# E-Companion for POM Journal Template

The additional results and proof of Section 3, Section 4 and Section 5 are respectively provided in Appendix EC.1, EC.2 and EC.3.

## EC.1. Additional Results and Proof in Section 3

We first present some additional results in Appendix EC.1.1. We provide some Auxiliary Results used to prove the results in Section 3 in Appendix EC.1.2 and we prove the results in Section 3 in Appendix EC.1.3.

### EC.1.1. Additional Results in Section 3

Proposition EC.1. **(existence and uniqueness of equilibrium)** *For any* $t \in \{1,...,T\}$, *given a commission profile* $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ *and the total mass of agents* $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$,

*(i) a competitive equilibrium* $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ *always exists;*

*(ii) all competitive equilibria share the same supply-demand vector* $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$, *and they share the same prices* $p_i(t)$ *for* $0 < q_i^s(t) < s_i(t)$ .

Lemma EC.1. **(commissions for feasible transactions)** *For any* $t \in \{1,...,T\}$, *given any positive population vector* $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ *and non-negative trading vector* $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ *that satisfy (i) the flow conservation conditions in* (2c)-(2e) *and (ii)* $\boldsymbol{q}^s(t) \le \boldsymbol{s}(t)$ *and* $\boldsymbol{q}^b(t) \le \boldsymbol{b}(t)$, *a commission profile* $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ *supports* $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ *in a competitive equilibrium if there exists a price vector* $\boldsymbol{p}(t) \in \mathbb{R}^{N_s}$ *that satisfies the following system of linear inequalities:*

$$p_i(t) - r_i^s(t) \; = \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; > \; 0, \tag{EC.1a}$$

$$p_i(t) - r_i^s(t) \; \le \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; = \; 0, \tag{EC.1b}$$

$$p_i(t) + r_j^b(t) \; = \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; > \; 0, \tag{EC.1c}$$

$$p_i(t) + r_j^b(t) \; \ge \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; = \; 0. \tag{EC.1d}$$

Consider the following convex optimization problem:

$$\mathcal{R}(T) \; = \; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \quad \sum_{t=1}^{T} \Bigg[ \sum_{j \in \mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) q_i^s(t) \Bigg] \tag{EC.2a}$$

$$
\begin{aligned}
\text{s.t.}\quad & q_i^s(t) \le s_i(t), && \forall i \in \mathcal{S}, t \in \{1,...,T\}, && \text{(EC.2b)}\\
& q_j^b(t) \le b_j(t), && \forall j \in \mathcal{B},\ t \in \{1,...,T\}, && \text{(EC.2c)}\\
& \sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), && \forall i \in \mathcal{S},\ t \in \{1,...,T\}, && \text{(EC.2d)}\\
& q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), && \forall j \in \mathcal{B},\ t \in \{1,...,T\}, && \text{(EC.2e)}\\
& x_{ij}(t) \ge 0, && \forall (i,j) \in E,\ t \in \{1,...,T\}, && \text{(EC.2f)}\\
& s_i(t+1) \le \mathcal{G}_i^s(s_i(t), q_i^s(t)), && \forall i \in \mathcal{S},\ t \in \{1,...,T-1\}, && \text{(EC.2g)}\\
& b_j(t+1) \le \mathcal{G}_j^b(b_j(t), q_j^b(t)), && \forall j \in \mathcal{B},\ t \in \{1,...,T-1\}. && \text{(EC.2h)}
\end{aligned}
$$

From Problem (EC.2), we can establish Proposition EC.2, which enables us to solve a concave maximization problem to obtain the optimal solution $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b})$ to Problem (EC.2), from which we can further establish the optimal commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ by solving a set of linear inequalities in (EC.1) of Lemma EC.1.

Proposition EC.2. **(tightness of relaxation)** *For any $T \ge 1$, Problem (EC.2) is a tight relaxation of Problem (3): $\mathcal{R}^*(T) = \mathcal{R}(T)$ and any optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}})$ to Problem (EC.2) is also optimal to Problem (3).*

#### EC.1.2. Auxiliary Results for Section 3

Lemmas EC.2 - EC.4 are needed to prove Proposition EC.1. In Lemma EC.4, we establish the connection between the equilibrium and the optimal solution to an optimization problem in (EC.4). Before that, we establish some properties for the optimization problem in Lemma EC.2. We also establish the existence of the optimal solution to this optimization problem in Lemma EC.3, and show that it is essentially unique. These lemmas enable us to establish the existence and uniqueness of the competitive equilibrium in Definition 1. The proof of Auxiliary Results follows a similar argument as the proof of Proposition EC.1 and Proposition 9 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

For simplicity of notation, we first define that

$$
W_{b_j}^t(q_j^b(t)) \;:=\; \int_0^{q_j^b(t)} F_{b_j}^{-1}\Big(1 - \frac{z}{b_j(t)}\Big)dz - r_j^b(t)q_j^b(t), \tag{EC.3a}
$$

$$W^t_{s_i}(q^s_i(t)) \;:=\; -\int_0^{q^s_i(t)} F^{-1}_{s_i}\Big(\frac{z}{s_i(t)}\Big)dz - r^s_i(t)q^s_i(t). \tag{EC.3b}$$

Note that the sum of $W^t_{b_j}(q^b_j(t))$ and $W^t_{s_i}(q^s_i(t))$ can be viewed as the total surplus of buyers and sellers trading in the platform, and is the objective function in Problem (EC.4). Let $W^{t'}_{b_j}(q)$ be the derivative of $W^t_{b_j}(q)$ at $q = q^b_j(t)$ for any $0 < q^b_j(t) < b_j(t)$, and abusing some notation, $W^{t'}_{b_j}(0) = \lim_{q^b_j(t)\downarrow 0} W^t_{b_j}(q^b_j(t))$ and $W^{t'}_{b_j}(b_j(t)) = \lim_{q^b_j(t)\uparrow b_j(t)} W^t_{b_j}(q^b_j(t))$ given Assumption 2(i). Similarly, we let $W^{t'}_{s_i}(q)$ be the derivative of $W^t_{s_i}(q)$ at $q = q^s_i(t)$ for any $0 < q^s_i(t) < s_i(t)$, and we let $W^{t'}_{s_i}(0) = \lim_{q^s_i(t)\downarrow 0} W^t_{s_i}(q^s_i(t))$ and $W^{t'}_{s_i}(s_i(t)) = \lim_{q^s_i(t)\uparrow s_i(t)} W^t_{s_i}(q^s_i(t))$ given Assumption 2(i). We consider the following properties of functions $W^t_{b_j}(q^b_j(t))$ and $W^t_{s_i}(q^s_i(t))$.

LEMMA EC.2. *For any $j \in \mathcal{B}$, $i \in \mathcal{S}$ and $t \in \{1, \ldots, T\}$,*

*(i) $W^t_{b_j}(q)$ is continuously differentiable and strictly concave in $q \in (0, b_j(t))$; moreover, both $W^t_{b_j}(q)$ and $W^{t'}_{b_j}(q)$ are right continuous at $q = 0$ and left continuous $q = b_j(t)$.*

*(ii) $W^t_{s_i}(q)$ is continuously differentiable and strictly concave in $q \in (0, s_i(t))$; moreover, both $W^t_{s_i}(q)$ and $W^{t'}_{s_i}(q)$ are right continuous at $q = 0$ and left continuous $q = s_i(t)$.*

For any $t \in \{1, ..., T\}$, we proceed to consider the following optimization problem:

$$W(t) = \max_{\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t)} \sum_{i\in\mathcal{B}} \left(\int_0^{q^b_j(t)} F^{-1}_{b_j}\left(1 - \frac{z}{b_j(t)}\right) dz - r^b_j(t)q^b_j(t)\right)$$

$$- \sum_{i\in\mathcal{S}} \left(\int_0^{q^s_i(t)} F^{-1}_{s_i}\left(\frac{z}{s_i(t)}\right) dz + r^s_i(t)q^s_i(t)\right) \tag{EC.4a}$$

$$\text{s.t. } q^b_j(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{EC.4b}$$

$$\sum_{j':(i,j')\in E} x_{i,j'}(t) = q^s_i(t), \qquad \forall i \in \mathcal{S}, \tag{EC.4c}$$

$$q^b_j(t) \le b_j(t), \qquad \forall j \in \mathcal{B}, \tag{EC.4d}$$

$$q^s_i(t) \le s_i(t), \qquad \forall i \in \mathcal{S}, \tag{EC.4e}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E. \tag{EC.4f}$$

From Problem (EC.4), we establish the result below. Before that, we define the notation "$a \le 0 \perp b \ge 0$" as $a \le 0, b \ge 0, ab = 0$.

LEMMA EC.3. *(i) There exists an optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ to Problem* (EC.4).

*(ii)* *Given any optimal primal solution* $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$, *there exists a dual multiplier vector* $(\boldsymbol{\theta^b}(t), \boldsymbol{\theta^s}(t), \boldsymbol{\eta}^b(t), \boldsymbol{\eta}^s(t), \boldsymbol{\pi}(t))$ *associated with constraints* (EC.4b)-(EC.4f) *that satisfy the KKT conditions below:*

$$F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big) - r_j^b(t) - \theta_j^b(t) - \eta_j^b(t) = 0, \qquad \forall j \in \mathcal{B}, \tag{EC.5a}$$

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) + r_i^s(t) - \theta_i^s(t) + \eta_i^s(t) = 0, \qquad \forall i \in \mathcal{S}, \tag{EC.5b}$$

$$\theta_j^b(t) - \theta_i^s(t) + \pi_{ij}(t) = 0, \qquad \forall (i,j) \in E, \tag{EC.5c}$$

$$q_j^b(t) - b_j(t) \le 0 \perp \eta_j^b(t) \ge 0, \qquad \forall j \in \mathcal{B}, \tag{EC.5d}$$

$$q_i^s(t) - s_i(t) \le 0 \perp \eta_i^s(t) \ge 0, \qquad \forall i \in \mathcal{S}, \tag{EC.5e}$$

$$x_{ij}(t) \ge 0 \perp \pi_{ij}(t) \ge 0, \qquad \forall (i,j) \in E, \tag{EC.5f}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{EC.5g}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}. \tag{EC.5h}$$

*In addition, these KKT conditions in* (EC.5) *are necessary and sufficient conditions for the optimality of solution* $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$.

*(iii)* *All primal optimal solution* $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ *share the same vector* $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$*;*

*(iv)* *The dual solution* $\theta_i^s(t)$ *for* $i \in \{i' : 0 < q_{i'}^s < s_{i'}\}$ *that satisfies* (EC.5) *is unique.*

The conditions in Lemma EC.4(i)-(ii) are sufficient and necessary conditions, while those in Lemma EC.4(iii) are only sufficient conditions for equilibrium, as the prices for type $i \in \{i' : q_{i'}^s(t) = 0 \text{ or } q_{i'}^s(t) = s_{i'}(t)\}$ are not necessarily unique.

Lemma EC.4. *In each period* $t \in \{1, ..., T\}$, *given any commission profile* $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$ *and population vector* $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$,

*(i)* $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ *satisfies the equilibrium conditions in Definition 1 if and only if it is an optimal solution to Problem* (EC.4)*;*

*(ii)* *for* $i \in \{i' : 0 < q_{i'}^s(t) < s_{i'}(t)\}$, $p_i(t)$ *satisfies the equilibrium conditions in Definition 1 if and only if*

$$p_i(t) = \theta_i^s(t). \tag{EC.6a}$$

*(iii) for* $i \in \{i' : q^s_{i'}(t) = 0$ *or* $q^s_{i'}(t) = s_{i'}(t)\}$, $p_i(t)$ *satisfies the equilibrium conditions in Definition 1 if*

$$p_i(t) = \theta^s_i(t). \tag{EC.6b}$$

Before proceeding, note that functions $F^{-1}_{s_i}(\cdot)$ and $F^{-1}_{b_j}(\cdot)$ have the following properties in an equilibrium:

(1) On the seller side, if $p_i(t) - r^s_i(t) \le 0$, then $q^s_i(t) = 0$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \;\ge\; p_i(t) - r^s_i(t), \tag{EC.7a}$$

if $0 < p_i(t) - r^s_i(t) < \overline{v}_{s_i}$, then $0 < q^s_i(t) < s_i(t)$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \;=\; p_i(t) - r^s_i(t), \tag{EC.7b}$$

if $\overline{v}_{s_i} \le p_i(t) - r^s_i(t)$, then $q^s_i(t) = s_i(t)$ and

$$F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big) \;\le\; p_i(t) - r^s_i(t). \tag{EC.7c}$$

(2) On the buyer side, if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \le 0$, then $q^b_j(t) = b_j(t)$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \;\ge\; \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{EC.8a}$$

if $0 < \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} < \overline{v}_{b_j}$, then $0 < q^b_j(t) < b_j(t)$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \;=\; \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{EC.8b}$$

if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \;\ge\; \overline{v}_{b_j}$, then $q^b_j(t) = 0$ and

$$F^{-1}_{b_j}\Big(1 - \frac{q^b_j(t)}{b_j(t)}\Big) \le \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}. \tag{EC.8c}$$

### EC.1.3. Proof of Results for Section 3

Based on Lemmas EC.2 - EC.4, Proposition EC.1 is proved as below:

**Proof of Proposition EC.1.** We establish the following two claims of this result.

Claim (i). Lemma EC.3(i) implies that the optimal primal solution to (EC.4) always exists, and Lemma EC.4(i) implies that the $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (EC.4). Therefore, the equilibrium transaction vector $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ exists.

Lemma EC.3(ii) implies that the optimal dual solution to (EC.4) always exists, and Lemma EC.4(ii) implies that $\boldsymbol{p}$ that satisfies the equality in (EC.6) is the equilibrium price vector. Therefore, there exists a corresponding equilibrium price vector.

Claim (ii). Lemma EC.3(iii) implies that the optimal primal solution $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ to (EC.4) is unique. Lemma EC.4(i) implies that the $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (EC.4). Therefore, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is unique.

Lemma EC.3(iv) implies that the optimal dual solution $\boldsymbol{\theta}^s$ to Problem (EC.4) is unique for $i \in \{i' : 0 < q^s_{i'} < s_{i'}\}$, and Lemma EC.4(ii) implies that $p_i(t) = \theta^s_i(t)$ for $i$ that satisfies $0 < q^s_i(t) < s_i(t)$. Therefore, the equilibrium price is unique for $i$ that satisfies $0 < q^s_i(t) < s_i(t)$. ■

**Proof of Lemma EC.1.** We establish the sufficiency of (EC.1) in Step 1 and construct a feasible commission profile in Step 2 to show that the feasible commission profile always exists.

Step 1: Sufficiency. We show that for any $(\boldsymbol{q}^b(t), \boldsymbol{q}^s(t), \boldsymbol{x}(t))$ that satisfies (2c)-(2e), if vector $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ satisfies the conditions in (EC.1), then it satisfies the conditions in Definition 1.

We first verify the conditions in Definition 1, in which (2c)-(2e) immediately follow from our conditions.

(2a) We consider the following two cases:

When $q^s_i(t) > 0$, $s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) \overset{(a)}{=} s_i(t)F_{s_i}(F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})) = q^s_i(t)$, (a) follows from (EC.1a).

When $q^s_i(t) = 0$, $0 \le s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) \overset{(b)}{\le} s_i(t)F_{s_i}(F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})) = q^s_i(t) = 0$, (b) follows from (EC.1b).This implies that the inequalities are all tight, then $s_i(t)F_{s_i}(p_i(t) - r^s_i(t)) = q^s_i(t)$.

(2b) We consider the following two cases:

When $q^b_j(t) = 0$, then $x_{ij}(t) = 0$ for any $i : (i, j) \in E$, then $0 \le b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r^b_j(t))\Big) \overset{(c)}{\le} b_j(t)\Big(1 - F_{b_j}(F^{-1}_{b_j}(1 - \frac{q^b_j(t)}{b_j(t)}))\Big) = q^b_j(t) = 0$, where (c) follows from (EC.1d).This implies that the inequalities are all tight, then $b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r^b_j(t))\Big) = q^b_j(t)$.

When $q_j^b(t)>0$, pick a $i_1$ such that $x_{i_1 j}(t)>0$ we have $p_{i_1}(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (EC.1c); if there exists any $i_2$ such that $x_{i_2 j}(t)=0$, we have $p_{i_2}(t)\geq F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ based on (EC.1d); then $\min_{i':(i',j)\in E}\{p_{i'}(t)\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ , then $b_j(t)\Big(1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))\Big)=b_j(t)\Big(1-F_{b_j}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)}))\Big)=q_j^b(t)$.

(2f) We consider two cases: When $q_j^b(t)=0$, then $x_{ij}(t)=0$ for any $i:(i,j)\in E$. When $q_j^b(t)>0$, we show in proof of (2b) that $p_i(t)\geq\min_{i':(i',j)\in E}\{p_{i'}\}=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-r_j^b(t)$ for $x_{ij}(t)=0$.

Step 2: construct an instance. In each period, given $(\boldsymbol{q}^b(t),\boldsymbol{q}^s(t),\boldsymbol{x}(t))$ that satisfies (2c)-(2e), consider the following one-period problem:

$$
\begin{aligned}
\widetilde{R}_t=\max_{\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}}&\Big[\sum_{j\in\mathcal{B}}q_j^b+\sum_{i\in\mathcal{S}}q_i^s\Big] && \\
\text{s.t. } & q_j^b\leq q_j^b(t), && \forall j\in\mathcal{B} \quad \text{(EC.9a)}\\
& q_i^s\leq q_i^s(t), && \forall i\in\mathcal{S} \quad \text{(EC.9b)}\\
& \sum_{j':(i,j')\in E}x_{i,j'}=q_i^s, && \forall i\in\mathcal{S} \quad \text{(EC.9c)}\\
& q_j^b=\sum_{i':(i',j)\in E}x_{i',j}, && \forall j\in\mathcal{B} \quad \text{(EC.9d)}\\
& x_{ij}\geq 0, && \forall (i,j)\in E. \quad \text{(EC.9e)}
\end{aligned}
$$

Note that the feasible solution set is not empty, as $q_j^b=q_j^b(t)$ for any $j\in\mathcal{B}$, $q_i^s=q_i^s(t)$ for any $i\in\mathcal{S}$ and $x_{ij}=x_{ij}(t)$ for any $(i,j)\in E$ is a feasible solution. Since the constraints are all linear, the KKT conditions are necessary for the optimal solution in (EC.9). Let $(\omega_i^s(t),\omega_j^b(t),\pi_{ij}(t))$ be the Lagrange multipliers corresponding to the constraint in (EC.9c)-(EC.9e), then we can write down the KKT conditions corresponding to $\boldsymbol{x}$:

$$
\begin{aligned}
&\omega_i^s(t)-\omega_j^b(t)-\pi_{ij}(t)=0, && \forall (i,j)\in E, \quad \text{(EC.10a)}\\
&x_{ij}(t)\geq 0\perp \pi_{ij}(t)\geq 0, && \forall i\in\mathcal{S},\forall (i,j)\in E. \quad \text{(EC.10b)}
\end{aligned}
$$

Then we consider the commission and equilibrium price as follows:

$$
\begin{aligned}
&p_i(t)=\omega_i^s(t), && \forall i\in\mathcal{S}, \quad \text{(EC.11a)}\\
&r_i^s(t)=\omega_i^s(t)-F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right), && \forall i\in\mathcal{S}, \quad \text{(EC.11b)}
\end{aligned}
$$

$$r_j^b(t) = F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) - \omega_j^b(t), \qquad \forall j \in \mathcal{B}. \tag{EC.11c}$$

then the conditions (EC.1a)-(EC.1b) immediately follow. For (EC.1c),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(a)}{=} \omega_j^b(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (a) follows from (EC.10a) and (EC.10b) that $\pi_{ij}(t) = 0$ when $x_{ij}(t) \geq 0$.

