambda_{i,t} + \eta_{s,t}^{pf}$ for all i. Since $V_{i,0}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = w_i I_{i,0}$ for all i, the initial condition is satisfied with $\beta_{s,0}^{pf} = 0$.

First, we observe that $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ and $\pi_{i,t}^{pf*} = \pi_{j,t}^{pf*}$ for all i and j follow directly from Proposition 4. Thus, we denote $y_{s,t}^{pf*} := y_{i,t}^{pf*}$ and $\pi_{s,t}^{pf*} := \pi_{i,t}^{pf*}$ for each firm i. Next, we show that if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) = p_{j,t}^{pf*}(\boldsymbol{\gamma}_t)$. Direct computation yields that, for the symmetric PF model, $\sum_{j=1}^N (\boldsymbol{A}^{-1})_{ij}$ is independent of i. Thus, if the value of $\gamma_{j,t}$ is independent of j,then $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) = \sum_{j=1}^N (\boldsymbol{A}^{-1})_{ij} f_j(\boldsymbol{\gamma}_t) = \sum_{j=1}^N [(\boldsymbol{A}^{-1})_{ij}(\phi_{s,t} + \theta_{sa}(w_s + \gamma_{j,t} - \delta_s\beta_{s,t-1}^{pf}k_s^2))] = (\phi_{s,t} + \theta_{sa}(w_s + \gamma_{j,t} - \delta_s\beta_{s,t-1}^{pf}k_s^2))\sum_{j=1}^N (\boldsymbol{A}^{-1})_{ij}$, which is independent of firm i, which we denote as $p_{s,t}^{pf*}(\boldsymbol{\gamma}_t)$.

The objective functions of $\mathcal{G}_t^{pf,1}$, $\{\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t) = \theta_{sa}(p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - w_s - \gamma_{i,t} + \delta_s\beta_{s,t-1}^{pf}k_s^2)\psi_{s,t}(\boldsymbol{\gamma}_t) : 1 \leq i \leq N\}$, are symmetric. Thus, the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$ is symmetric, which we denote as $\boldsymbol{\gamma}_{ss,t}^{pf*} = (\gamma_{s,t}^{pf*}, \gamma_{s,t}^{pf*}, \cdots, \gamma_{s,t}^{pf*})$. Hence, $\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_{ss,t}^{pf*}) = \Pi_{j,t}^{pf,1}(\boldsymbol{\gamma}_{ss,t}^{pf*}) = \Pi_{j,t}^{pf*,1} > 0$. Thus, we denote $\Pi_{i,t}^{pf*,1}$ of each firm i as $\Pi_{s,t}^{pf*,1}$. By Theorem 7(b) and (c), $\beta_{i,t}^{pf} = \delta_s\beta_{s,t-1}^{pf}\mu_{s,t} + \Pi_{i,t}^{pf*,1} = \delta_s\beta_{s,t-1}^{pf}\mu_{s,t} + \Pi_{j,t}^{pf*,1} = \beta_{j,t}^{pf} > 0$, and $\eta_{i,t}^{pf} = \delta_i\eta_{i,t-1}^{pf} + \pi_{i,t}^{pf*} = \delta_i\eta_{i,t-1}^{pf} + \pi_{j,t}^{pf*} = \eta_{j,t}^{pf}$. Thus, we denote the PF market size coefficient of each firm i as $\beta_{s,t}^{pf}$. This completes the induction and, thus, the proof of Proposition 8. *Q.E.D.*

Proof of Theorem 8: Part (a). Let $\zeta_{j,t}^{pf} = \delta_j\beta_{j,t-1}^{pf}k_j^2$. Since $\delta_j > 0$, $\beta_{j,t-1}^{pf} > 0$ and $k_j^2 > 0$, it is sufficient to show that $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t)$ is continuously decreasing in $\zeta_{j,t}^{pf}$. Because $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) = \sum_{j=1}^N (\boldsymbol{A}^{-1})_{ij} f_j(\boldsymbol{\gamma}_t) = \sum_{j=1}^N [(\boldsymbol{A}^{-1})_{ij}(\phi_j + \theta_{jj}(w_j + \gamma_{j,t} - \zeta_{j,t}^{pf}))]$, we have $\partial_{\zeta_{j,t}^{pf}} p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) = -\theta_{jj}(\boldsymbol{A}^{-1})_{ij} \leq 0$, where the inequality follows from Lemma 2(a). Thus, $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t)$ is continuously decreasing in $\zeta_{j,t}^{pf}$ for each j. Thus, part (a) follows.