For (EC.1d),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{ij}(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(c)}{\geq} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (b) follows from (EC.10a) and (c) follows from (EC.10b). In summary, (EC.1) holds for our construction in (EC.11). ■

**Proof of Proposition EC.2** We need to prove that the optimal solutions to (3) exist and that they achieve an objective value of $\mathcal{R}^* = \mathcal{R}$. We first show that $\mathcal{R}^* \leq \mathcal{R}$ in step 1, and construct a solution to (3) whose value equals to $\mathcal{R}$ in step 2, which implies that $\mathcal{R}^* = \mathcal{R}$ and the solution is optimal.

Step 1: Establish that $\mathcal{R}^* \leq \mathcal{R}$. We show that any feasible solution to (3) is feasible in Problem (EC.2) in Step 1.1, and we further show that it leads to a higher objective value in Problem (EC.2) in Step 1.2.

Step 1.1: Any feasible solution in (3) is feasible in (EC.2). To prove the claim, it is sufficient to verify the constraints (EC.2b)-(EC.2c), as other constraints immediately follow from the constraints in (3).

Based on (2a) and (2b), we have $q_i^s(t) = s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) \leq s_i(t)$ as $F_{s_i}(p_i(t) - r_i^s(t)) \in [0,1]$ and $q_j^b(t) = b_j(t)[1 - F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t))] \leq b_j(t)$ as $F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t)) \in [0,1]$. Therefore, the constraints (EC.2b)-(EC.2c) are satisfied.

Step 1.2: Any feasible solution in (3) results in a higher objective value in (EC.2). We first show that the optimal solution to Problem (3) satisfies the following:

$$\left(F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)\right) q_i^s(t) \leq \Big(p_i(t) - r_i^s(t)\Big) q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T\}, \tag{EC.12a}$$

$$\left(F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right)\right) q_j^b(t) \geq \left(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\right) q_j^b(t), \quad \forall j \in \mathcal{B}, t \in \{1, ..., T\}. \tag{EC.12b}$$

For (EC.12a), when $q_i^s(t)=0$, (EC.12a) immediately holds; when $q_i^s(t)>0$, (EC.12a) follows from (EC.7b) and (EC.7c) in the proof of Lemma EC.4. For (EC.12b), when $q_j^b(t)=0$, (EC.12b) immediately holds; when $q_j^b(t)>0$, (EC.12b) follows from (EC.8a) and (EC.8b) in the proof of Lemma EC.4.

Given (EC.12), the objective function in (3a) satisfies the following:

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t) + \sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t)\Big] \\
&\overset{(a)}{=} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} r_j^b(t)\sum_{i':(i',j)\in E} x_{i'j}(t) + \sum_{i\in\mathcal{S}} r_i^s(t)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\sum_{i':(i',j)\in E}\Big(p_{i'}(t)+r_j^b(t)\Big)x_{i'j}(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&\overset{(b)}{=} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)\sum_{i':(i',j)\in E} x_{i'j}(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)\sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&= \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t)\Big)q_j^b(t) - \sum_{i\in\mathcal{S}}\Big(p_i(t)-r_i^s(t)\Big)q_i^s(t)\Big] \\
&\overset{(c)}{\le} \sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\big(1-\frac{q_j^b(t)}{b_j(t)}\big)q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\big(\frac{q_i^s(t)}{s_i(t)}\big)q_i^s(t)\Big] = \mathcal{R},
\end{aligned}$$

where (a) follows from (2c)-(2d); (b) follows from (2f) that $x_{ij}=0$ for $i\notin \underset{i':(i',j)\in E}{argmin}\{p_i+r_i^s\}$; (c) follows from (EC.12).

Step 2: Establish that $\mathcal{R}^*=\mathcal{R}$. Given any feasible solution to (EC.2), we construct a feasible solution for (3) in Step 2.1, and we further obtain an objective value that equals $\mathcal{R}$ in Step 2.2.

Step 2.1: Construct a feasible solution for Problem (3).

In each period, given the solution for Problem (EC.2), we consider the construction from (EC.11) as in the proof of Lemma EC.1. We need to verify that all the constraints in (3) hold. Notice that we only need to verify that (2a) (2b) (2f) (3c) and (3d) hold, as other constraints exist in (EC.2) and automatically hold.

(2a) from the construction of $p_i(t)$ and $r_i^s(t)$, we can establish that

$$s_i(t)F_{s_i}(p_i(t)-r_i^s(t)) = s_i(t)F_{s_i}\Big(F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\Big) = q_i^s(t).$$

(2b) We consider the following two cases:

(i) if $q_j^b > 0$, we pick a $i'$ such that $(i', j) \in E$, then there are two further cases: (1) $x_{i'j} > 0$, then $p_{i'}(t) \overset{(a)}{=} \omega_{i'}^s(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{i'j}(t) \overset{(c)}{=} \omega_j^b(t)$, where (a) follows form the construction of $p_{i'}(t)$; (b) follows from (EC.10a); (c) follows from (EC.10b) for $x_{i'j} > 0$; (2) $x_{i'j} = 0$, then $p_{i'}(t) = \omega_{i'}^s(t) = \omega_j^b(t) + \pi_{i'j}(t) \overset{(d)}{\geq} \omega_j^b(t)$, where (d) follows from (EC.10b) for $x_{i'j} = 0$. In summary, $\min_{i':(i',j)\in E} \{p_{i'}(t)\} = \omega_j^b(t)$, then

$$b_j(t)[1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t))] = b_j(t)[1 - F_{b_j}(\omega_j^b(t) + r_j^b(t))] \overset{(e)}{=} b_j(t)[1 - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})] = q_j^b(t),$$

where (e) follows from the construction of $r_j^b(t)$;

(ii) if $q_j^b = 0$, we have $p_{i'}(t) = \omega_{i'}^s(t) = \omega_j^b(t) + \pi_{i'j}(t) \geq \omega_j^b(t)$, then $0 \overset{(f)}{\leq} b_j(t)[1 - F_{b_j}(\min\{p_i(t) + r_j^b(t)\})] \leq b_j(t)[1 - F_{b_j}(\omega_j^b(t) + r_j^b(t))] \overset{(g)}{=} b_j(t)[1 - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})] = q_j^b(t) = 0$,, where (f) follows from $F_{b_j}(\cdot) \leq 1$, (g) follows from the construction of $r_j^b(t)$. This implies that inequality must be tight. Therefore, (2b) holds.

(2f) We have verified in the proof of (2b) that for any $(i, j) \in E$, we have $p_i = \omega_j^b$ for $x_{ij} > 0$ and $p_i \geq \omega_j^b$ for $x_{ij} = 0$. Therefore, $x_{ij} = 0$ for $i \notin \arg\min_{i':(i',j)\in E} p_{i'}$.

(3c) We first prove (EC.2g) holds as equality by contradiction. Suppose that $s_i(t+1) < \mathcal{G}_i^s(s_i(t), q_i^s(t))$ in the optimal solution to (EC.2), then let $s_i'(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$, we can obtain higher objective value by replacing the $s_i(t+1)$ in the optimal solution with $s_i'(t+1)$ as (EC.2a) increases in $s_i(t+1)$; in addition, $s_i(t+2) \leq \mathcal{G}_i^s(s_i(t+1), q_i^s(t+1)) < \mathcal{G}_i^s(s_i'(t+1), q_i^s(t+1))$, which implies that the constraint in (EC.2g) still hold. This contradicts to our assumption that $s_i(t+1) < \mathcal{G}_i^s(s_i(t), q_i^s(t))$ is the optimal solution to (EC.2). Therefore, $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$ in the optimal solution to (EC.2), and (3c) immediately holds.

(3d) follows the same argument in (3c).

Step 2.2: Obtain a value that equals $\mathcal{R}$. We can deduce that

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t) q_j^b(t)\right] \\
&\overset{(a)}{=} \sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}} (\omega_i^s(t) - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})) q_i^s(t) + \sum_{j\in\mathcal{B}} (F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t)) q_j^b(t)\right] \\
&\overset{(b)}{=} \sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)}) q_i^s(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}\omega_i^s(t)\sum_{j':(i,j')\in E}x_{ij'}(t)-\sum_{j\in\mathcal{B}}\omega_j^b(t)\sum_{i':(i',j)\in E}x_{i'j}(t)\right]\\
&=\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]+\sum_{t=1}^{T}\left[\sum_{(i,j)\in E}\Big(\omega_i^s(t)-\omega_j^b(t)\Big)x_{ij}(t)\right]\\
&\overset{(c)}{=}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]=\mathcal{R},
\end{aligned}
$$

where (a) follows from the construction of $r_i^s(t)$ and $r_j^b(t)$, (b) follows from (EC.2d) and (EC.2e), (c) follows from (EC.10a) and (EC.10b) that when $x_{ij}>0$, $\omega_i^s=\omega_j^b$, while when $x_{ij}=0$, $\omega_i^s\geq\omega_j^b$. ■

## EC.2. Proof of Results in Section 4

We provide and prove some auxiliary results in Appendix EC.2.1 and prove the result in Section 4 in Appendix EC.2.2.

### EC.2.1. Auxiliary Results for Section 4

Given the definitions of the value functions $\widetilde{F}_{b_j}$ for any $j\in\mathcal{B}$ and $\widetilde{F}_{s_i}$ for any $i\in\mathcal{S}$ from Problem (5), we have the following lemma.

LEMMA EC.5. *$\widetilde{F}_{b_j}(q,b)$ is continuous at $(0,0)$ for $i\in\mathcal{S}$ and $\widetilde{F}_{s_i}(q,s)$ is continuous at $(0,0)$ for $j\in\mathcal{B}$.*

**Proof of Lemma EC.5.** We need to show that $\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b)=\widetilde{F}_{b_j}(0,0)=0$ and $\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s)=\widetilde{F}_{s_i}(0,0)=0$, which holds because

$$
\begin{aligned}
0 \;\leq\; \lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b) \;&=\; \lim_{(q,b)\downarrow(0,0)}F_{b_j}^{-1}\Big(1-\frac{q}{b}\Big)q \;\leq\; \overline{v}_{b_j}\times 0 \;=\; 0,\\
0 \;\leq\; \lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s) \;&=\; \lim_{(q,s)\downarrow(0,0)}F_{s_i}^{-1}\Big(\frac{q}{s}\Big)q \;\leq\; \overline{v}_{s_i}\times 0 \;=\; 0,
\end{aligned}
$$

where given Assumption 2, all of the inequalities above follow from $F_{b_j}^{-1}(x)\in[0,\overline{v}_{b_j}]$ for $x\in[0,1]$ where $\overline{v}_{b_j}<\infty$ and $F_{s_i}^{-1}(x)\in[0,\overline{v}_{s_i}]$ for $x\in[0,1]$ where $\overline{v}_{s_i}<\infty$. ■

We next develop an auxiliary result about the growth of populations. To simplify the notation, we let $\mathcal{N}:=\{1,...,|\mathcal{S}|,|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$, where the first $|\mathcal{S}|$ nodes represent the types from the seller side and the last $|\mathcal{B}|$ nodes represent the types from the buyer side. In addition, we use $n_i(t)$ and $q_i(t)$ to respectively denote the population and transaction quantity of type $i \in \mathcal{N}$ at time $t \in \{1,\ldots,T\}$. We define $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_i^s(\cdot,\cdot)$ for $i \in \{1,...,|\mathcal{S}|\}$ and $\mathcal{G}_i(\cdot,\cdot) := \mathcal{G}_{i-|\mathcal{S}|}^b(\cdot,\cdot)$ for $i \in \{|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$. In addition, we define $\mathcal{N}^+ := \{i \in \mathcal{N} : \overline{n}_i > 0\}$.

Recall that

$$m(t) = \min_{i \in \mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{EC.13}$$

Given the minimum population ratio $m(t)$ in (EC.13), we let $l(t)$ be the agent type with the lowest population ratio at time $t$ or "the lowest node at time $t$" for short:

$$l(t) := \arg\min_{i \in \mathcal{N}^+} \frac{n_i(t)}{\overline{n}_i}. \tag{EC.14}$$

If there is more than one $i$ such that $\frac{n_i(t)}{\overline{n}_i} = m(t)$, we can set $l(t)$ as any node with the minimum population ratio. After the population evolves in period $t$, it is worth noting that the lowest node can change. Let $\tau_0 := 0$ and $m(\tau_0)$ be a dummy agent type with the minimum ratio in period 0 with $m(\tau_0) \notin \mathcal{S} \cup \mathcal{B}$. Moreover, we let $X$ be the total number of times that the lowest node changes in Algorithm 1 for some $X \in \{1,\ldots,T\}$. we let $\tau_x := \min\{t : t > \tau_{x-1}, l(t) \neq l(\tau_{x-1})\}$ for $t \in \{1,\ldots,T\}$, in which $\tau_x$ is the $x^{th}$ time that the lowest node changes for $x \in \{1,\ldots,X\}$. For example, for $x \in \{0,1,...,X\}$, if node $i$ has the lowest ratio at time $\tau_x - 1$, then $n_{l(\tau_x-1)}(\tau_x)$ denotes the population ratio of the node $i$ at time $\tau_x$.

Given the lowest node $l(t) \in \mathcal{S} \cup \mathcal{B}$ we let

$$g_t(n) := \mathcal{G}_{l(t)}\left(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right), \tag{EC.15}$$

where $n \geq 0$. Then $g_t(n)$ is the transition equation for the lowest node in period $t$, as by the population transition specified in Algorithm 1 and the definition of $g_t(\cdot)$, we have that

$$n_{l(t)}(t+1) = \mathcal{G}_{l(t)}\left(n_{l(t)}(t), n_{l(t)}(t)\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right) = g_t(n_{l(t)}(t)). \tag{EC.16}$$

We have the following observation about function $g_t(\cdot)$.

Lemma EC.6. *$g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$. Moreover, its derivative satisfies $g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. Moreover, $g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$ and $g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$.*

**Proof of Lemma EC.6.** We divide the proof arguments into the following components.

Differentiability and monotonicity. From Assumption 1, we have that function $\mathcal{G}_i(n,q)$ is continuously differentiable, increasing and strictly concave in $n \geq 0$, which directly implies that $g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$.

$g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1,...,T\}$. By Algorithm 1, we have that $\overline{n}_{l(t)} > 0$. Since $g_t(n)$ is continuous in $n \in [0, \overline{n}_{l(t)}]$ and differentiable $(0, \overline{n}_{l(t)})$, by the mean value theorem, there exists a $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$ such that $g_t'(\widetilde{n}_{l(t)}) = \frac{g_t(\overline{n}_{l(t)}) - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(a)}{=} \frac{\overline{n}_{l(t)} - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(b)}{=} \frac{\overline{n}_{l(t)} - 0}{\overline{n}_{l(t)} - 0} = 1$, where (a) follows from Lemma 1(ii) and (b) follows from Assumption 1(i). Since $g_t(n)$ is strictly concave in $n \geq 0$, its derivative strictly decreases in $n \geq 0$, which implies that $g_t'(\overline{n}_{l(t)}) < 1$ given that $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$.

$g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$. we define that $y_t(n) := g_t(n) - n$, and it remains to show that $y_t(n) < 0$ for $n > \overline{n}_{l(t)}$. Since $y_t'(n_{l(t)}) = g_t'(n_{l(t)}) - 1 < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$ and $y_t(\overline{n}_{l(t)}) = 0$ based on Lemma 1(ii), $y_t(n_{l(t)}) < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$.

$g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$. It remains to show that $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. Note that $y_t(n)$ is concave in $n$. Since $y_t(0) = g_t(0) - 0 = 0$ and $y_t(\overline{n}_{l(t)}) = g_t(\overline{n}_{l(t)}) - \overline{n}_{l(t)} = 0$, we know $y_t((1-a) \times \overline{n}_{l(t)}) > a y_t(0) + (1-a) y_t(\overline{n}_{l(t)}) = 0 + 0 = 0$ for $a \in (0,1)$, therefore $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. ■

Lastly, we formally define the myopic policy and establish its tractability as a supporting result for our proof arguments for Section 4.

Definition EC.1. **(myopic policy)** For $t \in \{1,...,T\}$, given the current population $(\boldsymbol{s}^M(t), \boldsymbol{b}^M(t))$, the myopic policy solves the following optimization problem:

$$\mathcal{R}^{M*}(t) = \max_{\boldsymbol{r}(t),\boldsymbol{p}(t),\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t)} \sum_{i\in\mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t) q_j^b(t) \tag{EC.17a}$$

$$\text{s.t.} \quad (\boldsymbol{s^M}(t), \boldsymbol{b^M}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies (2)}, \forall t \in \{1,...,T\}. \tag{EC.17b}$$

To solve Problem (EC.17), we consider the following optimization problem:

$$\mathcal{R}^{M}(t) = \max_{\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{x}(t)} \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j^M(t)}\Big) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i^M(t)}\Big) q_i^s(t) \tag{EC.18a}$$

$$\text{s.t.} \quad q_i^s(t) \leq s_i^M(t), \quad \sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1,...,T\}, \tag{EC.18b}$$

$$q_j^b(t) \le b_j^M(t),\ q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B},\ t \in \{1,...,T\}, \tag{EC.18c}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E,\ t \in \{1,...,T\}. \tag{EC.18d}$$

Recalling the observations about Problem (EC.2), we can apply exactly the same arguments as in the proof of Proposition EC.2 to establish the following result about Problem (EC.18), whose proof will be neglected for avoiding repetition:

Corollary EC.1. *For any $t \in \{1,...,T\}$, Problem (EC.18) is a tight relaxation of Problem (EC.17), i.e., $\mathcal{R}^{M*}(t) = \mathcal{R}^M(t)$ and any optimal solution $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t))$ to Problem (EC.18) is also optimal to Problem (EC.17).*

### EC.2.2. Proof of Results for TRP

**Proof of Lemma 1.**

Show that AVG's optimal solution and objective value are finite. On the seller side, for any $i \in \mathcal{S}$, we first show that the optimal solution $(\overline{q}_i^s, \overline{s}_i)$ is finite for all $i \in \mathcal{S}$. We first show that $\overline{s}_i$ is finite. The constraint of AVG requires that $s_i \le \mathcal{G}_i^s(s_i, q_i^s) \le \mathcal{G}_i^s(s_i, s_i)$, which requires that $\mathcal{G}_i^s(s_i, s_i) - s_i \ge 0$. Given that $\lim_{x\to\infty}((\mathcal{G}_i^s)_1'(x,x) + (\mathcal{G}_i^s)_2'(x,x)) < 1$ and $\mathcal{G}_i^s(x,x)$ is continuously differentiable in $x \ge 0$ by Assumption 1, there exists a constant $a < 1$ and $\hat{s}_i > 0$ such that $(\mathcal{G}_i^s)_1'(\hat{s}_i, \hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i, \hat{s}_i) = a < 1$. Therefore, for any $s_i > \hat{s}_i$, the constraint requires that

$$\begin{aligned}\mathcal{G}_i^s(s_i, s_i) - s_i \le& \mathcal{G}_i^s(\hat{s}_i, \hat{s}_i) + (\mathcal{G}_i^s)_1'(\hat{s}_i, \hat{s}_i)(s_i - \hat{s}_i) + (\mathcal{G}_i^s)_2'(\hat{s}_i, \hat{s}_i)(s_i - \hat{s}_i) - s_i \\ =& \mathcal{G}_i^s(\hat{s}_i, \hat{s}_i) + a(s_i - \hat{s}_i) - s_i\end{aligned}$$

which indicates that for any $s_i > \max\{\hat{s}_i, \frac{\mathcal{G}_i^s(\hat{s}_i,\hat{s}_i) - a\hat{s}_i}{1-a}\}$, we have $\mathcal{G}_i^s(s_i, s_i) - s_i < 0$ and therefore is not feasible. Therefore, it is without loss of optimality to focus on the compact set $[0, \hat{s}_i]$ for the optimal solution $\overline{s}_i$. Since $q_i \le s_i$, this suggests that the optimal solution $\overline{q}_i^s \in [0, \hat{s}_i]$, which is also finite. The same arguments hold for the buyer side.

Show that optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists. For any $u \in [0,1]$, we have that $F_{s_i}^{-1}(u) \le \overline{v}_{s_i} < \infty$ for any $i \in \mathcal{S}$ and $F_{b_j}^{-1}(u) \le \overline{v}_{b_j} < \infty$ for all $j \in \mathcal{B}$. Therefore, the objective value of AVG is also finite. We have already shown that the feasible set of $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ is closed and bounded. The constraints in (5b)-(5c) also ensure that the feasible set of $\boldsymbol{x}$ is closed and bounded. In summary, the feasible set characterized by constraint (5b)-(5f) is compact. In addition, the feasible set is not empty, as solution $\boldsymbol{0}$ is feasible. Furthermore, the objective function in (5a) is continuous in this compact set based on Assumption 2(i). By the extreme value theorem, an optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists.

We proceed to prove the lemma.

(i). By the extreme value theorem, the optimal solution to (5) exists. Since the objective function is strictly concave and the feasible region is a convex set, the optimal solution to (5) is unique.

(ii). If there exists a $i \in \mathcal{S}$ such that $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$, then given that $\mathcal{G}_i^s(s_i, q_i^s)$ is continuous on $s_i$, we can always find a $\epsilon > 0$ small enough such that $\mathcal{G}_i^s(\overline{s}_i + \epsilon, \overline{q}_i^s) - (\overline{s}_i + \epsilon) > 0$. In addition, $\overline{s}_i + \epsilon > \overline{s}_i \geq \overline{q}_i^s$. By replacing $\overline{s}_i$ with $\overline{s}_i + \epsilon$, we obtain a higher objective value since the objective function strictly increases in $s_i$. Therefore, the assumption $\mathcal{G}_i^s(\overline{s}_i, \overline{q}_i^s) - \overline{s}_i > 0$ contradicts the optimality of $(\overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to Problem (5). The same proof arguments can be applied to the buyer side. ■

**Proof of Proposition 1.** By Proposition EC.2, $\mathcal{R}(T) = \mathcal{R}^*(T)$. So it suffices to show that there exists a constant $C_1$ such that $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$. To prove the result, we establish the following two claims.

Claim 1: $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq -C_1'$. We delay the proof to the proof of Theorem 1 that there exists a constant $C_1'$ and a policy $\pi$ such that $\mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$, which further implies that $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq \mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$ given that $\mathcal{R}(T) \geq \mathcal{R}^\pi(T)$.

Claim 2: $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$. Before proving the claim, we first consider the following optimization problem for any $T > 0$:

$$
\begin{aligned}
\widetilde{\mathcal{R}} = \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} \quad & \sum_{j\in\mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i\in\mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i) && \text{(EC.19a)} \\
\text{s.t.} \quad & q_i^s \leq s_i, \quad \forall i \in \mathcal{S}, && \text{(EC.19b)} \\
& q_j^b \leq b_j, \quad \forall j \in \mathcal{B}, && \text{(EC.19c)} \\
& \sum_{j:(i,j)\in E} x_{ij} = q_i^s, \quad \forall i \in \mathcal{S}, && \text{(EC.19d)} \\
& q_j^b = \sum_{i:(i,j)\in E} x_{ij}, \quad \forall j \in \mathcal{B}, && \text{(EC.19e)} \\
& x_{ij} \geq 0, \quad \forall (i,j) \in E, && \text{(EC.19f)} \\
& s_i \leq \mathcal{G}_i^s(s_i, q_i^s) + \frac{s_i(1)}{T}, \quad \forall i \in \mathcal{S}, && \text{(EC.19g)} \\
& b_j \leq \mathcal{G}_j^b(b_j, q_j^b) + \frac{b_j(1)}{T}, \quad \forall j \in \mathcal{B}. && \text{(EC.19h)}
\end{aligned}
$$

Note that the only difference between Problem (EC.19) and Problem (5) is the right-hand side of the constraints (EC.19g)-(EC.19h). Given that $s_i(1)>0$ for all $i\in\mathcal{S}$ and $b_j(1)>0$ for all $j\in\mathcal{B}$, Problem (EC.19) could be viewed as a relaxation of Problem (5). We first show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$ and then show that there exists a positive constant $C_1''$ such that $T\widetilde{\mathcal{R}}-T\overline{\mathcal{R}}\leq C_1''$ for any $T>0$. Consequently, we can have $\mathcal{R}(T)-T\overline{\mathcal{R}}\leq C_1''$ for any $T>0$.

Step 2.1: Show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$. For any optimal solution $(\boldsymbol{s}(t),\boldsymbol{b}(t),\boldsymbol{q}^{\boldsymbol{s}}(t),\boldsymbol{q}^{\boldsymbol{b}}(t),\boldsymbol{x}(t):t=1,\dots,T)$ to Problem (EC.2), we construct the following alternative solution vector $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}},\overline{\boldsymbol{q}}^{\boldsymbol{s}},\overline{\boldsymbol{q}}^{\boldsymbol{b}},\overline{\boldsymbol{x}})$ for Problem (EC.19):

$$\begin{aligned}
&\overline{s}_i=\frac{1}{T}\sum_{t=1}^{T}s_i(t)\text{ and }\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t), && \forall i\in\mathcal{S},\\
&\overline{b}_j=\frac{1}{T}\sum_{t=1}^{T}b_j(t)\text{ and }\overline{q}_j^b=\frac{1}{T}\sum_{t=1}^{T}q_j^b(t), && \forall j\in\mathcal{B},\\
&\overline{x}_{ij}=\frac{1}{T}\sum_{t=1}^{T}x_{ij}(t), && \forall (i,j)\in E
\end{aligned}$$

We establish the feasibility of $(\overline{\boldsymbol{s}},\overline{\boldsymbol{b}},\overline{\boldsymbol{q}}^{\boldsymbol{s}},\overline{\boldsymbol{q}}^{\boldsymbol{b}},\overline{\boldsymbol{x}})$ for Problem (EC.19) in Step 2.1.1 and then show that $T\widetilde{\mathcal{R}}\geq\mathcal{R}(T)$ in Step 2.1.2.

Step 2.1.1: Feasibility. First, from the constraints in Problem (EC.2), we can easily show (EC.19b) - (EC.19f) hold. In particular, $\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\overset{(a)}{\leq}\frac{1}{T}\sum_{t=1}^{T}s_i(t)=\overline{s}_i$. The same argument applies for $\overline{q}_j^b$ and $\overline{b}_j$ on the buyer side. For (EC.19d)-(EC.19e), $\overline{q}_i^s=\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\overset{(b)}{=}\frac{1}{T}\sum_{j':(i,j')\in E}\sum_{t=1}^{T}x_{ij'}(t)=\sum_{j':(i,j')\in E}\overline{x}_{ij}$. and $\overline{q}_j^b=\frac{1}{T}\sum_{t=1}^{T}q_j^b(t)\overset{(c)}{=}\frac{1}{T}\sum_{i':(i',j)\in E}\sum_{t=1}^{T}x_{i'j}(t)=\sum_{i':(i',j)\in E}\overline{x}_{ij}$. For (EC.19f), $\overline{x}_{ij}=\frac{1}{T}\sum_{t=1}^{T}x_{ij}(t)\overset{(e)}{\geq}0$.

For constraints in (EC.19g)-(EC.19h), we show that

$$\begin{aligned}
\overline{s}_i-\mathcal{G}_i^s(\overline{s}_i,\overline{q}_i^s)-\frac{s_i(1)}{T}&\overset{(a)}{=}\frac{1}{T}\sum_{t=1}^{T}s_i(t)-\mathcal{G}_i^s\Big(\frac{1}{T}\sum_{t=1}^{T}s_i(t),\frac{1}{T}\sum_{t=1}^{T}q_i^s(t)\Big)-\frac{s_i(1)}{T}\\
&\overset{(b)}{\leq}\frac{1}{T}\sum_{t=1}^{T}\Big[s_i(t)-\mathcal{G}_i^s(s_i(t),q_i^s(t))\Big]-\frac{s_i(1)}{T}\\
&=\frac{1}{T}\sum_{t=1}^{T-1}\Big[s_i(t+1)-\mathcal{G}_i^s(s_i(t),q_i^s(t))\Big]+\frac{1}{T}(s_i(1)-\mathcal{G}_i^s(s_i(T),q_i^s(T)))-\frac{s_i(1)}{T}\\
&\leq 0+\frac{1}{T}\Big(-\mathcal{G}_i^s(s_i(T),q_i^s(T))\Big)\ \leq\ 0,
\end{aligned}$$

where (a) follows from the construction of $\overline{s}_i$ and $\overline{q}_i^s$ at the beginning of Step 2.1; (b) follows the Assumption 1(ii) that $\mathcal{G}_i^s(\cdot)$ is concave. This proves that Constraint (EC.19g) holds. Following the same argument, we can show that Constraint (EC.19h) holds.

Step 2.1.2: $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. Given the construction of $\bar{s}_i$ and $\bar{b}_j$, we obtain that $\bar{s}_i > 0$ and $\bar{b}_j > 0$. Given the definitions of $\widetilde{F}_b(\bar{q}^b_j, \bar{b}_j)$ and $\widetilde{F}_s(\bar{q}^s_i, \bar{s}_i)$ in Problem (5), the objective value in (5a) is given by $\sum_{j\in\mathcal{B}} F^{-1}_{b_j}(1-\frac{\bar{q}^b_j}{\bar{b}_j})\bar{q}^b_j - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}(\frac{\bar{q}^s_i}{\bar{s}_i})\bar{q}^s_i$. This allows us to establish that

$$T\widetilde{\mathcal{R}} \overset{(a)}{=} T\Big[\sum_{j\in\mathcal{B}} F^{-1}_{b_j}\Big(1-\frac{\frac{1}{T}\sum_{t=1}^T q^b_j(t)}{\frac{1}{T}\sum_{t=1}^T b_j(t)}\Big)\frac{1}{T}\sum_{t=1}^T q^b_j(t) - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\Big(\frac{\frac{1}{T}\sum_{t=1}^T q^s_i(t)}{\frac{1}{T}\sum_{t=1}^T s_i(t)}\Big)\frac{1}{T}\sum_{t=1}^T q^s_i(t)\Big]$$
$$\overset{(b)}{\geq} T\times\frac{1}{T}\sum_{t=1}^T\Big[\sum_{j\in\mathcal{B}} F^{-1}_{b_j}\left(1-\frac{q^b_j(t)}{b_j(t)}\right)q^b_j(t) - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\left(\frac{q^s_i(t)}{s_i(t)}\right)q^s_i(t)\Big] = \mathcal{R}(T).$$

where (a) follows from the construction of $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}}, \bar{\boldsymbol{x}})$ in Step 2-1; (b) follows from the concavity of $F^{-1}_{b_j}(1-\frac{a}{b})a$ and $-F^{-1}_{s_i}(\frac{a}{b})a$ by Assumption 3.

Summarizing the arguments in these two steps, we have $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$.

Step 2.2: Show that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C''_1$ for some $C''_1 > 0$. Let $(\boldsymbol{\mu}^s, \boldsymbol{\mu}^b)$ be the dual optimal solution corresponding to the constraint $s_i \leq \mathcal{G}^s_i(s_i, q^s_i)$ and $b_j \leq \mathcal{G}^b_j(b_j, q^b_j)$ in Problem (5), then $\mu^s_i \geq 0$ for any $i\in\mathcal{S}$ and $\mu^b_j \geq 0$ for any $j\in\mathcal{B}$ according to duality theory. Note that the only difference between Problem (5) and Problem (EC.19) is the right-hand side of the constraints in (EC.19g)-(EC.19h). Therefore, based on (5.57) in Boyd et al. (2004), we can establish that

$$\widetilde{\mathcal{R}} \;\leq\; \overline{\mathcal{R}} + \sum_{i\in\mathcal{S}} \mu^s_i \times \frac{1}{T}s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j \times \frac{1}{T}b_j(1),$$

which further implies that

$$T(\widetilde{\mathcal{R}} - \overline{\mathcal{R}}) \;\leq\; T\Big(\sum_{i\in\mathcal{S}} \mu^s_i \times \frac{1}{T}s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j \times \frac{1}{T}b_j(1)\Big) \;=\; \sum_{i\in\mathcal{S}} \mu^s_i s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j b_j(1).$$

We let $C''_1 := \sum_{i\in\mathcal{S}} \mu^s_i s_i(1) + \sum_{j\in\mathcal{B}} \mu^b_j b_j(1)$, and obtain the desired result.

In summary, $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \;\leq\; C_1$, where $C_1 = \max\{|C'_1|, |C''_1|\}$. ■

**Proof of Theorem 1.** We divide the proof arguments for the first claim into the following steps: in Step 1, we show that the solution generated by the TRP is feasible to Problem (EC.2); in Step 2, we show when $w = 0$, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1)-1| \leq \gamma|m(t)-1|$ for any $t\in\{1,...,T-1\}$; in Step 3, we show that when $w = 0$, there exists a constant $C'_1$ such that $T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T) \leq C'_1$ for all $w$. Then, together with Step 2.1.2 and Step 2.2 of Proposition 1, we conclude that there exists a constant $C_2 := C'_1 + C''_1$ such that $\mathcal{L}^{TR}(T) = \mathcal{R}^*(T) - \mathcal{R}^{TR}(T) \leq T\tilde{\mathcal{R}} - \mathcal{R}^{TR}(T) = \left(T\tilde{\mathcal{R}} - T\overline{\mathcal{R}}\right) + \left(T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right) \leq C_2$. Finally, in Step 4, we generalize the result to the case when $w > 0$.

The claim $|m^w(t) - 1| \leq |m^0(t) - 1|$ directly follows from Step 4.1 and Step 4.2.1.

Step 1: Show that the solution generated by the TRP is feasible to Problem (EC.2).

(EC.2b)-(EC.2c). $q_i^s(t) \overset{(a)}{=} \overline{q}_i^s \hat{m}(t) \overset{(b)}{\leq} s_i(t)\frac{\overline{q}_i^s}{\overline{q}_i^s} = s_i(t)$, where (a) follows from Algorithm 1; (b) follows directly from the definition of $\hat{m}(t)$. The same argument follows for the buyer side.

(EC.2d)-(EC.2e). $q_i^s(t) = \overline{q}_i^s \hat{m}(t) \overset{(a)}{=} \sum_{j':(i,j')\in E} \overline{x}_{i,j'}\hat{m}(t) \overset{(b)}{=} \sum_{j':(i,j')\in E} x_{i,j'}(t)$, where (a) follows from (5b); (b) follows from Algorithm 1. The same argument follows for the buyer side.

(EC.2f) . $x_{i,j} = \overline{x}_{i,j}\hat{m}(t) \geq 0$ follows from (5d).

(EC.2g)-(EC.2h). Given $s_i(t+1) = \mathcal{G}_i^s(s_i(t), q_i^s(t))$, the inequality is a relaxation, which directly follows. A similar argument holds for the buyer side.

In summary, the solution generated by the TRP is feasible for Problem (EC.2).

Step 2: Show that when $w = 0$, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma |m(t) - 1|$ for $t \in \{1, ..., T-1\}$. Recall the definition of $l(t)$ and $g_t(n)$ in (EC.14) and (EC.15), respectively. We discuss three cases: (1) $m(1) > 1$, (2) $m(1) < 1$ and (3) $m(1) = 1$. In each case, we will first show that $m(t)$ gets closer to 1 as $t$ increases, and then we show that the convergence rate can be upper bounded by $\gamma < 1$.

Step 2 - Case 1: $m(1) > 1$.

Step 2 - Case 1 - Step 2.1: Show that $m(1) > m(2) > ... > m(T-1) > m(T) > 1$. To prove the claim of this case, we show that for any $t \in \{1, \ldots, T-1\}$, if $m(t) > 1$, then $m(t) > m(t+1) > 1$. Let $X > 0$ denote the number of times the agent type with the lowest ratio changes. We consider the following two cases for any $t \in \{1, \ldots, T\}$: (1) the lowest node does not change in the next period, i.e., $\tau_x \leq t \leq \tau_{x+1} - 2$ for $x \in \{0, ..., X-1\}$ ; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1} - 1$ for $x \in \{0, ..., X-1\}$.

(1) For any $\tau_x \leq t \leq \tau_{x+1} - 2$ with $x \in \{0, ..., X-1\}$, we show that if $m(t) > 1$, then $m(t) > m(t+1) > 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds given that $l(t) = l(t+1)$ for $\tau_x \leq t \leq \tau_{x+1} - 2$ and $x \in \{0, \ldots, X-1\}$. Then, to show that $m(t) > m(t+1) > 1$, it is equivalent to establish that $n_{l(t)}(t) > n_{l(t)}(t+1) > \overline{n}_{l(t)}$. First, we have

$$n_{l(t)}(t+1) - n_{l(t)}(t) \overset{(b)}{=} g_t(n_{l(t)}(t)) - n_{l(t)}(t) \overset{(c)}{<} 0,$$

where (b) follows from (EC.16); (c) follows directly from Lemma EC.6. Second, we deduce that

$$n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(d)}{=} g_t(n_{l(t)}(t))-\overline{n}_{l(t)} \overset{(e)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)}) \overset{(f)}{>} 0,$$

where (d) follows from (EC.16); (e) follows from Lemma 1(ii); (f) follows from $n_{l(t)}(t) > \overline{n}_{l(t)}$ given that $m(t)=\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}>1$ and that $g_t(n)$ increases in $n\geq 0$ from Lemma EC.6.

In summary, for $\tau_x \leq t \leq \tau_{x+1}-2$, if $m(t)>1$, then $m(t)>m(t+1)>1$.

(2) For $t=\tau_x-1$ with $x\in\{1,\dots,X\}$, we want to show that if $m(\tau_x-1)>1$, then $m(\tau_x-1)>m(\tau_x)>1$. To prove this, we can deduce that

$$m(\tau_x) = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(a)}{\leq} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(b)}{<} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x-1),$$

where (a) follows directly from the definition that $l(\tau_x)$ in (EC.14); (b) follows from $n_{l(\tau_x-1)}(\tau_x)=g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))<n_{l(\tau_x-1)}(\tau_x-1)$, where the second inequality follows from $n_{l(\tau_x-1)}(\tau_x-1)>\overline{n}_{l(\tau_x-1)}$ given that $m(\tau_x-1)=\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}>1$ and Lemma EC.6. Therefore, $m(\tau_x)<m(\tau_x-1)$.