Part (b). We denote the objective function of each firm i in $\mathcal{G}_{s,t}^{pf,1}$ as $\Pi_{i,t}^{pf,1}(\cdot|\zeta_{s,t}^{pf*})$ to capture its dependence on $\zeta_{s,t}^{pf*}$. The unique symmetric pure strategy Nash equilibrium in $\mathcal{G}_{s,t}^{pf,1}$ is denoted as $\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})$ to capture the dependence of the equilibrium on $\zeta_{s,t}^{pf*}$, where $\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*}) = (\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}), \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}), \cdots, \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}))$. We first show that, if $\bar{\zeta}_{s,t}^{pf*} > \zeta_{s,t}^{pf*}$, $\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) \geq \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$. To the contrary, $\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) < \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$, Lemma 1 yields that $\partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})|\bar{\zeta}_{s,t}^{pf*})) \leq \partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})|\zeta_{s,t}^{pf*}))$, i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - \frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii})}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*}} \\
\leq\ &\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})) - \frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii})}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&[p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*}] - [p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}] \\
=&\left(1 - \sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}\theta_{sa}\right)(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) + \left(1 - \sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}\theta_{sa}\right)(\bar{\zeta}_{s,t}^{pf*} - \zeta_{s,t}^{pf*}) > 0
\end{aligned} \tag{38}$$

where the inequality follows from Lemma 2(c). Thus, $p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*} > p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*} > 0$. Lemma 2(b) implies that $1 - \theta_{sa}(\boldsymbol{A}^{-1})_{ii} > 0$. Hence,

$$-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii}))}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*}} \geq -\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii})}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}}.$$

Thus, we have $\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) \le \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}))$. By (3) and Newton-Leibniz formula,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log(\psi_1(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}))) - \partial_{\gamma_{1,t}} \log(\psi_1(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}))) \\
&= \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}))[\frac{\partial^2 \log(\psi_s(s\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + (1-s)\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}] \, \mathrm{d}s < 0,
\end{aligned}$$

which leads to a contradiction. Therefore, $\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$ is increasing in $\zeta_{s,t}^{pf*}$. The continuity of $\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$ in $\zeta_{s,t}^{pf*}$ follows directly from that $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t|\zeta_{s,t}^{pf*})$ is twice continuously differentiable in $(\boldsymbol{\gamma}_t, \zeta_{s,t}^{pf*})$ and the implicit function theorem.

Part(c). Next we show that if (16) holds, $\Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*}) := \Pi_{s,t}^{pf*,1}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})|\zeta_{s,t}^{pf*})$ is increasing in $\beta_{s,t-1}^{pf}$ and k_s^2. Since $\delta_s > 0$, $\beta_{s,t-1}^{pf} > 0$ and $k_s^2 > 0$, it suffices to show that $\Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*})$ is increasing in $\zeta_{s,t}^{pf*}$. Assume that $\bar{\zeta}_{s,t}^{pf*} > \zeta_{s,t}^{pf*}$. Since we have just shown $\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) \ge \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$, (16) implies that $\psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) \ge \psi_{s,t}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) > 0$. We discuss two cases to prove that $\Pi_{s,t}^{pf*,1}(\bar{\zeta}_{s,t}^{pf*}) > \Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*})$.