Next, we show that $m(\tau_x)>1$. Since

$$\begin{aligned} m(\tau_x) &= \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(c)}{=} \frac{\mathcal{G}_{l(\tau_x)}\left(n_{l(\tau_x)}(\tau_x-1), \overline{q}_{l(\tau_x)}\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\right)}{\overline{n}_{l(\tau_x)}} \\ &\overset{(d)}{\geq} \frac{\mathcal{G}_{l(\tau_x)}(n_{l(\tau_x)}(\tau_x-1),\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(e)}{>} \frac{\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)},\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} = 1, \end{aligned}$$

where (c) follows from Algorithm 1; (d) follows from the condition that $\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x-1)>1$ and $\mathcal{G}_{l(\tau_x)}(n,q)$ increases in $q\geq 0$; (e) follows from $\frac{n_{l(\tau_x)}(\tau_x-1))}{\overline{n}_{l(\tau_x)}} \geq m(\tau_x-1)>1$. Therefore, $m(\tau_x)>1$.

Based on the arguments above, if $m(t)>1$, then $m(t)>m(t+1)>1$, which holds for any $t\in\{1,...,T-1\}$. Thus, we can conclude that if $m(1)>1$, then $m(1)>m(2)>...>m(T-1)>m(T)>1$.

<u>Step 2 - Case 1 - Step 2.2: Show that there exists a constant $\gamma_1\in(0,1)$ such that $|m(t+1)-1|\leq\gamma_1|m(t)-1|$ for any $t\in\{1,\dots,T\}$.</u> Again, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x\leq t\leq\tau_{x+1}-2$ for any $x\in\{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t=\tau_{x+1}-1$ for any $x\in\{0,...,X-1\}$. For both cases, we first show that $\left|m(t+1)-1\right|\leq g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$. Then we show that there exists a $\gamma_1\in(0,1)$ independent from $T$ such that for any positive integer $T$, $\max_{t=1,...,T} g_t'(\overline{n}_{l(t)})\leq\gamma_1<1$.

(1) For $\tau_x \le t \le \tau_{x+1}-2$, we observe that

$$\left|n_{l(t)}(t+1)-\overline{n}_{l(t)}\right| \overset{(a)}{=} n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(b)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)})$$
$$\overset{(c)}{<} (n_{l(t)}(t)-\overline{n}_{l(t)})g_t'(\overline{n}_{l(t)}) \overset{(d)}{=} \left|n_{l(t)}(t)-\overline{n}_{l(t)}\right|g_t'(\overline{n}_{l(t)}),$$

where (a) follows from $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} \ge m(t+1)>1$ for any $t\in\{1,...,T-1\}$; (b) follows from (EC.16) and Lemma 1(ii); (c) follows from Lemma EC.6 given that $g_t(n)$ is strictly concave in $n\ge 0$; (d) follows from $m(t)=\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}>1$ for any $t\in\{1,...,T\}$. Therefore, $\left|m(t+1)-1\right| = \left|\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}-1\right| < g_t'(\overline{n}_{l(t)})\left|\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}-1\right| = g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$.

(2) For $t=\tau_x-1$,

$$\left|m(\tau_x)-1\right| \overset{(a)}{=} m(\tau_x)-1 = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}-1 \overset{(b)}{\le} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}-1$$
$$\overset{(c)}{=} \frac{g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))-g_{\tau_x-1}(\overline{n}_{l(\tau_x-1)})}{\overline{n}_{l(\tau_x-1)}} \overset{(d)}{<} \Big(\frac{n_{l(\tau_x-1)}(\tau_x-1)-\overline{n}_{l(\tau_x-1)}}{\overline{n}_{l(\tau_x-1)}}\Big)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)})$$
$$= (m(\tau_x-1)-1)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}) \overset{(e)}{=} \left|m(\tau_x-1)-1\right|g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}),$$

where (a) follows from $m(t)\ge 1$ for any $t\in\{1,...,T\}$; (b) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \le \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$; (c) follows from $g_t(\cdot)$ in (EC.15) and Lemma 1(ii); (d) follows from the strict concavity of $g_t(\cdot)$ in Lemma EC.6; (e) follows from $m(\tau_x-1)=\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} > 1$.

In summary, $\left|m(t+1)-1\right| \le g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$ for any $t\in\{1,\dots,T\}$. Define $\gamma_1 := \max\limits_{i\in\mathcal{N}^+}\frac{\partial\mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})$, then

$$\max_{t=1,...,T} g_t'(\overline{n}_{l(t)}) \overset{(a)}{=} \max_{t=1,...,T}\frac{\partial\mathcal{G}_{l(t)}}{\partial n}(n, n\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}})|_{n=\overline{n}_{l(t)}} \le \max_{i\in\mathcal{N}^+}\frac{\partial\mathcal{G}_i}{\partial n}(n, n\frac{\overline{q}_i}{\overline{n}_i})|_{n=\overline{n}_i} = \gamma_1 \overset{(b)}{<} 1,$$

where (a) follows from the definition of $g_t(\cdot)$ in (EC.15) and (b) follows from the finite network $G(\mathcal{S}\cup\mathcal{B},E)$ and discussion in Lemma EC.6. This allows us to conclude the contraction arguments for the case of $m(1)>1$.

Step 2 - Case 2: $m(1)<1$.

Step 2 - Case 2 - Step 2.1: Show that $m(1)<m(2)<...<m(T-1)<m(T)<1$. Similar to the discussions in Step 2 - Case 1, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x \le t\le \tau_{x+1}-2$ for any $x\in\{0,...,X-1\}$ ; (2) the lowest node changes in next step, i.e., $t=\tau_{x+1}-1$ for any $x\in\{0,...,X-1\}$.

(1) For $\tau_x \le t \le \tau_{x+1} - 2$, we want to show that if $m(t) < 1$, then $m(t) < m(t+1) < 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds as $l(t) = l(t+1)$ for $\tau_x \le t \le \tau_{x+1} - 2$. Therefore, $m(t) < 1$ implies that $n_{l(t)}(t) < \overline{n}_{l(t)}$. We observe that $m(t) < m(t+1) < 1$ is then equivalent to $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$, which holds because

$$n_{l(t)}(t+1) - n_{l(t)}(t) \;=\; g_t(n_{l(t)}(t)) - n_{l(t)}(t) \;>\; 0,$$

where the equality follows from (EC.16) and the inequality follows from the condition that $0 < n_{l(t)}(t) < \overline{n}_{l(t)}$ and Lemma EC.6. In addition,

$$n_{l(t)}(t+1) - \overline{n}_{l(t)} = g_t(n_{l(t)}(t)) - g_t(\overline{n}_{l(t)}) < 0,$$

given that $n_{l(t)}(t) < \overline{n}_{l(t)}$ and that $g_t(n)$ increases in $n \ge 0$ based on Lemma EC.6. The derivations above allow us to establish that $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$.

(2) For $t = \tau_x - 1$, we show that $m(\tau_x - 1) < m(\tau_x) < 1$ if $m(\tau_x - 1) < 1$, then

$$\begin{aligned} m(\tau_x) &\overset{(a)}{=} \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = \frac{\mathcal{G}_{l(\tau_x)}\big(n_{l(\tau_x)}(\tau_x - 1), \overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\big)}{\overline{n}_{l(\tau_x)}} \overset{(b)}{\ge} \frac{\mathcal{G}_{l(\tau_x)}\big(\overline{n}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}, \overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\big)}{\overline{n}_{l(\tau_x)}} \\ &\overset{(c)}{>} \frac{\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} \mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(d)}{=} \frac{n_{l(\tau_x-1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x - 1), \end{aligned}$$

where (a) follows the definition of $m(\tau_x)$ in (EC.13) and $l(\tau_x)$ in (EC.14); (b) follows from $\frac{n_{l(\tau_x)}(\tau_x-1)}{\overline{n}_{l(\tau_x)}} \ge m(\tau_x - 1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x - 1)$ in (EC.13); (c) follows from

$$\mathcal{G}_i(a\overline{n}_i, a\overline{q}_i) = \mathcal{G}_i(a\overline{n}_i + (1-a)0, a\overline{q}_i + (1-a)0) > a\mathcal{G}_i(\overline{n}_i, \overline{q}_i) + (1-a)\mathcal{G}_i(0,0) = a\mathcal{G}_i(\overline{n}_i, \overline{q}_i), \tag{EC.20}$$

for $0 < a < 1$ given that $\mathcal{G}_i(0,0) = 0$ and $\mathcal{G}_i(n_i, q_i)$ is strictly concave in $(n_i, q_i)$; in addition, (d) follows from $\mathcal{G}_{l(\tau_x)}(\overline{n}_{l(\tau_x)}, \overline{q}_{l(\tau_x)}) = \overline{n}_{l(\tau_x)}$ . In summary, we have $m(\tau_x) > m(\tau_x - 1)$.

To proceed, we further observe that

$$m(\tau_x) \;=\; \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(d)}{\le} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(e)}{<} 1,$$

where (d) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \le \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$ given the definition of $m(\tau_x)$ in (EC.13); (e) follows from Lemma EC.6 that $n_{l(\tau_x-1)}(\tau_x) = g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x - 1)) < \overline{n}_{l(\tau_x-1)}$ for $n_{l(\tau_x-1)}(\tau_x - 1) < \overline{n}_{l(\tau_x-1)}$. Thus, we have that $m(\tau_x) < 1$.

In summary, $m(t) < m(t+1) < 1$ if $m(t) < 1$ for any $t \in \{1, ..., T-1\}$. Since $m(t) < 1$, we obtain that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$.

Step 2 - Case 2 - Step 2.2: Show that there exists a constant $\gamma_2 \in (0,1)$ such that $|m(t+1)-1| \leq \gamma_2 |m(t)-1|$ for any $t \in \{1, \ldots, T\}$. Following a similar argument in the previous step, we can obtain the desired results.

Step 2 - Case 3: $m(1) = 1$. When $m(1) = 1$, we want to show that $m(t) = 1$ for any $t \in \{1, ..., T\}$. To establish the claim, we show that inductively, if $m(t) = 1$ then $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$. We observe that

$$n_{l(t)}(t+1) \overset{(a)}{=} \mathcal{G}_{l(t)}(n_{l(t)}(t), \overline{q}_{l(t)} m(t)) \overset{(b)}{=} \mathcal{G}_{l(t)}(\overline{n}_{l(t)}, \overline{q}_{l(t)}) \overset{(c)}{=} \overline{n}_{l(t)},$$

where (a) follows from the population transition induced by Algorithm 1; (b) holds given that $m(t) = 1$, which further implies that $n_{l(t)}(t) = \overline{n}_{l(t)}$; (c) follows from Lemma 1(ii). Thus, $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} = 1$. In addition, for $i \in \mathcal{N}^+$ with $i \neq l(t)$, we can deduce that

$$n_i(t+1) \;=\; \mathcal{G}_i(n_i(t), \overline{q}_i m(t)) \;\overset{(d)}{\geq}\; \mathcal{G}_i(\overline{n}_i, \overline{q}_i) \;=\; \overline{n}_i,$$

where (d) follows from $\frac{n_i(t)}{\overline{n}_i} \geq m(t) = 1$ given the definition of $m(t)$ in (EC.13) and the condition that $i \neq l(t)$. The observation above implies that $\frac{n_i(t+1)}{\overline{n}_i} \geq 1$ for $i \in \mathcal{N}^+$ with $i \neq l(t)$. Therefore, we can establish that

$$m(t+1) \;=\; \min\left\{\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}, \min_{\substack{i \in \mathcal{N}^+, \\ i \neq l(t)}} \left\{\frac{n_i(t+1)}{\overline{n}_i}\right\}\right\} \;=\; 1.$$

Given that $m(1) = 1$, by inductively establishing that $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$, we have that $m(t) = 1$ for any $t \in \{1, ..., T\}$. Thus, we obtain that $\big|m(t+1) - 1\big| = 0 \leq \gamma_3 \big|m(t) - 1\big| = 0$ for any $\gamma_3 \in (0,1)$.

In summary of the three cases above for $m(t) < 1$, $m(t) > 1$ and $m(t) = 1$, by letting $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$, We have that for some $\gamma \in (0,1)$, $\big|m(t+1) - 1\big| \leq \gamma \big|m(t) - 1\big|$, for any $t = \{1, ..., T-1\}$.

Next, we use the superscript $w$ to denote the value under policy with parameter $w$.

Step 3: Show that when $w = 0$. there exists a constant $C_1'$ such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$. We prove this by the following steps. Given $\boldsymbol{q}(t)$ and $\boldsymbol{n}(t)$ induced by TRP, we show in Step 3.1 that there exists a positive constant $C_{q_i}$ such that $\lim_{T\to\infty} \sum_{t=1}^{T} |q_i(t) - \overline{q}_i| \leq C_{q_i}$; In Step 3.2, we show that the previous two steps induce a positive constant $C_{\frac{q_i}{n_i}}$ that satisfies $\lim_{T\to\infty} \sum_{t=1}^{T} |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$; In Step 3.3, based on Steps 4.1 - 4.2, we conclude that there exists a constant $C_1'$ such that $\left|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right| \leq C_1'$.

Step 3.1: Show that there exists constants $C_{q_i}$ such that $\lim_{T\to\infty} \sum_{t=1}^{T} |q_i(t) - \overline{q}_i| < C_{q_i}$ for any $i \in \mathcal{N}^+$. Notice that

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \left|q_i(t) - \overline{q}_i\right| &\overset{(a)}{=} \lim_{T\to\infty} \sum_{t=1}^{T} \overline{q}_i \left|m(t) - 1\right| \overset{(b)}{\leq} \lim_{T\to\infty} \sum_{t=1}^{T} \overline{q}_i \left|m(1) - 1\right| \gamma^{t-1} \\
&= \lim_{T\to\infty} \overline{q}_i \left|m(1) - 1\right| \frac{1-\gamma^T}{1-\gamma} \overset{(c)}{=} \frac{1}{1-\gamma} \overline{q}_i \left|m(1) - 1\right|,
\end{aligned}$$

where (a) follows from $q_i(t) = \overline{q}_i m(t)$ in Algorithm 1; (b) follows from the contraction arguments in Step 2; (c) follows from $\gamma < 1$ in Step 2. Let $C_{q_i} = \frac{\overline{q}_i |m(1)-1|}{1-\gamma}$, and then the result follows.

Before proceeding, we provide some supporting results whose proofs will be provided towards the end of this section:

Lemma EC.7. *For any $i \in \mathcal{N}^+$ with $n_i(1) \geq \overline{n}_i$, there exists a positive constant $C_{n_i}$ such that $\lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| < C_{n_i}$. Moreover, for any $i \in \mathcal{N}^+$ with $n_i(1) < \overline{n}_i$, if $m(1) < 1$, then $n_i(t) < \overline{n}_i$ for $t \in \{1, ..., T\}$.*

Step 3.2: Show that there exists positive constants $C_{\frac{q_i}{n_i}}$ such that $\lim_{T\to\infty} \sum_{t=1}^{T} |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$. To show the claim for this step, we notice that for any $i \in \mathcal{N}_+$,

$$\left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \overset{(a)}{=} \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{\overline{q}_i m(t)}{n_i(t)}\right| = \frac{\overline{q}_i}{\overline{n}_i}\left|1 - \frac{\overline{n}_i m(t)}{n_i(t)}\right| \overset{(b)}{\leq} \frac{\overline{q}_i}{\overline{n}_i}\left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)}\left|1 - m(t)\right|\right),$$

where (a) follows from the population transition induced by Algorithm 1, and (b) follows directly from the triangle inequality. Therefore,

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| &\leq \lim_{T\to\infty} \sum_{t=1}^{T} \frac{\overline{q}_i}{\overline{n}_i}\left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)}\left|1 - m(t)\right|\right) \\
&= \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T} \frac{\overline{n}_i}{n_i(t)}\left|1 - \frac{n_i(t)}{\overline{n}_i}\right| + \sum_{t=1}^{T} \frac{\overline{n}_i}{n_i(t)}\left|1 - m(t)\right|\right) \\
&\overset{(c)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T} \frac{1}{m(t)}\left|1 - \frac{n_i(t)}{\overline{n}_i}\right| + \sum_{t=1}^{T} \frac{1}{m(t)}\left|1 - m(t)\right|\right), \quad (*)
\end{aligned} \tag{EC.21}$$

where (c) follow from the definition of $m(t)$ in (EC.13).

Notice that if $m(1)=\min_{i\in\mathcal{N}^+}\frac{n_i(1)}{\overline{n}_i}\geq 1$, then $n(1)\geq\overline{n}_i$ for any $i\in\mathcal{N}^+$. Thus, it is without loss of generality to consider the following three cases for any $i\in\mathcal{N}^+$ to further relax the term in the RHS of (EC.21), which we denote by "(*)".

(1) When $n_i(1)\geq\overline{n}_i$ and $m(1)\geq 1$, we show that

$$(*)\overset{(d)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\left|1-\frac{n_i(t)}{\overline{n}_i}\right|+\sum_{t=1}^{T}\left|1-m(t)\right|\right)\overset{(e)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1}\right)$$
$$=\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i}+\left|1-m(1)\right|\frac{1}{1-\gamma}\right),$$

where (d) follows from the result in Step 2 - Case 1- Step 2.1 and Step 2 - Case 3 that if $m(1)>1$, then $m(1)\geq m(2)\geq\ldots\geq m(T)\geq 1$; (e) follows from Lemma EC.7 that $\lim_{T\to\infty}\sum_{t=1}^{T}|n_i(t)-\overline{n}_i|\leq C_{n_i}$ given that $n_i(1)\geq\overline{n}_i$, and we also have $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,\ldots,T\}$ by Step 2. Therefore, by letting $C_{\frac{q_i}{n_i}}:=\frac{\overline{q}_i}{\overline{n}_i}\big(\frac{C_{n_i}}{\overline{n}_i}+|1-m(1)|\frac{1}{1-\gamma}\big)$, we obtain the desired result.

(2) When $n_i(1)<\overline{n}_i$ and $m(1)<1$, we show that

$$(*)\overset{(f)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\left|1-m(t)\right|+\sum_{t=1}^{T}\frac{1}{m(t)}\left|1-m(t)\right|\right)$$
$$\overset{(g)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1}+\frac{1}{m(1)}\sum_{t=1}^{T}\left|1-m(1)\right|\gamma^{t-1}\right)\leq\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\right),$$

where (f) follows from the observation that $m(t)\leq\frac{n_i(t)}{\overline{n}_i}<1$, where the first inequality follows from the definition of $m(t)$ in (EC.13) and the second inequality follows from Lemma EC.7 that if $n_i(1)<\overline{n}_i$ and $m(1)<1$, then $n_i(t)<\overline{n}_i$ for $t\in\{1,\ldots,T\}$; (g) follows from the observation that $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,\ldots,T\}$ by Step 2, and therefore $|m(t)-1|\leq\gamma^{t-1}|m(1)-1|$; in addition, we show in Step 2 - Case 2- Step 2.1 that when $m(1)<1$, we have $m(1)\leq m(t)$ for any $t\in\{1,\ldots,T\}$. Therefore, we can let $C_{\frac{q_i}{n_i}}:=\frac{\overline{q}_i}{\overline{n}_i}\big(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\big)$, and then obtain the desired result.

(3) When $n_i(1)\geq\overline{n}_i$ and $m(1)<1$, we show that

$$(*)\overset{(h)}{<}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\frac{1}{m(1)}\left|1-m(t)\right|\right)$$
$$\overset{(i)}{\leq}\lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\sum_{t=1}^{T}\frac{1}{m(1)}\left|1-m(1)\right|\gamma^{t-1}\right)\overset{(j)}{=}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+\left|\frac{1}{m(1)}-1\right|\frac{1}{1-\gamma}\right),$$

where (h) follows from the observation in Step 2 -Case 2- Step 2.1 that $m(1) < m(2) < ... < m(T) < 1$ when $m(1) < 1$ and the result in Lemma EC.7 that $\lim_{T\to\infty} \sum_{t=1}^T |n_i(t) - \overline{n}_i| \le C_{n_i}$ when $n_i(1) \ge \overline{n}_i$; (i) follows from the results in Step 2 that $|m(t+1)-1| \le \gamma |m(t)-1|$; (j) follows from the observation in Step 2 that $\gamma < 1$. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\big(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + |\frac{1}{m(1)} - 1|\frac{1}{1-\gamma}\big)$, we can establish the desired result.

In summary, we have that for any $i \in \mathcal{N}^+$, there exists a positive constant $C_{\frac{q_i}{n_i}}$ such that

$$\lim_{T\to\infty} \sum_{t=1}^T \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \;\le\; C_{\frac{q_i}{n_i}}.$$

Step 3.3: Show that there exists a constant $C_1'$ such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \le C_1'$. Note that for $j \in \mathcal{B}$ with $\overline{b}_j = 0$, we have $\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) = 0$ based on the definition of $\widetilde{F}_{b_j}$ before the formulation of (5). Since $\overline{q}_j^b \le \overline{b}_j = 0$, we have $q_j^b(t) = \overline{q}_j^b m(t) = 0$ induced by Algorithm 1, which further implies that $F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t) = 0$. Therefore,

$$\lim_{T\to\infty} \sum_{t=1}^T \sum_{j\in\mathcal{B}:\overline{b}_j=0} \left(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t)\right) \;=\; 0.$$

Similarly, we can establish that for any $i \in \mathcal{S}$ with $\overline{s}_i = 0$, we have that $\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) = 0$, which further implies that $q_i^s(t) = \overline{q}_i^s m(t) = 0$. Thus, we have that

$$\lim_{T\to\infty} \quad \sum_{t=1}^T \sum_{i\in\mathcal{S}:\overline{s}_i=0} \left(\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right) \;=\; 0.$$

Based on the two observations above, with $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{s}(t), \boldsymbol{b}(t) : t = 1, \ldots T)$ induced by the TRP, we can deduce that

$$\begin{aligned}
&\lim_{T\to\infty} \quad \left|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right| \\
=& \lim_{T\to\infty} \sum_{t=1}^T \left[\sum_{j\in\mathcal{B}} \left(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\right) - \sum_{i\in\mathcal{S}} \left(\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\right)\right] \\
=& \lim_{T\to\infty} \sum_{t=1}^T \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(F_{b_j}^{-1}\Big(1 - \frac{\overline{q}_j^b}{\overline{b}_j}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\right) - \sum_{i\in\mathcal{S}:\overline{s}_i>0} \left(F_{s_i}^{-1}\Big(\frac{\overline{q}_i^s}{\overline{s}_i}\Big)\overline{q}_i^s - F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right)\right] \\
\overset{(a)}{\le}& \lim_{T\to\infty} \sum_{t=1}^T \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(\left|F_{b_j}^{-1}\Big(1 - \frac{\overline{q}_j^b}{\overline{b}_j}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)\overline{q}_j^b\right| + \left|F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)\overline{q}_j^b - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\right|\right)\right. \\
&\left. + \sum_{i\in\mathcal{S}:\overline{s}_i>0} \left(\left|F_{s_i}^{-1}\Big(\frac{\overline{q}_i^s}{\overline{s}_i}\Big)\overline{q}_i^s - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\overline{q}_i^s\right| + \left|F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\overline{q}_i^s - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\right|\right)\right] \\
\overset{(b)}{\le}& \lim_{T\to\infty} \sum_{t=1}^T \left[\sum_{j\in\mathcal{B}:\overline{b}_j>0} \left(\overline{q}_j^b \frac{1}{d_j^b}\left|\frac{\overline{q}_j^b}{\overline{b}_j} - \frac{q_j^b(t)}{b_j(t)}\right| + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})\left|\overline{q}_j^b - q_j^b(t)\right|\right)\right.
\end{aligned}$$

$$+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}\Big|\frac{\overline{q}_i^s}{\overline{s}_i}-\frac{q_i^s(t)}{s_i(t)}\Big|+F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\Big|\overline{q}_i^s-q_i^s(t)\Big|\right)\Bigg]$$

$$\leq\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}_j^b\frac{1}{d_j^b}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}_j^b}{\overline{b}_j}-\frac{q_j^b(t)}{b_j(t)}\Big|+\max_t F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\overline{q}_j^b-q_j^b(t)\Big|\right)$$

$$+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}_i^s}{\overline{s}_i}-\frac{q_i^s(t)}{s_i(t)}\Big|+\max_t F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\overline{q}_i^s-q_i^s(t)\Big|\right)$$

$$\overset{(c)}{\leq}\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}_j^b\frac{1}{d_j^b}C_{q_j^b/b_j}+\overline{v}_j^bC_{q_j^b}\right)+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}_i^s\frac{1}{d_i^s}C_{q_i^s/s_i}+\overline{v}_i^sC_{q_i^s}\right):=C_1'.$$

where (a) follows from the triangle inequality; (b) follows from Assumption 2(ii) that the derivative of $F_{b_j}$ ($F_{s_i}$) is lower bounded by a positive constant $d_j^b$ ($d_i^s$), and therefore the derivative of $F_{b_j}^{-1}$ ($F_{s_i}^{-1}$) is upper bounded by $\frac{1}{d_j^b}$ ($\frac{1}{d_i^s}$), then $|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|\leq\frac{1}{d_j^b}|x_1-x_2|$ for any $x_1,x_2$ in the domain, otherwise $\frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|}>\frac{1}{d_j^b}$ implies that there exists a $x_3\in(x_1,x_2)$ such that $f'(x_3)=\frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|}>\frac{1}{d_j^b}$ by mean value theorem, which contradicts to the fact that the derivative of $F_{b_j}^{-1}$ is upper bounded by $\frac{1}{d_j^b}$; following the same argument, $|F_{s_i}^{-1}(x_1)-F_{s_i}^{-1}(x_2)|\leq\frac{1}{d_i^s}|x_1-x_2|$ for any $x_1,x_2$ in the domain. (c) follows from the results in Step 3.1- Step 3.2 that $\lim_{T\to\infty}\sum_{t=1}^T|q_i(t)-\overline{q}_i|<C_{q_i}$ and $\lim_{T\to\infty}\sum_{t=1}^T|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}|\leq C_{\frac{q_i}{n_i}}$ for any $i\in\mathcal{N}^+$; in addition, $F_{b_j}^{-1}\leq\overline{v}_{b_j}$ and $F_{s_i}^{-1}\leq\overline{v}_{s_i}$. Note that we have $\overline{v}_{b_j}<\infty$ for $j\in\mathcal{B}$ and $\overline{v}_{s_i}<\infty$ for $i\in\mathcal{S}$ and $\frac{1}{d_j^b}<\infty$ for $j\in\mathcal{B}$ and $\frac{1}{d_i^s}<\infty$ for $i\in\mathcal{S}$ given Assumption 2(ii).

Step 4: Show that when $0<w\leq1$. there exists a constant $C_2'$ such that $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\leq C_2'$. We consider the case with $m(1)\geq1$ and $m(1)<1$ respectively in Step 4.1 and Step 4.2.

Step 4.1. $m(1)\geq1$. We show that in this case, $Overexpansion=True$ from the beginning.

$$\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(1),\overline{q}_i\hat{m}(1))}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i\Big(n_i(1),\overline{q}_i((1-w)\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{n}_i}\}+w\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{q}_i}\})\Big)}{\overline{n}_i}\right\}$$

$$\overset{(a)}{>}\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i\Big(n_i(1),\overline{q}_i\min_{i\in\mathcal{N}^+}\{\frac{n_i(1)}{\overline{n}_i}\}\Big)}{\overline{n}_i}\right\}\overset{(b)}{\geq}\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(\overline{n}_i,\overline{q}_i)}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\overline{n}_i}{\overline{n}_i}\right\}=1,$$

where (a) follows from $0\leq\overline{q}_i\leq n_i$ and $0<w\leq1$; (b) follows from $m(1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i(1)}{\overline{n}_i}\right\}\geq1$. As a result, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(1),\overline{q}_i\hat{m}(1))}{\overline{n}_i}\right\}>1$, which means that $OverExpension=True$ from the beginning, and the update rule when $w>0$ the same as that when $w=0$. Therefore, $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\leq C_2'$ by Step 3.

Step 4.2. $0<m(1)<1$. We will show in Step 4.2.2 that $OverExpansion$ occurs within a finite period. After that, the policy with $w>0$ becomes identical to the policy with $w=0$.

Prior to this, we must show that the system converges faster under $w>0$ compared to $w=0$ in Step 4.2.1 to facilitate our later proof. We next use the superscript $w$ to denote the value under policy with parameter $w$.

Step 4.2.1. Show that if $0<m(1)<1$, then $0\le m^0(t)\le m^w(t)\le 1$ for all $w\in(0,1]$ and for $t=\{1,...,T\}$. We already know $0\le m^0(t)\le 1$ from Step 2-Case 2. We will respectively show that $m^0(t)\le m^w(t)$ and $m^w(t)\le 1$ for $t=\{1,...,T\}$.

Step 4.2.1(i). Show that $m^0(t)\le m^w(t)$ for $t=\{1,...,T\}$. Based on the definition that $m^w(t):=\min_{i\in\mathcal{N}}\{\frac{n_i^w(t)}{\overline{n}_i}\}$, it is sufficient to show that $n_i^0(t)\le n_i^w(t)$ for $t\in\{1,...,T\}$ and $i\in\mathcal{N}$. We show it by induction.

We already know that $n_i(1)$ is the same under different $w$ as they are exogenously given. We then show that if $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$, then $n_i^0(t+1)\le n_i^w(t+1)$ for any $i\in\mathcal{N}$. Since the update rule of TRP depends on the state of the system, we need to consider the following two cases:

(1). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}\le 1$ and $OverExpension=False$, then for any $i\in\mathcal{N}$,

$$n_i^0(t+1)\overset{(a)}{=}\mathcal{G}_i^s\Big(n_i^0(t),\overline{q}_i m^0(t)\Big)\overset{(b)}{\le}\mathcal{G}_i^s\Big(n_i^w(t),\overline{q}_i\hat{m}^w(t)\Big)\overset{(c)}{=}n_i^w(t+1).$$

where (a) and (c) follow from the construction of two policies, (b) follows from $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$ and $m^0(t)=\min_{i:\overline{n}_i>0}\left\{\frac{n_i^0(t)}{\overline{n}_i}\right\}\le(1-w)\min_{i:\overline{q}_i>0}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}+w\min_{i:\overline{q}_i>0}\left\{\frac{n_i^w(t)}{\overline{q}_i}\right\}=\hat{m}^w(t)$ as $\overline{q}_i\le\overline{n}_i$.

(2). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}>1$ or and $OverExpension=True$, then for any $i\in\mathcal{N}$,

$$n_i^0(t+1)=\mathcal{G}_i^s\Big(n_i^0(t),\overline{q}_i m^0(t)\Big)\le\mathcal{G}_i^s\Big(n_i^w(t),\overline{q}_i m^w(t)\Big)=n_i^w(t+1).$$

where the inequality follows from $n_i^0(t)\le n_i^w(t)$ for any $i\in\mathcal{N}$ and $m^0(t)=\min_{i:\overline{n}_i>0}\left\{\frac{n_i^0(t)}{\overline{n}_i}\right\}\le\min_{i:\overline{n}_i>0}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}=m^w(t)$.

Step 4.2.1(ii): Show that $m^w(t)\le 1$ for $t=\{1,...,T\}$. It is equivalent to show that $\min_{i\in\mathcal{N}^+}\left\{\frac{n_i^w(t)}{\overline{n}_i}\right\}<1$ for any $t\in\{1,...,T\}$. We show it by induction. We already know that $m(1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i(1)}{\overline{n}_i}\right\}<1$. Then we show that given $m(t)<1$, we have $m(t+1)<1$. Consider the following two cases:

(1). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\}\le 1$ and $OverExpension=False$, then $m^w(t+1)=\min_{i\in\mathcal{N}^+}\left\{\frac{n_i^w(t+1)}{\overline{n}_i}\right\}=\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i m^w(t))}{\overline{n}_i}\right\}\le 1$;

(2). If $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}(t))}{\overline{n}_i}\right\} > 1$ or $OverExpension = True$, then the update rule when $w > 0$ is the same as that when $w = 0$. We need to show that given $m(t) < 1$, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i m(t))}{\overline{n}_i}\right\} < 1$, which is already shown in Step 2-Case 2 of Theorem 1.

Step 4.2.2. Show that there exists a constant $\tilde{t}$ such that if $t > \tilde{t}$, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$.

We first show that there exists a constant $\tilde{m}$ such that if $m^w(t) > \tilde{m}$, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$. For all $i\in\mathcal{N}^+$, define $\tilde{m}_i = \{0 < m < 1 | \overline{n}_i = \mathcal{G}_i\Big(m\overline{n}_i, \overline{q}_i((1-w)m + wm\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}\Big)\}$. Since the RHS of the equation increases in $m$, and $\mathcal{G}_i(0,0) = 0 < \overline{n}_i$ and $\mathcal{G}_i\Big(\overline{n}_i, \overline{q}_i((1-w) + w\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big)\} > \mathcal{G}_i(\overline{n}_i, \overline{q}_i)\} = \overline{n}_i$, we know $\tilde{m}_i$ is well-defined.

If $m^w(t) > \tilde{m} := \max_{i\in\mathcal{N}^+}\tilde{m}_i$, then for all $i\in\mathcal{N}^+$,

$$\begin{aligned}
\mathcal{G}_i\Big(n_i(t), \overline{q}_i\hat{m}^w(t)\Big) &\overset{(a)}{=} \mathcal{G}_i\Big(n_i(t), \overline{q}_i((1-w)m^w(t) + w\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{q}_{i'}}\})\Big) \\
&\overset{(b)}{\geq} \mathcal{G}_i\Big(m^w(t)\overline{n}_i, \overline{q}_i((1-w)m^w(t) + wm^w(t)\min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big) \\
&\overset{(c)}{>} \mathcal{G}_i\Big(\tilde{m}_i\overline{n}_i, \overline{q}_i((1-w)\tilde{m}_i + w\tilde{m}_i\min_{i'}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\})\Big) \overset{(d)}{=} \overline{n}_i,
\end{aligned}$$

where (a) follows from the definition of $\hat{m}^w(t)$; (b) follows from $m^w(t) \leq \frac{n_i^w(t)}{\overline{n}_i}$ based on its definition, and $\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{q}_{i'}}\} > \min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}\min_{i'\in\mathcal{N}}\{\frac{n_{i'}^w(t)}{\overline{n}_{i'}}\} = \min_{i'\in\mathcal{N}}\{\frac{\overline{n}_{i'}}{\overline{q}_{i'}}\}m^w(t)$; (c) follows from $m^w(t) > \tilde{m} := \max_{i\in\mathcal{N}^+}\tilde{m}_i$; (d) follows from the definition of $\tilde{m}_i$. In conclusion, $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i^w(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$.

We then show that there exists a constant $\tilde{t}$ such that if $t > \tilde{t}$, we have $m^w(t) > \tilde{m}$. Define $\tilde{t} = \frac{\log(1-\tilde{m})/(1-m(1))}{\log\gamma} + 1$, then when $t > \tilde{t}$, we have

$$1 - m^w(t) \overset{(a)}{\leq} 1 - m^0(t) \overset{(b)}{\leq} \gamma^{t-1}(1 - m^0(1)) \overset{(c)}{<} \gamma^{\tilde{t}-1}(1 - m^0(1)) \overset{(d)}{=} 1 - \tilde{m},$$

where (a) follows from Step 4.2.1; (b) follows from Step 2; (c) follows from $t > \tilde{t}$ and $0 < \gamma < 1$; (d) follows from the definition of $\tilde{t}$. Therefore, $m^w(t) > \tilde{m}$ for $t > \tilde{t}$.

In summary of the above two claims, we have $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}^w(t))}{\overline{n}_i}\right\} > 1$ for $t > \tilde{t}$, which suggests that the system is in the state of overexpansion after a finite period.

Step 4.3: Conclude the case.

$$\begin{aligned}
&\lim_{T\to\infty} \quad \left|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right| \\
&= \sum_{t=1}^{\tilde{t}}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b, \overline{b}_j) - F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big) - \sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s, \overline{s}_i) - F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{T\to\infty}\sum_{t=\tilde{t}+1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big)-\sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)-F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]\\
\le\ &\tilde{t}\Big[\sum_{j\in\mathcal{B}}\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-\sum_{i\in\mathcal{S}}\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)\Big]\\
&+\lim_{T\to\infty}\sum_{t=\tilde{t}+1}^{T}\Big[\sum_{j\in\mathcal{B}}\Big(\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t)\Big)-\sum_{i\in\mathcal{S}}\Big(\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)-F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big)\Big]\\
\le\ &\tilde{t}\Big[\sum_{j\in\mathcal{B}}\widetilde{F}_{b_j}(\overline{q}_j^b,\overline{b}_j)-\sum_{i\in\mathcal{S}}\widetilde{F}_{s_i}(\overline{q}_i^s,\overline{s}_i)\Big]+C_1':=C_2',
\end{aligned}
$$

where the last inequality follows from Step 3. ■

Summarizing Step 1-4, we conclude the claim of this result.

**Proof of Lemma EC.7.** We prove the two claims of this result separately. Given that the supporting lemma is located in Step 3 in the proof of Theorem 1, we would borrow some observations from Step 2 in the proof of Theorem 1 in the proof arguments below.

Claim 1. For $i\in\mathcal{N}^+$, when $n_i(1)\ge\overline{n}_i$, we further consider the following two cases: (1) $m(1)\ge 1$; (2) $m(1)<1$.

(1) When $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we first show that $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$. Given that $n_i(1)\ge\overline{n}_i$ for any $i\in\mathcal{N}^+$, we assume for induction purpose that $n_i(t)\ge\overline{n}_i$, and then we can establish that

$$
n_i(t+1)\overset{(a)}{=}\mathcal{G}_i(n(t),\overline{q}_i m(t))\overset{(b)}{\ge}\mathcal{G}_i(n(t),\overline{q}_i)\ge\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\overset{(c)}{=}\overline{n}_i,
$$

where (a) follows from Algorithm 1; (b) follows from our observations in Step 2 Case 1 in the proof of Theorem 1 that if $m(1)>1$, then we have $m(1)>m(2)>...>m(T)>1$, and in Step 2 Case 3 that if $m(1)=1$, then we have $m(1)=m(2)=...=m(T)=1$; (c) follows directly from Lemma 1(ii). By induction, with $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we obtain that $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$.