First, if $\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) = \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$, $p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*} > p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}$, and, hence,

$$\begin{aligned}
\Pi_{s,t}^{pf*,1}(\bar{\zeta}_{s,t}^{pf*}) &= \theta_{sa}(p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*})^2 \psi_s(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) \\
&> \theta_{sa}(p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*})^2 \psi_s(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) = \Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*}).
\end{aligned}$$

Second, if $\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) > \boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})$, Lemma 1 implies that $\partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})|\bar{\zeta}_{s,t}^{pf*})) \ge \partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})|\zeta_{s,t}^{pf*}))$, i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - \frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii}))}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*}} \\
\ge\; &\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})) - \frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii}))}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}}.
\end{aligned}$$

By (3) and Newton-Leibniz formula,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log(\psi_1(\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}))) - \partial_{\gamma_{1,t}} \log(\psi_1(\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}))) \\
&= \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})) \left[\frac{\partial^2 \log(\psi_s((1-s)\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + s\gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}\right] \mathrm{d}s < 0,
\end{aligned}$$

Hence,

$$-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii}))}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*}} > -\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii}))}{p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}}.$$

Because, by Lemma 2(b), and $1-\theta_{sa}(\boldsymbol{A}^{-1})_{ii} > 0$, we have $p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*} > p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*}$. Therefore,

$$\begin{aligned}
\Pi_{s,t}^{pf*,1}(\bar{\zeta}_{s,t}^{pf*}) &= \theta_{sa}(p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*}) + \bar{\zeta}_{s,t}^{pf*})^2 \psi_s(\boldsymbol{\gamma}_{ss,t}^{pf*}(\bar{\zeta}_{s,t}^{pf*})) \\
&> \theta_{sa}(p_{s,t}^{pf*}(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) - w_s - \gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*}) + \zeta_{s,t}^{pf*})^2 \psi_s(\boldsymbol{\gamma}_{ss,t}^{pf*}(\zeta_{s,t}^{pf*})) = \Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*}).
\end{aligned}$$

We have, thus, shown that $\Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*})$ is increasing in $\zeta_{s,t}^{pf*}$. The continuity of $\Pi_{s,t}^{pf*,1}(\zeta_{s,t}^{pf*})$ in $\zeta_{s,t}^{pf*}$ follows directly from that of $\gamma_{s,t}^{pf*}(\zeta_{s,t}^{pf*})$ and that $\Pi_{i,t}^{pf,1}(\boldsymbol{\gamma}_t|\zeta_{s,t}^{pf*})$ is continuous in $(\boldsymbol{\gamma}_t, \zeta_{s,t}^{pf*})$. This concludes the proof of part (d).

Part (d). By part (c), we have that $\Pi_{s,t}^{pf*,1}$ is continuously increasing in $\beta_{s,t-1}^{pf}$ and k_s^2. By Theorem 7(b), thus, we have that $\beta_{s,t}^{pf}$ is continuously increasing in $\beta_{s,t-1}^{pf}$, k_s^1 and k_s^2 since $k_s^1 > 0$. *Q.E.D.*

Proof of Theorem 9: Part (a). For each $\boldsymbol{\gamma}_t$, $\hat{p}_{s,t}^{pf*}(\boldsymbol{\gamma}_t) \le p_{s,t}^{pf*}(\boldsymbol{\gamma}_t)$ directly follows from Theorem 8(a) if $\hat{k}_s^2 \ge k_s^2$. By Proposition 8(d), $\hat{p}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t) = \hat{p}_{s,t}^{pf*}(\boldsymbol{\gamma}_t) \le p_{s,t}^{pf*}(\boldsymbol{\gamma}_t) = p_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t)$. This completes the proof of part (a).