To proceed, we further notice that for any $t\in\{1,...,T\}$,

$$
\begin{aligned}
n_i(t)-\overline{n}_i\overset{(d)}{=}&\mathcal{G}_i(n_i(t-1),\overline{q}_i m(t-1))-\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\\
=&\mathcal{G}_i(n_i(t-1),\overline{q}_i m(t-1))-\mathcal{G}_i(n_i(t-1),\overline{q}_i)+\mathcal{G}_i(n_i(t-1),\overline{q}_i)-\mathcal{G}_i(\overline{n}_i,\overline{q}_i)\\
\overset{(e)}{\le}&\overline{q}_i(m(t-1)-1)(\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i)+(\overline{n}_i(t-1)-\overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i),
\end{aligned}
$$

where (d) follows from Algorithm 1 and Lemma 1(ii); (e) follows from the concavity of $\mathcal{G}_i(\cdot,\cdot)$ by Assumption 1. Since $n_i(t) \geq \overline{n}_i$, the LHS of the inequality for (e) is nonnegative, and we can take the absolute values and obtain the following inequality:

$$\begin{aligned}\sum_{t=2}^{T}\Big|n_i(t)-\overline{n}_i\Big| &\leq \sum_{t=2}^{T}\Big[\Big|\overline{q}_i(m(t-1)-1)(\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i)\Big| + \Big|(\overline{n}_i(t-1)-\overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)\Big|\Big]\\ &\overset{(f)}{\leq} \sum_{t=2}^{T}\Big|\overline{q}_i(m(t-1)-1)\Big| + \sum_{t=2}^{T}\Big|(\overline{n}_i(t-1)-\overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)\Big|\\ &\leq \overline{q}_i\sum_{t=2}^{T}\gamma^{t-2}\Big|(m(1)-1)\Big| + \sum_{t=2}^{T}\Big|(\overline{n}_i(t-1)-\overline{n}_i)(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)\Big|.\end{aligned}$$

For (f), we show that $(\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i) < 1$. Define $y(n) := \mathcal{G}(n, n\frac{\overline{q}_i}{n_i(t-1)})$, by the mean value theorem, there must exist a $\hat{n} \in (0, n_i(t-1))$ such that $y'(\hat{n}) = \frac{y(n_i(t-1))-y(0)}{n_i(t-1)-0} = \frac{\mathcal{G}(n_i(t-1),\overline{q}_i)}{n_i(t-1)} < 1$ for $n_i(t-1) > \overline{n}_i$. Therefore, given the concavity of $y(n)$, $y'(n_i(t-1)) < 1$, which suggest that $(\mathcal{G}_i)_1'(n_i(t-1),\overline{q}_i) + (\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i)\frac{\overline{q}_i}{n_i(t-1)} < 1$, which suggest that $(\mathcal{G}_i)_2'(n_i(t-1),\overline{q}_i) < 1$. Then

$$\begin{aligned}\sum_{t=1}^{T}\Big|n_i(t)-\overline{n}_i\Big| &\leq \frac{\overline{q}_i\sum_{t=2}^{T}\gamma^{t-2}\Big|(m(1)-1)\Big|}{1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)} - \frac{(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)}{1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)} \times \Big|n_i(T)-\overline{n}_i\Big| + \frac{\Big|n_i(1)-\overline{n}_i\Big|}{1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)}\\ &\leq \frac{\overline{q}_i\sum_{t=2}^{T}\gamma^{t-2}\Big|(m(1)-1)\Big|}{1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)} + \frac{\Big|n_i(1)-\overline{n}_i\Big|}{1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i)}.\end{aligned}$$

Therefore, $\lim_{T\to\infty}\sum_{t=1}^{T}\Big|n_i(t)-\overline{n}_i\Big| \leq \frac{\overline{q}_i\Big|(m(1)-1)\Big|}{(1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i))(1-\gamma)} + \frac{\Big|n_i(1)-\overline{n}_i\Big|}{1-(\mathcal{G}_i)_1'}$. In the end, we define the positive constant

$$C_{n_i} := \frac{\overline{q}_i\Big|(m(1)-1)\Big|}{(1-(\mathcal{G}_i)_1'(\overline{n}_i,\overline{q}_i))(1-\gamma)} + \frac{\Big|n_i(1)-\overline{n}_i\Big|}{1-(\mathcal{G}_i)_1'},$$

which allows us to obtain the desired result.

(2) Given that $m(1) < 1$ and that $n_i(1) \geq \overline{n}_i$, we consider two cases. In the first case, we consider the scenario where there exists a $\widetilde{t} \in \{2,...,T\}$ such that $n_i(\widetilde{t}) \geq \overline{n}_i$. In the second case, we consider the scenario where $n_i(t) \geq \overline{n}_i$ for all $t \in \{1,\dots,T\}$.

In the first case, given $\widetilde{t} \in \{2,\dots,T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we want to show that $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We prove the claim by induction. Given that $n_i(\widetilde{t}) < \overline{n}_i$, for any $t \geq \widetilde{t}$, suppose towards an induction purpose that $n_i(t) < \overline{n}_i$, and we can establish that

$$n_i(t+1) \overset{(a)}{=} \mathcal{G}_i(n(t),\overline{q}_i m(t)) \overset{(b)}{<} \mathcal{G}_i(n(t),\overline{q}_i) < \mathcal{G}_i(\overline{n}_i,\overline{q}_i) \overset{(c)}{=} \overline{n}_i, \qquad \text{(EC.22)}$$

where (a) follows from Algorithm 1; (b) follows from the condition that $\mathcal{G}_i(q)$ strictly increases in $q \geq 0$ and from the observation in Step 2.1 from the proofs of Theorem 1 that if $m(1) < 1$, then $m(1) < m(2) < ... < m(T) < 1$; (c) follows directly from Lemma 1(ii). Therefore, we obtain that if there exists a $\widetilde{t} \in \{2, \dots, T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we have $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We then show that $\widetilde{t}$ is independent of $T$. Given the definition of $\widetilde{t}$ as the first time that $n_i(t) < \overline{n}_i$, it is equivalent to show that the value of $n_i(t)$ for $0 \leq t \leq \widetilde{t}$ is independent of $T$. This is true as given $n_i(1)$ and $m(1)$, for $t \in \{1, ..., \widetilde{t} - 1\}$, $n_i(t+1) = \mathcal{G}_i(n(t), \overline{q}_i m(t))$, where $m(t) = \min_{i' \in \mathcal{N}^+} \{\frac{n_{i'}(t)}{\overline{n}_{i'}}\}$ is independent of $T$ for $1 \leq t \leq \widetilde{t} - 1$.

The observations above allow us to deduce that in the first case,

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=1}^{T} \left|n_i(t) - \overline{n}_i\right| &= \sum_{t=1}^{\widetilde{t}-1} \left|n_i(t) - \overline{n}_i\right| + \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \left|n_i(t) - \overline{n}_i\right| \\
&\overset{(d)}{=} \sum_{t=1}^{\widetilde{t}-1} \left|n_i(t) - \overline{n}_i\right| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \left|m(t) - 1\right| \overset{(e)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \left|n_i(t) - \overline{n}_i\right| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\widetilde{t}}^{T} \left|m(\widetilde{t}) - \right. \\
&= \sum_{t=1}^{\widetilde{t}-1} \left|n_i(t) - \overline{n}_i\right| + \overline{n}_i \left|m(\widetilde{t}) - 1\right| \frac{1}{1-\gamma} \overset{(f)}{\leq} \sum_{t=1}^{\widetilde{t}-1} \left|n_i(t) - \overline{n}_i\right| + \overline{n}_i \left|m(1) - 1\right| \frac{1}{1-\gamma} \\
&\overset{(g)}{\leq} \frac{\overline{q}_i \left|(m(1) - 1)\right|}{(1 - (\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\left|n_i(1) - \overline{n}_i\right|}{1 - (\mathcal{G}_i)'_1} + \overline{n}_i \left|m(1) - 1\right| \frac{1}{1-\gamma},
\end{aligned}$$

where (d) follows from the definition of $m(t)$, (e) follows from Step 2, and (f) follows from $m(1) < m(2) < ... < m(T) < 1$ if $m(1) < 1$ in Step 2.1; (g) follows from the Case (1). Then let $C_{n_i} = \frac{\overline{q}_i \left|(m(1)-1)\right|}{(1-(\mathcal{G}_i)'_1(\overline{n}_i, \overline{q}_i))(1-\gamma)} + \frac{\left|n_i(1)-\overline{n}_i\right|}{1-(\mathcal{G}_i)'_1} + \overline{n}_i \left|m(1) - 1\right| \frac{1}{1-\gamma}$, we obtain the desired result.

In the second case, if $n_i(t) \geq \overline{n}_i$ for all $t \in \{1, ..., T\}$, we can apply the same upper bound as in Case (1) above under Claim 1.

Claim 2. To establish the second claim of this result, when $n_i(1) \leq \overline{n}_i$ and $m(1) < 1$, by applying the same induction arguments as in (EC.22) from the previous claim, we can establish that $n_i(t) \leq \overline{n}_i$ for any $t \in \{1, \dots, T\}$.

Summarizing the arguments above, we complete the proofs of the two claims in this result. ■

**Proof of Proposition 2.** Claim (i). Let $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ denote the commission in period $t \in \{1, ..., T\}$ when $w = 0$; in addition, given the optimal solution to AVG in (5) $(\overline{x}, \overline{q}^s, \overline{q}^b, \overline{s}, \overline{b})$, we define $\overline{r}_{ij} := F_{b_j}^{-1}(1-\frac{\overline{q}_j^b}{\overline{b}_j}) - F_{s_i}^{-1}(\frac{\overline{q}_i^s}{\overline{s}_i})$, which can be seen as the total commission for each transaction at AVG in (5). We will respectively show that $\overline{r}_{ij} \geq 0$ and $r_i^s(t) + r_j^b(t) \geq \overline{r}_{ij}$ for any $(i,j)$ with $x_{ij} > 0$ in Step 1.2 and Step 1.3. Before that, we need to establish an auxiliary result in Step 1.1.

Abusing some notations, given any $q > 0$, we use $\overline{s}_i(q)$ to denote the population level at which the transition remains stable, i.e., $\overline{s}_i(q) := \{s > 0 | s = \mathcal{G}_i^s(s,q)\}$. Given that $\mathcal{G}_i^s(s_i,q)$ is increasing and strictly concave in $s \in [q,\infty]$ for any given $q > 0$ and $\mathcal{G}_i^s(0,q) > 0$ (see Assumption 1), it can be easily shown that $\overline{s}_i(q)$ is well-defined. Similarly, we define $\overline{b}_j(q) := \{b > 0 | b = \mathcal{G}_j^b(b,q)\}$.

Step 1.1: show that for any $i \in \mathcal{S}, j \in \mathcal{B}$, if $0 < q_1 < q_2$, then $\frac{q_1}{\overline{s}_i(q_1)} \leq \frac{q_2}{\overline{s}_i(q_2)}$ and $\frac{q_1}{\overline{b}_j(q_1)} \leq \frac{q_2}{\overline{b}_j(q_2)}$. Suppose towards a contradiction that $\frac{q_1}{\overline{s}_i(q_1)} > \frac{q_2}{\overline{s}_i(q_2)}$, then we have

$$\begin{aligned}\overline{s}_i(q_1) = &\frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2) \overset{(a)}{=} \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\mathcal{G}_i^s\Big(q_2, \overline{s}_i(q_2)\Big) \\ &\overset{(b)}{<} \mathcal{G}_i^s\Big(\frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}q_2, \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2)\Big) \overset{(c)}{<} \mathcal{G}_i^s\Big(\frac{q_1}{q_2}q_2, \frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)}\overline{s}_i(q_2)\Big) = \mathcal{G}_i^s\Big(q_1, \overline{s}_i(q_1)\Big),\end{aligned}$$

where (a) follows from the definition of $\overline{s}_i(q_2)$; (b) follows from the strict concavity of $\mathcal{G}_i^s$ (see (EC.20)); (c) follows from $\frac{\overline{s}_i(q_1)}{\overline{s}_i(q_2)} = \frac{q_1}{q_2}\frac{\frac{q_2}{\overline{s}_2(q_2)}}{\frac{q_1}{\overline{s}_1(q_1)}} < \frac{q_1}{q_2}$ when $\frac{q_1}{\overline{s}_i(q_1)} > \frac{q_2}{\overline{s}_i(q_2)}$, and $\mathcal{G}_i^s(s,q)$ is increasing in $(s,q)$ for $0 \leq q \leq s$ (see Assumption 1). As a result, we have $\overline{s}_i(q_1) < \mathcal{G}_i^s\Big(q_1, \overline{s}_i(q_1)\Big)$, which contradicts to the definition of $\overline{s}_i(q_1)$. Therefore, if $0 < q_1 < q_2$, then $\frac{q_1}{\overline{s}_i(q_1)} \leq \frac{q_2}{\overline{s}_i(q_2)}$. The same argument holds for the buyer side.

Step 1.2: $\overline{r}_{ij} \geq 0$ for $(i,j)$ with $\overline{x}_{ij} > 0$. Suppose towards a contradiction that for the optimal solution to AVG in (5) $\overline{\boldsymbol{x}}$, there exists $(i_0,j_0)$ with $\overline{x}_{i_0j_0} > 0$ such that $\overline{r}_{i_0,j_0} < 0$; based on Lemma 1, we can plug in constraints (5b)(5c)(5d)(5e) and have $\overline{r}_{i_0,j_0} = F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_{i_0}}^{-1}(\frac{\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'})}) < 0$. Then we construct another feasible solution $\tilde{\boldsymbol{x}}$ in the following way: let $\tilde{x}_{ij} := \overline{x}_{ij}$ for $(i,j) \neq (i_0,j_0)$ and $\tilde{x}_{i_0j_0} := 0$. We can show that $\tilde{\boldsymbol{x}}$ leads to a higher objective value of AVG:

$$\begin{aligned}\overline{\mathcal{R}}(\overline{\boldsymbol{x}}) \overset{(a)}{=} &\sum_{(i,j)\in E} \overline{x}_{ij}\Big(F_{b_j}^{-1}(1 - \frac{\sum_{i' \in N_E(j)} \overline{x}_{i'j}}{\overline{b}_j(\sum_{i' \in N_E(j)} \overline{x}_{i'j})}) - F_{s_i}^{-1}(\frac{\sum_{j' \in N_E(i)} \overline{x}_{ij'}}{\overline{s}_i(\sum_{j' \in N_E(i)} \overline{x}_{ij'})})\Big) \\ =&\overline{x}_{i_0j_0}\Big(F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_{i_0}}^{-1}(\frac{\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j' \in N_E(i_0)} \overline{x}_{i_0j'})})\Big) \\ &+ \sum_{i \in N_E(j_0)} \overline{x}_{ij_0}\Big(F_{b_{j_0}}^{-1}(1 - \frac{\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i' \in N_E(j_0)} \overline{x}_{i'j_0})}) - F_{s_i}^{-1}(\frac{\sum_{j' \in N_E(i)} \overline{x}_{ij'}}{\overline{s}_i(\sum_{j' \in N_E(i)} \overline{x}_{ij'})})\Big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\in N_E(i_0)}\overline{x}_{i_0j}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\overline{x}_{i'j}}{\overline{b}_j(\sum_{i'\in N_E(j)}\overline{x}_{i'j})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'})})\Big)\\
&+\sum_{(i,j)\in E, i\neq i_0, j\neq j_0}\overline{x}_{ij}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\overline{x}_{i'j}}{\overline{b}_j(\sum_{i'\in N_E(j)}\overline{x}_{i'j})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\overline{x}_{ij'}}{\overline{s}_i(\sum_{j'\in N_E(i)}\overline{x}_{ij'})})\Big)\\
&\overset{(b)}{<}\sum_{i\in N_E(j_0)}\tilde{x}_{ij_0}\Big(F_{b_{j_0}}^{-1}(1-\frac{\sum_{i'\in N_E(j_0)}\tilde{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i'\in N_E(j_0)}\tilde{x}_{i'j_0})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\tilde{x}_{ij'}}{\overline{s}_i(\sum_{j'\in N_E(i)}\tilde{x}_{ij'})})\Big)\\
&+\sum_{j\in N_E(i_0)}\tilde{x}_{i_0j}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\tilde{x}_{i'j}}{\overline{b}_j(\sum_{i'\in N_E(j)}\tilde{x}_{i'j})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\tilde{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j'\in N_E(i_0)}\tilde{x}_{i_0j'})})\Big)\\
&+\sum_{(i,j)\in E, i\neq i_0, j\neq j_0}\tilde{x}_{ij}\Big(F_{b_j}^{-1}(1-\frac{\sum_{i'\in N_E(j)}\tilde{x}_{i'j}}{\overline{b}_j(\sum_{i'\in N_E(j)}\tilde{x}_{i'j})})-F_{s_i}^{-1}(\frac{\sum_{j'\in N_E(i)}\tilde{x}_{ij'}}{\overline{s}_i(\sum_{j'\in N_E(i)}\tilde{x}_{ij'})})\Big)=\overline{\mathcal{R}}(\tilde{\boldsymbol{x}}),
\end{aligned}
$$

where in (a), we plug in the constraint (5b)(5c)(5d)(5e) into the objective function, where the inequalities in (5d)(5e) hold based on Lemma 1; (b) follows from $\Big(F_{b_{j_0}}^{-1}(1-\frac{\sum_{i'\in N_E(j_0)}\overline{x}_{i'j_0}}{\overline{b}_{j_0}(\sum_{i'\in N_E(j_0)}\overline{x}_{i'j_0})})-F_{s_{i_0}}^{-1}(\frac{\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'}}{\overline{s}_{i_0}(\sum_{j'\in N_E(i_0)}\overline{x}_{i_0j'})})\Big)<0$ and furthermore, $\sum_{i\in N_E(j)}\overline{x}_{ij}\leq\sum_{i\in N_E(j)}\tilde{x}_{ij}$ for any $j\in\mathcal{B}$ and $\sum_{j\in N_E(i)}\overline{x}_{ij}\leq\sum_{j\in N_E(i)}\tilde{x}_{ij}$ for any $i\in\mathcal{S}$ based on the construction of $\tilde{\boldsymbol{x}}$ and the result in Step 1. As a result, $\overline{\mathcal{R}}(\overline{\boldsymbol{x}})<\overline{\mathcal{R}}(\tilde{\boldsymbol{x}})$, which contradicts to the optimality of $\overline{\boldsymbol{x}}$. Therefore, $\overline{r}_{ij}\geq 0$ for $(i,j)$ with $\overline{x}_{ij}>0$.

Step 1.3: $r_i^s(t)+r_j^b(t)\geq\overline{r}_{ij}$ for $(i,j)$ with $x_{ij}>0$. When $w=0$,

$$
\begin{aligned}
r_i^s(t)+r_j^b(t)=&F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\\
&\overset{(a)}{=}F_{b_j}^{-1}(1-\frac{\overline{q}_j^b\hat{m}(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{\overline{q}_i^s\hat{m}(t)}{s_i(t)})\overset{(b)}{\geq}F_{b_j}^{-1}(1-\frac{\overline{q}_j^b}{\overline{b}_j})-F_{s_i}^{-1}(\frac{\overline{q}_i^s}{\overline{s}_i})=\overline{r}_{ij}.
\end{aligned}
$$

where (a) follows from the policy rule, (b) holds because when $w=0$, $\hat{m}(t)=m(t)\leq\frac{b_j(t)}{\overline{b}_j}$ for any $j\in\mathcal{B}$ and $\hat{m}(t)=m(t)\leq\frac{s_i(t)}{\overline{s}_i}$ for any $i\in\mathcal{S}$ by definition.

In summary of Step 1.2 and 1.3, $r_i^s(t)+r_j^b(t)\geq\overline{r}_{ij}>0$ for $(i,j)$ with $x_{ij}>0$.

Claim (ii). We define $\tilde{t}$ such that $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(t),\overline{q}_i\hat{m}(t))}{\overline{n}_i}\right\}\leq 1$ for $t\in\{1,...,\tilde{t}\}$ and $\min_{i\in\mathcal{N}^+}\left\{\frac{\mathcal{G}_i(n_i(\tilde{t}),\overline{q}_i\hat{m}(\tilde{t}))}{\overline{n}_i}\right\}>1$. Then for $t\in\{\tilde{t}+1,...,T\}$, the update rule when $w>0$ is the same as that when $w=0$, and we have already shown that $r_i^s(t)+r_j^b(t)>0$ for $(i,j)$ with $x_{ij}>0$ (see Step 1). We next establish the result for $t\in\{1,...,\tilde{t}-1\}$.

Define $\kappa_{ij}:=\min\{\frac{\overline{q}_j^b}{\max_{l\in\mathcal{N}}\overline{q}_l},\frac{\overline{q}_i^s}{\max_{l\in\mathcal{N}}\overline{q}_l}\}$ for $(i,j)\in E$ and define $z_{ij}:=\Big\{z\in(\kappa_{ij},+\infty)|F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=0\Big\}$. Given that $F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=-\overline{v}_{s_i}<0$ when $z=\kappa_{ij}$ and

$F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})=\overline{v}_{b_j}>0$ when $z=\infty$, and $F_{b_j}^{-1}(1-\frac{\kappa_{ij}}{z})-F_{s_i}^{-1}(\frac{\kappa_{ij}}{z})$ is increasing in $z$, we know $z_{ij}$ is well-defined. Then

$$\begin{aligned}
r_i^s(t)+r_j^b(t)=&F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})\\
\overset{(a)}{\leq}&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{b_j(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\overline{q}_l}\})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{s_i(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\overline{q}_l}\})\\
\leq&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{b_j(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{s_i(t)}\min_{l\in\mathcal{N}^+}\{\frac{n_l(t)}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\})\\
=&F_{b_j}^{-1}(1-w\frac{\overline{q}_j^b}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\frac{\min_{l\in\mathcal{N}^+}n_l(t)}{b_j(t)})-F_{s_i}^{-1}(w\frac{\overline{q}_i^s}{\max_{l'\in\mathcal{N}}\overline{q}_{l'}}\frac{\min_{l\in\mathcal{N}^+}n_l(t)}{s_i(t)})\\
\overset{(b)}{<}&F_{b_j}^{-1}(1-\frac{w\kappa_{ij}}{wz_{ij}})-F_{s_i}^{-1}(\frac{w\kappa_{ij}}{wz_{ij}})\overset{(c)}{=}0,
\end{aligned}$$

where (a) follows the definition of $\hat{m}(t)$; (b) follows from the definition of $\kappa_{ij}$ and the condition that $\max\{s_i(t),b_j(t)\}<wz_{i,j}\min_{i\in\mathcal{N}^+}n_i(t)$; (c) follows from the definition of $z_{ij}$. ■

**Proof of Corollary 1.** Under TRP,

$$Y_j^b(t):=r_j^b(t)+\min_{i\in N_E(j)}p_i^s(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})=F_{b_j}^{-1}(1-\frac{\overline{q}_j^b\hat{m}(t)}{b_j(t)})=F_{b_j}^{-1}(1-\hat{m}(t)\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}).$$

Since $\hat{m}(t)$ are the same across types and $F_{b_j}$ are assumed to be homogeneous across types, the $Y_j^b(t)$ only differ when $\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}$ are different. Similarly, we can show that $I_i^s(t)$ depends only on $\frac{\frac{\overline{q}_i}{\overline{s}_i}}{\frac{s_i(t)}{\overline{s}_i}}$.

For any $t\in\{1,...,T\}$, for any positive constant $p$, by constructing $p_i(t)=p$ and $r_i^s(t)=p-F_{s_i}^{-1}(1-\frac{q_i^s(t)}{s_i(t)})$ for any $i\in\mathcal{S}$ ; $r_j^b(t)=F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-p$ for any $j\in\mathcal{B}$, we obtain a feasible commission (see Lemma EC.1). For this solution, we can see that $r_j^b(t)$ depends only on and deceases in $\frac{\frac{\overline{q}_j^b}{\overline{b}_j}}{\frac{b_j(t)}{\overline{b}_j}}$, while $r_i^s(t)$ depends only on and decreases in $\frac{\frac{\overline{q}_i}{\overline{s}_i}}{\frac{s_i(t)}{\overline{s}_i}}$. ■

### EC.2.3. Proof of Results for MP

**Proof of Proposition 3.** We denote by $(\boldsymbol{r}^{\boldsymbol{MP}}(t),\boldsymbol{p}^{\boldsymbol{MP}}(t),\boldsymbol{q}^{s,MP}(t),\boldsymbol{q}^{b,MP}(t),\boldsymbol{x}^{MP}(t))$ the optimal solution to the optimization problem for the MP in Definition EC.1. We consider the following problem instance: Consider a simple network in which there is only one buyer type and one seller type with initial population $s(1)=b(1)>0$. Given the commissions $\boldsymbol{r}^{\boldsymbol{MP}}(t)$ induced by the MP, we let the populations for the next period be

$(\boldsymbol{s}^{MP}(t+1), \boldsymbol{b}^{MP}(t+1))$ is updated by $s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{s,MP}(t))^{\xi}$ and $b^{MP}(t+1) = \alpha b^{MP}(t) + \beta(q^{b,MP}(t))^{\xi}$, where we assume $\beta > 0$ and $0 < \xi < 1$ so that the Assumption 1 holds. In addition, we let $F_s(\cdot)$ and $F_b(\cdot)$ be the distribution functions over $[0,1]$ from the uniform distribution.

We establish two claims to complete the proof.

Claim 1: $\lim_{t\to\infty} R^{MP}(t)$ exists . We divide the proof arguments into the following steps. In Step 1.1, we show that if a steady state induced by the MP exists, we characterize the properties of the steady state. In Step 1.2, we show that the populations converge to the steady state under the platform's MP. For simplicity of notations, we let $\mathcal{R}^{MP}(t)$ denote the profit in period $t$ under the MP.

Step 1.1. Characterize the quantity $\overline{q}^{MP}$ and the profit $\overline{R}^{MP}$ in a steady state. We first define a steady state as such that the populations and transaction quantities remain unchanged after the population transition in each period. Given the definition of a steady state, under the platform's myopic policy, the steady-state population vector $(\overline{s}^{MP}, \overline{b}^{MP}, \overline{q}^{MP})$ should satisfy the following three conditions:

$$\overline{q}^{MP} = \underset{0\le q\le \min\{\overline{s}^{MP}, \overline{b}^{MP}\}}{\arg\max} \left[\left(1 - \frac{q}{\overline{s}^{MP}} - \frac{q}{\overline{b}^{MP}}\right) q\right], \tag{EC.23a}$$

$$\overline{s}^{MP} = \alpha \overline{s}^{MP} + \beta(\overline{q}^{MP})^{\xi}, \tag{EC.23b}$$

$$\overline{b}^{MP} = \alpha \overline{b}^{MP} + \beta(\overline{q}^{MP})^{\xi}. \tag{EC.23c}$$

Condition (EC.23a) ensures that given the population in each period $(\overline{s}^{MP}, \overline{b}^{MP})$, the platform's commissions $r$ could induce the equilibrium quantity $\overline{q}^{MP}$ to maximize its profit in the current period (see Corollary EC.1 for the formulation of optimization problem); (EC.23b) and (EC.23c) ensure that the population vector $(\overline{s}^{MP}, \overline{b}^{MP})$ remains unchanged after the update in each period.

For Problem (EC.23a), from the first-order-condition $\frac{\partial}{\partial q}\left[\left(1 - \frac{q}{\overline{s}^{MP}} - \frac{q}{\overline{b}^{MP}}\right)q\right] = 0$, we can obtain that $\overline{q}^{MP} = \frac{\overline{s}^{MP}\overline{b}^{MP}}{2\overline{s}^{MP} + 2\overline{b}^{MP}}$, which falls in the region $(0, \min\{\overline{s}^{MP}, \overline{b}^{MP}\})$. Thus, the optimal solution to (EC.23a) is an interior point. Together with the equations in (EC.23b)-(EC.23c), we obtain that

$$\overline{q}^{MP} = \left(\frac{k}{4}\right)^{\frac{1}{1-\xi}}, \overline{b}^{MP} = k\left(\frac{k}{4}\right)^{\frac{\xi}{1-\xi}}, \overline{s}^{MP} = k\left(\frac{k}{4}\right)^{\frac{\xi}{1-\xi}}.$$

where we let $k = \frac{\beta}{1-\alpha}$ for simplicity of notations. This allows us to show that the profit induced by the platform's MP satisfies that

$$\overline{\mathcal{R}}^{MP} = \Big(1 - \frac{\overline{q}^{MP}}{\overline{s}^{MP}} - \frac{\overline{q}^{MP}}{\overline{b}^{MP}}\Big)\overline{q}^{MP} = \frac{1}{2}\Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}.$$

Step 1.2: For the seller side, show that there exists a $\gamma \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \leq \gamma|(\overline{s}^{MP} -$

Next, we establish the convergence of the platform's MP. Without loss of generality, we prove the convergence on the seller side, and notice that the same argument would hold for the buyer side as well.

Since we have $s^{MP}(1) = b^{MP}(1)$ in the problem instance, and in each iteration we have $s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{MP}(t))^{\xi}$ and $b^{MP}(t+1) = \alpha b^{MP}(t) + \beta(q^{MP}(t))^{\xi}$, we obtain that $s^{MP}(t) = b^{MP}(t)$ for any $t \in \{1, ..., T\}$. Based on this observation, we can obtain that

$$\begin{aligned} q^{MP}(t) &= \operatorname*{arg\,max}_{0<q<\min\{s^{MP}(t), b^{MP}(t)\}} \Big\{\Big(1 - \frac{q}{s^{MP}(t)} - \frac{q}{b^{MP}(t)}\Big)q\Big\} \\ &= \operatorname*{arg\,max}_{0<q<s^{MP}(t)} \Big\{\Big(1 - \frac{q}{s^{MP}(t)} - \frac{q}{s^{MP}(t)}\Big)q\Big\} = \frac{s^{MP}(t)}{4}. \end{aligned}$$

From the optimal solution $q^{MP}(t)$ above, we obtain that

$$s^{MP}(t+1) = \alpha s^{MP}(t) + \beta(q^{MP}(t))^{\xi} = \alpha s^{MP}(t) + \beta\Big(\frac{s^{MP}(t)}{4}\Big)^{\xi}.$$

Abusing some notations, we let $g_s(s) := \alpha s + \beta(\frac{s}{4})^{\xi}$ for any $s \geq 0$ such that $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$ based on the condition in (EC.23b). To proceed, we consider the following two cases that $s^{MP}(1) \geq \overline{s}^{MP}$ and $s^{MP}(1) < \overline{s}^{MP}$:

(1) When $s^{MP}(1) \geq \overline{s}^{MP}$, we want to show that $s^{MP}(t) \geq \overline{s}^{MP}$ for $t \in \{1, .., T\}$. By induction, if $s^{MP}(t) \geq \overline{s}^{MP}$, we have $s^{MP}(t+1) = g_s(s^{MP}(t)) \geq g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the inequality follows from the fact that $g_s(\cdot)$ is an increasing function. Since $s^{MP}(1) \geq \overline{s}^{MP}$, we obtain that $s^{MP}(t) \geq \overline{s}^{MP}$ for $t \in \{1, .., T\}$.

Based on the observation above, we can establish that

$$\Big|s^{MP}(t+1) - \overline{s}^{MP}\Big| = \Big|g_s(s^{MP}(t)) - \overline{s}^{MP}\Big| \overset{(a)}{=} g_s(s^{MP}(t)) - g_s(\overline{s}^{MP}) \overset{(b)}{\leq} \Big|s^{MP}(t) - \overline{s}^{MP}\Big| g_s'(\overline{s}^{MP}), \tag{EC.24}$$

where (a) follows from the observation that $s^{MP}(t) \geq \overline{s}^{MP}$ for $t \in \{1, .., T\}$ in this case; (b) follows from the condion that $g_s$ is concave given that $g_s(s) = \alpha s + \beta(\frac{s}{4})^{\xi}$ with

$a \in (0,1)$. Moreover, we have $g_s'(\overline{s}^{MP}) < 1$ given that $g_s(0) = 0$ and $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, and so by the mean value theorem, there exists a $\widetilde{s} \in (0, \overline{s}^{MP})$ such that $g_s'(\widetilde{s}) = \frac{g_s(\overline{s}^{MP}) - g_s(0)}{\overline{s}^{MP} - 0} = 1$. Since $g_s(\cdot)$ is concave, we have that $g_s'(\overline{s}^{MP}) < g_s'(\widetilde{s}) = 1$ given that $\overline{s}^{MP} > \widetilde{s}$. By letting $\gamma_1 := g_s'(\overline{s}^{MP})$, we establish that there exists $\gamma_1 \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \le \gamma_1 |(\overline{s}^{MP} - s^{MP}(t))|$ for $t \in \{1, ..., T-1\}$ if $s^{MP}(1) \ge \overline{s}^{MP}$. From the definition of $g_s(\cdot)$ and $\overline{s}^{MP}$, we see that $\gamma_1$ is independent of $T$.

(2) When $s^{MP}(1) < \overline{s}^{MP}$, we want to show that $s^{MP}(t) < \overline{s}^{MP}$ for $t \in \{1, .., T\}$. If $s^{MP}(t) < \overline{s}^{MP}$, we have $s^{MP}(t+1) = g_s(s^{MP}(t)) < g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the inequality follows from that $g_s(\cdot)$ is an increasing function given that $s^{MP}(t) < \overline{s}^{MP}$. Since $s^{MP}(1) < \overline{s}^{MP}$, by induction we obtain that $s^{MP}(t) < \overline{s}^{MP}$ for any $t \in \{1, .., T\}$.

Then, we can establish that

$$\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)} \overset{(c)}{<} \frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)} \overset{(d)}{<} 1,$$

where in Step (c), we establish the following set of observations: (c-i) we first establish that $\frac{\overline{s}^{MP} - g_s(s)}{\overline{s}^{MP} - s}$ decreases in $s \ge 0$ by showing that $\frac{\partial}{\partial s}\left(\frac{\overline{s}^{MP} - g_s(s)}{\overline{s}^{MP} - s}\right) = \frac{(s - \overline{s}^{MP}) g_s'(s) - g_s(s) + \overline{s}^{MP}}{(s - \overline{s}^{MP})^2} < 0$, with the inequality following as $g_s(s)$ is strictly concave in $s \ge 0$ such that $\overline{s}^{MP} = g_s(\overline{s}^{MP}) < g_s(s) + (\overline{s}^{MP} - s) g_s'(s)$; (c-ii) we then show that $s^{MP}(t) > s^{MP}(1)$ for $t \in \{2, .., T\}$. Note that $g_s(0) = 0$ and $g_s(\overline{s}^{MP}) = \overline{s}^{MP}$. Since $g_s(s) - s$ is strictly concave in $s \ge 0$, by the Jensen's inequality, we obtain that $g_s(a\overline{s}^{MP}) - a\overline{s}^{MP} > a(g_s(\overline{s}^{MP}) - \overline{s}^{MP}) + (1-a)\big(g_s(0) - 0\big) = 0$ for $0 < a < 1$. Therefore, we have $g_s(a\overline{s}^{MP}) > a\overline{s}^{MP}$ for $0 < a < 1$, which further implies that $s^{MP}(t+1) = g_s(s^{MP}(t)) > s^{MP}(t)$ given that $0 < s^{MP}(t) < \overline{s}^{MP}$. Thus, we can obtain that $s^{MP}(t) < s^{MP}(t+1) < \overline{s}^{MP}$ for $t \in \{1, .., T-1\}$. Combining the observations in (c-i) and (c-ii), since $\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)}$ decreases in $s^{MP}(t)$ and $s^{MP}(t+1) > s^{MP}(t) > s^{MP}(1)$ for $t \in \{2, .., T-1\}$, we have that Step (c) holds. For Step (d), we have $s^{MP}(1) < s^{MP}(2) = g_s(s^{MP}(1)) < g_s(\overline{s}^{MP}) = \overline{s}^{MP}$, where the first inequality follows from $s^{MP}(t+1) = g_s(s^{MP}(t)) > s^{MP}(t)$ for $0 < s^{MP}(t) < \overline{s}^{MP}$ based on previous discussion; the second inequality follows from the condition that $s^{MP}(1) < \overline{s}^{MP}$ in this case and $g_s(.)$ is a increasing function; the last equation follows directly from the observation in (EC.23b). Therefore, we have that $\frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)} < 1$.

By letting $\gamma_2 = \frac{\overline{s}^{MP} - g_s(s^{MP}(1))}{\overline{s}^{MP} - s^{MP}(1)}$, we obtain that $\frac{\overline{s}^{MP} - g_s(s^{MP}(t))}{\overline{s}^{MP} - s^{MP}(t)} \le \gamma_2$, which implies that

$$\left|\overline{s}^{MP} - g_s(s^{MP}(t))\right| \overset{(e)}{=} \overline{s}^{MP} - g_s(s^{MP}(t)) \le \gamma_2 \Big(\overline{s}^{MP} - s^{MP}(t)\Big) \overset{(f)}{=} \gamma_2 \left|\overline{s}^{MP} - s^{MP}(t)\right|$$

where (e) and (f) follow from the observations that $s^{MP}(t) < \overline{s}^{MP}$ for $t \in \{1,..,T\}$. In summary, there exists a $\gamma_2 \in (0,1)$ such that $|\overline{s}^{MP} - s^{MP}(t+1)| \le \gamma_2 |(\overline{s}^{MP} - s^{MP}(t))|$ for $t \in \{1,...,T-1\}$ if $s^{MP}(1) < \overline{s}^{MP}$. Again, from the definition of $g_s(\cdot)$, we see that $\gamma_2$ is independent of $T$.

In summary of the two cases above, we let $\gamma := \max\{\gamma_1, \gamma_2\}$, which allows us to obtain the desired result.