Part (b). By part (a), $\hat{p}_{i,t}^{pf*}(\boldsymbol{\gamma}_t) \le p_{i,t}^{pf*}(\boldsymbol{\gamma}_t)$ for each firm i and each $\boldsymbol{\gamma}_t$. When the PF model is symmetric, $\sum_{j=1}^N \theta_{jj}(\boldsymbol{A}^{-1})_{ij}$ is independent of i. Direct computation yields that $p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - \hat{p}_{i,t}^{pf*}(\boldsymbol{\gamma}_t) = (\sum_{j=1}^N \theta_{jj}(\boldsymbol{A}^{-1})_{ij})\delta_s(\hat{\beta}_{s,t-1}^{pf}\hat{k}_s^2 - \beta_{s,t-1}^{pf}k_s^2) \ge 0$, for all $\boldsymbol{\gamma}_t$, which is independent of i. Thus, (16) and Newton-Leibniz formula imply that

$$\rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t)) - \rho_s(\hat{p}_t^{pf*}(\boldsymbol{\gamma}_t)) = \int_{s=0}^1 \sum_{i=1}^N (p_{i,t}^{pf*}(\boldsymbol{\gamma}_t) - \hat{p}_{i,t}^{pf*}(\boldsymbol{\gamma}_t)) \frac{\partial \rho_s((1-s)\hat{p}_t^{pf*}(\boldsymbol{\gamma}_t) + s\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))}{\partial p_{i,t}} \,\mathrm{d}s \le 0.$$

Hence, $\rho_s(\hat{p}_t^{pf*}(\boldsymbol{\gamma}_t)) \ge \rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))$. For any $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \in \mathcal{S}$ and $\boldsymbol{\gamma}_t \in [0, \bar{\gamma}_{s,t}]^N$,

$$\hat{x}_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t) = \hat{y}_{s,t}^{pf*} + \Lambda_{s,t}\rho_s(\hat{p}_t^{pf*}(\boldsymbol{\gamma}_t))\psi_s(\boldsymbol{\gamma}_t) \ge y_{s,t}^{pf*} + \Lambda_{s,t}\rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}_t))\psi_s(\boldsymbol{\gamma}_t) = x_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t, \boldsymbol{\gamma}_t).$$

This completes the proof of part (b).

Part (c). We prove part (c) first then . Because $\hat{k}_s^2 \ge k_s^2$, Theorem 8(b) yields that $\hat{\gamma}_{s,t}^{pf*} \ge \gamma_{s,t}^{pf*}$ and, hence, $\gamma_{i,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) = \gamma_{s,t}^{pf*} \ge \tilde{\gamma}_{s,t}^{pf*} = \tilde{\gamma}_{s,t}^{pf*}(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t)$ for each i and $(\boldsymbol{I}_t, \boldsymbol{\Lambda}_t) \in \mathcal{S}$. This completes the proof of part (c). *Q.E.D.*

Proof of Theorem 10: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta_{s,t-1}^{pf} \ge \beta_{s,t-2}^{pf}$, (1) $p_{i,t}^{pf*}(\boldsymbol{\gamma}) \le p_{i,t-1}^{pf*}(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} \in [0, \bar{\gamma}_{s,t}]^N$, (2) $p_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) \le p_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$ for each i, $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, and $\boldsymbol{\gamma} \in [0, \bar{\gamma}_{s,t}]^N$, (3) $\gamma_{s,t}^{pf*} \ge \gamma_{s,t-1}^{pf*}$, (4) $\gamma_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}) \ge \gamma_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, (5) $x_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) \ge x_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$ for each i, $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, and $\boldsymbol{\gamma} \in [0, \bar{\gamma}_{s,t}]^N$, and (6) $\beta_{s,t}^{pf} \ge \beta_{s,t-1}^{pf}$. Since, by Theorem 7(a), $\beta_{s,1}^{pf} \ge \beta_{s,0}^{pf} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 8(a), $\beta_{s,t-1}^{pf} \ge \beta_{s,t-2}^{pf}$ suggests that $p_{s,t}^{pf*}(\boldsymbol{\gamma}) \le p_{s,t-1}^{pf*}(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} \in [0, \bar{\gamma}_{s,t}]^N$. Theorem 8(b) implies that $\gamma_{s,t}^{pf*} \ge \gamma_{s,t-1}^{pf*}$. Hence, $p_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) = p_{i,t}^{pf*}(\boldsymbol{\gamma}) \le p_{i,t-1}^{pf*}(\boldsymbol{\gamma}) = p_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$ for each i, $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, and $\boldsymbol{\gamma} \in [0, \bar{\gamma}_{s,t}]^N$, and $\gamma_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}) = \gamma_{s,t}^{pf*} \ge \gamma_{s,t-1}^{pf*} = \gamma_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda})$ for each i and $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$.

We now show that $x_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) \ge x_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma})$ for each i, $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$, and $\boldsymbol{\gamma} \in [0, \bar{\gamma}]^N$. Because the PF model is symmetric, $\sum_{j=1}^N \theta_{jj}(\boldsymbol{A}^{-1})_{ij}$ is independent of i. Direct computation yields that $p_{i,t-1}^{pf*}(\boldsymbol{\gamma}) - p_{i,t}^{pf*}(\boldsymbol{\gamma}) = (\sum_{j=1}^N \theta_{jj}(\boldsymbol{A}^{-1})_{ij})\delta_s k_s^2(\beta_{s,t}^{pf} - \beta_{s,t-1}^{pf}) \ge 0$, for all $\boldsymbol{\gamma}$, which is independent of i. Thus, (16) and the Newton-Leibniz formula implies that

$$\rho_s(\boldsymbol{p}_{t-1}^{pf*}(\boldsymbol{\gamma})) - \rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma})) = \int_{s=0}^1 \sum_{i=1}^N (p_{i,t-1}^{pf*}(\boldsymbol{\gamma}) - p_{i,t}^{pf*}(\boldsymbol{\gamma})) \frac{\partial \rho_s((1-s)\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}) + s\boldsymbol{p}_{t-1}^{pf*}(\boldsymbol{\gamma}))}{\partial p_i} \,\mathrm{d}s \le 0.$$

Hence, $\rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma})) \ge \rho_s(\boldsymbol{p}_{t-1}^{pf*}(\boldsymbol{\gamma}))$ for all $\boldsymbol{\gamma}$. Since $y_{s,t}^{pf*} = y_{s,t-1}^{pf*}$, Theorem 7(b) implies that, for any $(\boldsymbol{I}, \boldsymbol{\Lambda}) \in \mathcal{S}$ and $\gamma \in [0, \bar{\gamma}_{s,t}]^N$,

$$x_{i,t}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}) = y_{s,t}^{pf*} + \Lambda_{s,t}\rho_s(\boldsymbol{p}_t^{pf*}(\boldsymbol{\gamma}))\psi_s(\boldsymbol{\gamma}) \ge y_{s,t-1}^{pf*} + \Lambda_{s,t}\rho_s(\boldsymbol{p}_{t-1}^{pf*}(\boldsymbol{\gamma}))\psi_s(\boldsymbol{\gamma}) = x_{i,t-1}^{pf*}(\boldsymbol{I}, \boldsymbol{\Lambda}, \boldsymbol{\gamma}).$$

Finally, $\beta^{pf}_{s,t} \geq \beta^{pf}_{s,t-1}$ follows from Theorem 8(d). This completes the induction and, thus, the proof of Theorem 10. *Q.E.D.*

Proof of Theorem 11: Part (a). We now show that, if $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, $\gamma^{pf*}_{s,t} \geq \gamma^{sc*}_{s,t}$. Proposition 5 implies that $\boldsymbol{p}^{pf*}_t(\boldsymbol{\gamma}^{pf*}_{ss,t}) = \boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t})$. By Proposition 1, $\boldsymbol{p}^{sc*}_{ss,t} = \boldsymbol{A}^{-1}f(\boldsymbol{\gamma}^{sc*}_{ss,t})$. We assume, to the contrary, that $\gamma^{pf*}_{s,t} < \gamma^{sc*}_{s,t}$. Lemma 1 implies that $\partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\boldsymbol{\gamma}^{pf*}_{ss,t})) \leq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}, \boldsymbol{p}^{sc*}_{ss,t}))$, i.e.,

$$\begin{aligned}
&-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11})}{\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{pf*}_{s,t} - \delta_s \beta^{pf}_{s,t-1} k_s^2)] - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2} + \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{pf*}_{ss,t})) \\
&\leq -\frac{1}{\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{sc*}_{s,t} - \delta_s \beta^{sc}_{s,t-1} k_s^2)] - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2} + \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{sc*}_{ss,t})).
\end{aligned} \tag{39}$$

Inequality (3) and the Newton-Leibniz formula imply that

$$\partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{sc*}_{ss,t})) - \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{pf*}_{ss,t})) = \int_{s=0}^1 \sum_{j=1}^N (\gamma^{sc*}_{s,t} - \gamma^{pf*}_{s,t}) \left[\frac{\partial^2 \log(\psi_s((1-s)\gamma^{pf*}_{s,t} + s\gamma^{sc*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}\right] \mathrm{d}s < 0.$$

By (39),

$$\begin{aligned}
&-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11})}{\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{pf*}_{s,t} - \delta_s \beta^{pf}_{s,t-1} k_s^2)] - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2} \\
&< -\frac{1}{\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{sc*}_{s,t} - \delta_s \beta^{sc}_{s,t-1} k_s^2)] - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2}.
\end{aligned}$$

Lemma 2(b) suggests that $0 \leq 2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11}) \leq 1$. Hence,

$$\begin{aligned}
&\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{pf*}_{s,t} - \delta_s \beta^{pf}_{s,t-1} k_s^2)] - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2 \\
<&\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{sc*}_{s,t} - \delta_s \beta^{sc}_{s,t-1} k_s^2)] - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2.
\end{aligned} \tag{40}$$

Since $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$ and $\gamma^{pf*}_{s,t} < \gamma^{sc*}_{s,t}$, $\delta_s \beta^{pf}_{s,t-1} k_s^2 - \gamma^{pf*}_{s,t} > \delta_s \beta^{sc}_{s,t} k_s^2 - \gamma^{sc*}_{s,t}$. Lemma 2(c) implies that $1 - \sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}\theta_{sa} > 0$. Therefore,

$$\begin{aligned}
&\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{pf*}_{s,t} - \delta_s \beta^{pf}_{s,t-1} k_s^2)] - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2 \\
>&\sum_{j=1}^N (\boldsymbol{A}^{-1})_{1j}[\phi_{sa} + \theta_{sa}(w_s + \gamma^{sc*}_{s,t} - \delta_s \beta^{sc}_{s,t-1} k_s^2)] - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2,
\end{aligned}$$

which contradicts the inequality (40). Therefore, $\gamma^{pf*}_{s,t} \geq \gamma^{sc*}_{s,t}$. This completes the proof of part (a).

Part (b). We first show, by backward induction, that, if $\theta_{sb} = 0$ for each t, $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$ for each t. Since $\beta^{pf}_{s,0} = \beta^{sc}_{s,0} = 0$, the initial condition is satisfied. Now we prove that if $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$ and $\theta_{sb} = 0$, we have $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$.

First, we observe that if $\theta_{sb} = 0$, $(\boldsymbol{A}^{-1})_{11}\theta_{sa} = \frac{1}{2}$ and, thus, $2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11}) = 1$. Part (a) shows that $\gamma^{pf*}_{s,t} \geq \gamma^{sc*}_{s,t}$. If $\gamma^{pf*}_{s,t} = \gamma^{sc*}_{s,t}$,

$$\begin{aligned}
\Pi^{pf*,1}_{s,t} = &\ \theta_{sa}((\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t}))_i - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2)^2 \psi_{s,t}(\boldsymbol{\gamma}^{pf*}_{ss,t}) \\
\geq &\ \theta_{sa}((\boldsymbol{A}^{-1}f(\boldsymbol{\gamma}^{sc*}_{ss,t}))_i - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2)^2 \psi_{s,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}) = \Pi^{sc*}_{s,t},
\end{aligned}$$

where the inequality follows from $\beta^{pf}_{s,t-1} \geq \beta^{pf}_{s,t-1}$.

If $\gamma^{pf*}_{s,t} > \gamma^{sc*}_{s,t}$, Lemma 1 implies that $\partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\boldsymbol{\gamma}^{pf*}_{ss,t})) \geq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}, \boldsymbol{p}^{sc*}_{ss,t}))$, i.e.,

$$\begin{aligned} &-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11})}{(\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t}))_1 - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2} + \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{pf*}_{ss,t})) \\ \geq &-\frac{1}{(\boldsymbol{A}^{-1} f(\boldsymbol{\gamma}^{sc*}_{ss,t}))_1 - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2} + \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{sc*}_{ss,t})). \end{aligned} \tag{41}$$

Inequality (3) and the Newton-Leibniz formula imply that

$$\partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{pf*}_{ss,t})) - \partial_{\gamma_{1,t}} \log(\psi_s(\boldsymbol{\gamma}^{sc*}_{ss,t})) = \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma^{pf*}_{s,t} - \gamma^{sc*}_{s,t}) \left[\frac{\partial^2 \log(\psi_s((1-s)\gamma^{sc*}_{s,t} + s\gamma^{pf*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}} \right] \mathrm{d}s < 0.$$

By (41), we have

$$-\frac{2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11})}{(\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t}))_1 - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2} > -\frac{1}{(\boldsymbol{A}^{-1} f(\boldsymbol{\gamma}^{sc*}_{ss,t}))_1 - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2}.$$

Because $2(1-\theta_{sa}(\boldsymbol{A}^{-1})_{11}) = 1$, we have $(\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t}))_1 - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2 > ((\boldsymbol{A}^{-1} f(\boldsymbol{\gamma}^{sc*}_{ss,t}))_1 - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2 > 0$. By inequality (16), $\gamma^{pf*}_{s,t} > \gamma^{sc*}_{s,t}$ implies that $\psi_{s,t}(\boldsymbol{\gamma}^{pf*}_{ss,t}) > \psi_{s,t}(\boldsymbol{\gamma}^{sc*}_{ss,t})$. Thus, we have

$$\begin{aligned} \Pi^{pf*,1}_{s,t} = &\quad \theta_{sa}((\boldsymbol{A}^{-1}\boldsymbol{f}(\boldsymbol{\gamma}^{pf*}_{ss,t}))_1 - w_s - \gamma^{pf*}_{s,t} + \delta_s \beta^{pf}_{s,t-1} k_s^2)^2 \psi_{s,t}(\boldsymbol{\gamma}^{pf*}_{ss,t}) \\ &> \theta_{sa}((\boldsymbol{A}^{-1} f(\boldsymbol{\gamma}^{sc*}_{ss,t}))_1 - w_s - \gamma^{sc*}_{s,t} + \delta_s \beta^{sc}_{s,t-1} k_s^2)^2 \psi_{s,t}(\boldsymbol{\gamma}^{sc*}_{ss,t}) = \Pi^{sc*}_{s,t}. \end{aligned}$$

We have thus shown that if $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, $\Pi^{pf*,1}_{s,t} \geq \Pi^{sc*}_{s,t}$. By Theorem 2(b) and Theorem 7(b), $\beta^{pf}_{s,t} = \delta_s \beta^{pf}_{s,t-1} k^1_s + \Pi^{pf*,1}_{s,t} \geq \delta_s \beta^{sc}_{s,t-1} k^1_s + \Pi^{sc*}_{s,t} = \beta^{sc}_{s,t}$. This completes the induction and, by part (a), the proof of part (b) for the case $\theta_{sb} = 0$.

For any fixed θ_{sa}, both $\beta^{pf}_{s,t}$ and $\beta^{sc}_{s,t}$ are continuous in θ_{sb}. Thus, for each period t, there exists a $\epsilon_t \geq 0$, such that, if $\theta_{sb} \leq \epsilon \theta_{sa}$, $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$. It remains to show that $\epsilon \leq \frac{1}{N-1}$. This inequality follows from the diagonal dominance condition that $\theta_{sa} > (N-1)\theta_{sb}$. This completes the proof of part (b). *Q.E.D.*

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