Claim 2: For any $\epsilon > 0$, there exists $a \in (0,1)$ for the population transition in this problem instance such that $\overline{\mathcal{R}}^{MP} < \epsilon \overline{\mathcal{R}}$. For the AVG in (5) given the problem instance before Step 1, we have that

$$\begin{aligned} \overline{\mathcal{R}} \; &= \max_{s,b,q} \quad \Big(1 - \frac{q}{s} - \frac{q}{b}\Big) q \\ \text{s.t. } 0 \le q \le s, &\quad 0 \le q \le b, \quad s \le \alpha s + \beta q^{\xi}, \quad b \le \alpha b + \beta q^{\xi}. \end{aligned}$$

In addition, based on Lemma 1(ii), the inequalities in the last two constraints are both tight. Note that $s = \alpha s + \beta q^{\xi}$ and $b = \alpha b + \beta q^{\xi}$ are equivalent to $s = b = kq^{\xi}$, where $k = \frac{\beta}{1-\alpha}$. By plugging $s = b = kq^{\xi}$ into the objective function we obtain $\overline{\mathcal{R}} = \max_{0 \le q \le kq^{\xi}} (1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}}) q$. Since $(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}}) q$ is concave in $q \ge 0$ for $0 < \xi < 1$, from the first-order condition, we have $\overline{q} = \big(\frac{k}{2(2-\xi)}\big)^{\frac{1}{1-\xi}}$, which satisfy $0 < \overline{q} < kq^{\xi}$. Thus, the optimal commission $\overline{r}$ and the optimal profit $\overline{R}$ for the instance of the AVG in (5) satisfies that

$$\begin{aligned} \overline{r} \; &= \; 1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}} = \frac{1-\xi}{2-\xi}, \\ \overline{\mathcal{R}} \; &= \; \Big(1 - \frac{q}{kq^{\xi}} - \frac{q}{kq^{\xi}}\Big) q = \frac{1-\xi}{2-\xi} \Big(\frac{k}{2(2-\xi)}\Big)^{\frac{1}{1-\xi}}, \end{aligned}$$

which further implies that $\frac{\overline{\mathcal{R}}^{MP}}{\overline{\mathcal{R}}} = (\frac{2-\xi}{2})^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)}$. Therefore, we can obtain that

$$\lim_{\xi \to 1} \frac{\overline{\mathcal{R}}^{MP}}{\overline{\mathcal{R}}} \; = \; \lim_{\xi \to 1} \Big(\frac{2-\xi}{2}\Big)^{\frac{1}{1-\xi}} \frac{2-\xi}{2(1-\xi)} \; = \; 0.$$

■

## EC.3. Proof of Results in Section 5

In this section, we develop some auxiliary results that are needed for the proofs of results in Section 5 in EC.3.1. We then respectively prove the results from Section 5.2 in EC.3.2 and those from 5.1 in EC.3.3.

### EC.3.1. Auxiliary Results for Section 5.

In this section, we first develop a simper formulation for Problem (5) in (EC.30). To do that, we first characterize the properties of Problem (5) in Lemma EC.8 and Lemma EC.9. Next, we reformulate it in Lemma EC.10, and will further simplify its formulation into (EC.30) in Lemma EC.11. We then show the connection between the optimal solution to (EC.30) $\boldsymbol{w}^*$ and $(\mathcal{S}_\tau, \mathcal{B}_\tau)$ constructed in (10) in Lemma EC.12. The proof of the auxiliary results follows a similar argument to the proof of Lemma 1, Lemma 2 and Proposition 10 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

To develop an equivalent reformulation in $(\boldsymbol{q}, \boldsymbol{x})$ for **AVG**, recall from Lemma 1(ii) that the relaxed population dynamics constraints $s_i \le \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ and $b_j \le \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ with the optimal solutions to **AVG** are tight. Together with (7), on the seller side, we have $s_i = \frac{\beta_i^s (q_i^s)^{\xi_s}}{1-\alpha_i^s}$ for any $i \in \mathcal{S}$. We further let $k_i^s := \frac{\beta_i^s}{1-\alpha_i^s}$, which allows us to obtain that $s_i = k_i^s (q_i^s)^{\xi_s}$ for any $i \in \mathcal{S}$. Similarly, on the buyer side, we have $b_j = k_j^b (q_j^b)^{\xi_b}$ for any $j \in \mathcal{B}$, where $k_j^b = \frac{\beta_j^b}{1-\alpha_j^b}$. Plugging the expressions of $s_i = k_i^s (q_i^s)^{\xi_s}$ and $b_j = k_j^b (q_j^b)^{\xi_b}$ into **AVG**, we obtain the following reformulation of **AVG**:

$$\overline{\mathcal{R}} = \max_{\boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \Big[ \sum_{j \in \mathcal{B}} \widetilde{F}_b(q_j^b, k_j^b (q_j^b)^{\xi_b}) - \sum_{i \in \mathcal{S}} \widetilde{F}_s(q_i^s, k_i^s (q_i^s)^{\xi_s}) \Big] \tag{EC.25a}$$

$$\text{s.t. } q_i^s \le k_i^s (q_i^s)^{\xi_s}, \quad \forall i \in \mathcal{S}, \tag{EC.25b}$$

$$q_j^b \le k_j^b (q_j^b)^{\xi_b}, \quad \forall j \in \mathcal{B}, \tag{EC.25c}$$

$$\sum_{j:(i,j)\in E} x_{ij} = q_i^s, \quad \forall i \in \mathcal{S}, \tag{EC.25d}$$

$$q_j^b = \sum_{i:(i,j)\in E} x_{ij}, \quad \forall j \in \mathcal{B}, \tag{EC.25e}$$

$$x_{ij} \ge 0, \quad \forall (i,j) \in E. \tag{EC.25f}$$

where $\widetilde{F}_b(\cdot)$ and $\widetilde{F}_s(\cdot)$ are defined before Problem (5).

For $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, define $y_b(q) := F_b^{-1}\left(1 - (q)^{1-\xi_b}\right) q$ for $0 \le q \le 1$. Define $y_s(q,u) := -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. For simplicity of notations, we let $y_b'(q) := \frac{dy_b(q)}{dq}$ for $0 < q < 1$ and $(y_s)_1'(q,u) := \frac{\partial y_s(q,u)}{\partial q}$ for $0 < q < u$. Furthermore, we let $y_b'(0) := \lim_{q\downarrow 0} y_b'(q)$, $y_b'(1) := \lim_{q\uparrow 1} y_b'(q)$; for $u > 0$, we let $(y_s)_1'(0,u) := \lim_{q\to 0}(y_s)_1'(q,u)$, $(y_s)_1'(u,u) := \lim_{q\to u}(y_s)_1'(q,u)$; for $q > 0$, we let $(y_s)_2'(q,q) := \lim_{u\to q}(y_s)_2'(q,u)$. We show in the following lemma that all of the limiting values are finite.

Lemma EC.8. *(i)* $y_b(q)$ *is continuously differentiable and strictly concave in* $q \in [0,1]$*;*

*(ii)* $y_s(q,u)$ *is continuous and strictly concave in* $(q,u) \in \{(q',u') : 0 \le q' \le u'\}$*; moreover,* $y_s(q,u)$ *is continuously differentiable in* $(q,u) \in \{(q',u') : 0 \le q' \le u', u' > 0\}$*;*

*(iii) for any* $0 < \xi_s < 1$*,* $-(1-\xi_s)\left[F_s^{-1}\right]'(x)x - F_s^{-1}(x)$ *strictly decreases in* $x \in [0,1]$*.*

Before the next auxiliary result, we define

$$\rho(u) := \operatorname*{arg\,max}_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0, \tag{EC.26}$$

$$h(u) = \max_{0 \le q \le \min\{1,u\}} \Big(y_b(q) + y_s(q,u)\Big), \qquad \text{for } u \ge 0. \tag{EC.27}$$

Given the definition of $\rho(u)$ and $h(u)$ above, we proceed to consider the following auxiliary result about $(\rho(u), h(u))$ for $u \ge 0$. Notice that $-(y_s)_1'(u,u) = (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s > 0$, which is a constant. To support our proof arguments below, when $u > 0$, if $y_b'(0) > (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$; if $y_b'(0) \le (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := 0$.

Lemma EC.9. *(i)* $\rho(u)$ *is a well-defined and strictly increasing in* $u \ge 0$*; moreover, given* $\widetilde{u} \ge 0$ *defined before the lemma statement,* $\frac{\rho(u)}{u} = 1$ *for* $u \in (0, \widetilde{u}]$ *and* $\frac{\rho(u)}{u}$ *strictly decreases in* $u \ge \widetilde{u}$*;*

*(ii)* $h(u)$ *is continuous, strictly increasing and strictly concave in* $u \ge 0$*.*

We next develop an alternative optimization for Problem (EC.25). Consider the following optimization problem:

$$\overline{\mathcal{V}} = \max_{\boldsymbol{w},\boldsymbol{z}} \sum_{j \in \mathcal{B}} \left[ (k_j^b)^{\frac{1}{1-\xi_b}} h\left( \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} \right) \right] \tag{EC.28a}$$

$$\text{s.t.} \quad (w_j)^{\frac{1}{1-\xi_b}} = \sum_{i:(i,j) \in E} z_{ij}, \qquad j \in \mathcal{B} \tag{EC.28b}$$

$$\sum_{j:(i,j) \in E} z_{ij} = (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad i \in \mathcal{S}, \tag{EC.28c}$$

$$z_{ij} \ge 0, \qquad \forall (i,j) \in E. \tag{EC.28d}$$

where

$$h(u) = \max_{0 \le \widetilde{q}_j \le \min\{1,u\}} F_b^{-1}(1 - (\widetilde{q}_j)^{1-\xi_b})\widetilde{q}_j - F_s^{-1}\left( \frac{(\widetilde{q}_j)^{1-\xi_s}}{u^{1-\xi_s}} \right)\widetilde{q}_j \text{ for any } u > 0 \tag{EC.29}$$

and $h(0) = 0$. We consider the following result:

Lemma EC.10. *We have the following equivalence properties between Problem (EC.28) and Problem (EC.29):*

*(i) let* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *be the optimal solution to Problem (EC.25), and construct* $(\boldsymbol{w}, \boldsymbol{z})$ *such that* $w_j = (\frac{q_j^b}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}})^{1-\xi_b}$ *for any* $i: x_{ij} > 0$ *and* $z_{ij} = \frac{x_{ij}}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}}$, $\widetilde{q}_j = \frac{q_j^b}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, *then* $(\boldsymbol{w}, \boldsymbol{z})$ *is the optimal solution to Problem (EC.28) and* $\widetilde{q}_j$ *is the optimal solution to Problem (EC.29) with* $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$;

*(ii) let* $(\boldsymbol{w}, \boldsymbol{z})$ *be the optimal solution to Problem (EC.28) and* $\widetilde{q}_j$ *is the optimal solution to Problem (EC.29) with* $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, *then construct* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *such that* $x_{ij} = \frac{z_{ij}(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j}{(w_j)^{\frac{1}{1-\xi_b}}}$ *and* $q_i^s = \frac{(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j(k_i^s)^{\frac{1}{1-\xi_s}}}{w_j^{\frac{1}{1-\xi_b}}}$ *for* $j: z_{ij} > 0$, $q_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j$, *then* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *is the optimal solution to (EC.25);*

*(iii) Problem (EC.25) and Problem (EC.28) share the same optimal objective value, i.e.,* $\overline{\mathcal{R}} = \overline{\mathcal{V}}$.

We can further simplify the formulation in (EC.28) in the following Lemma EC.11.

Lemma EC.11. *Problem (EC.28) and the following problem share the same optimal solution vector* $\boldsymbol{w}$,

$$\overline{\mathcal{Y}} = \max_{\boldsymbol{w}} \sum_{j\in\mathcal{B}} \Big[(k_j^b)^{\frac{1}{1-\xi_b}} h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big] \tag{EC.30a}$$

$$s.t. \quad \sum_{j\in\widetilde{\mathcal{B}}}(w_j)^{\frac{1}{1-\xi_b}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}}, \qquad \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \tag{EC.30b}$$

$$w_j \ge 0, \qquad \forall j \in \mathcal{B}, \tag{EC.30c}$$

*and moreover,* $\overline{\mathcal{Y}} = \overline{\mathcal{V}}$ *where* $\overline{\mathcal{V}}$ *is the optimal objective value for Problem (EC.28).*

The next lemma establishes the connection between the optimal solution $\boldsymbol{w}^*$ to Problem (EC.30) and the network components $G(\mathcal{S}_\tau \cup \mathcal{B}_\tau, E_\tau)$ constructed in (10). Given the finiteness of the network $G(\mathcal{S}\cup\mathcal{B}, E)$, the iteration in (10) yields a maximum index $\overline{\tau}$.

Lemma EC.12. *For any* $\tau \in \{1, ..., \overline{\tau}\}$ *and any* $j' \in \mathcal{B}_\tau$, *we have* $\frac{(w_{j'}^*)^{\frac{1}{1-\xi_b}}}{(k_{j'}^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in\mathcal{S}_\tau}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$.

#### EC.3.2. Proof of Results in Section 5.2.

**Proof of Proposition 4.** Recall that we have established the connection for the optimal solution and the optimal objective value of Problem (EC.25) with those of Problem

(EC.28) and Problem (EC.30) in Lemma EC.10 and Lemma EC.11. Therefore, we focus on characterizing the properties of optimization problems in (EC.28) and (EC.30) instead of (EC.25) in this proof. We have already shown that (EC.28) and (EC.30) share the same optimal solution $\boldsymbol{w}^*$ in Lemma EC.11. To prove the claim, we consider the buyer side in Step 1 and the seller side in Step 2.

Step 1: Establish the ranking of buyers' service levels and payments. Based on Lemma EC.10(ii), we let $(\boldsymbol{w},\boldsymbol{z})$ be the optimal solution to Problem (EC.28) and $\widetilde{q}_j$ is the optimal solution to Problem (EC.29) with the parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$. We know the optimal solution to Problem (EC.25) satisfies

$$\frac{q_j^b}{b_j} \overset{(a)}{=} \frac{(q_j^b)^{1-\xi_b}}{k_j^b} \overset{(b)}{=} (\widetilde{q}_j)^{1-\xi_b} \overset{(c)}{=} \rho^{1-\xi_b}\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big),$$

where Step (a) follows from the observation that $b_j = k_j^b (q_j^b)^{\xi_b}$ in Problem (EC.25); Step (b) follows from the solution property of $\widetilde{q}_j$ in Problem (EC.29) by Lemma EC.10(ii); Step (c) follows from the definition of the optimal solution $\rho$ to Problem (EC.26). Therefore, the ranking of service levels $(\frac{q_j^b}{b_j})_{j\in\mathcal{B}}$ is the same as that of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

For buyers' payments, we know that

$$\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b = F_b^{-1}\Big(1-\frac{q_j^b}{b_j}\Big) = F_b^{-1}\Big(1-\rho\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\Big)\Big).$$

Therefore, the ranking of buyers' payments $(\min_{i':(i',j)\in E}\{p_{i'}^s\} + r_j^b)_{j\in\mathcal{B}}$ is the opposite of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

By Lemma EC.9(i), we have that $\rho(u)$ strictly increases in $u>0$. From Lemma EC.12, we know that $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ and $\tau = 1,\dots\overline{\tau}$. Furthermore, the definition in (10) implies that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau = 1,\dots,\overline{\tau}$. Therefore, we have

$$\begin{aligned}
&\frac{q_{j_1}^b}{b_{j_1}} = \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1, j_2\in\mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},\\
&\frac{q_{j_1}^b}{b_{j_1}} < \frac{q_{j_2}^b}{b_{j_2}}, && \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

and

$$\min_{i':(i',j_1)\in E}\{p_{i'}^s\} + r_{j_1}^b = \min_{i':(i',j_2)\in E}\{p_{i'}^s\} + r_{j_2}^b, \quad \text{for } j_1, j_2\in\mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},$$

$$\min_{i':(i',j_1)\in E}\{p^s_{i'}\}+r^b_{j_1} > \min_{i':(i',j_2)\in E}\{p^s_{i'}\}+r^b_{j_2}, \quad \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.$$

Step 2: Establish the ranking of sellers' service levels and incomes. To establish the ranking of sellers' service levels, given the optimal solution $\boldsymbol{w}$ to Problem (EC.30) and the optimal solution $\widetilde{q}_j$ to Problem (EC.29) with parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}$, we have that for any $i\in\mathcal{S}$ and $j: x_{ij}>0$,

$$\frac{q^s_i}{s_i} \overset{(a)}{=} \frac{(q^s_i)^{1-\xi_s}}{k^s_i} \overset{(b)}{=} \left(\frac{\rho\big((w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}\big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k^b_j)^{\frac{1}{1-\xi_b}}}\right)^{1-\xi_s}, \tag{EC.31}$$

where (a) follows from our discussion before Problem (EC.25) that $s_i = k^s_i(q^s_i)^{\xi_s}$; (b) follows from Lemma EC.10(ii) for $j: x_{ij}>0$.

We next show that for any $\tau_1\neq\tau_2$, we have $x_{ij}=0$ with $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. Based on Lemma EC.10(ii), it is equivalent to show the optimal solution to Problem (EC.28) satisfies that for any $\tau_1\neq\tau_2$, $z_{ij}=0$ for $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. We show it by induction. Again, to simplify the notation in Problem (EC.28), we let $W_j := (w_j)^{\frac{1}{1-\xi_b}}$ and $\psi^b_j := (k^b_j)^{\frac{1}{1-\xi_b}}$ for any $j\in\mathcal{B}$ and let $\psi^s_i := (k^s_i)^{\frac{1}{1-\xi_s}}$ for any $i\in\mathcal{S}$. We first consider $\tau=1$. The buyers in $\mathcal{B}_1$ can only trade with the sellers in $\mathcal{S}_1$ given that they are not connected to any other seller types. It remains to show that the sellers in $\mathcal{S}_1$ only trade with the buyers in $\mathcal{B}_1$ at the platform's optimal commissions. Suppose towards a contradiction that there exist $\tau_1\neq 1$ such that $z_{ij}>0$ for some $i\in\mathcal{S}_1$ and $j\in B_{\tau_1}$, then

$$\begin{aligned}\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} &= \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\in\mathcal{B}_1} z_{ij} + \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\notin\mathcal{B}_1} z_{ij} \\ &\overset{(a)}{>} \sum_{j\in\mathcal{B}_1}\sum_{i:(i,j)\in E, i\in\mathcal{S}_1} z_{ij} \overset{(b)}{=} \sum_{j\in\mathcal{B}_1} W_j \overset{(c)}{=} \sum_{j\in\mathcal{B}_1}\psi^b_j\frac{\sum_{i\in\mathcal{S}_1}\psi^s_i}{\sum_{j\in\mathcal{B}_1}\psi^b_j} = \sum_{i\in\mathcal{S}_1}\psi^s_i\end{aligned} \tag{EC.32}$$

where (a) follows from the assumption that $z_{ij}>0$ for some $i\in\mathcal{S}_1$ and some $j\in B_{\tau_1}$ with $\tau_1\neq 1$; (b) follows from (EC.28b); (c) follows from the observation in Lemma EC.12. In summary, $\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_1}\psi^s_i$, which violate Constraint (EC.28c). In summary, we have that $z_{ij}=0$ for all $i\in\mathcal{S}_1$ and $j\in B_{\tau_1}$ if $\tau_1\neq 1$. Assuming that $\mathcal{B}_\tau$ only trade with $\mathcal{S}_\tau$ and vice versa, we proceed to show that $\mathcal{B}_{\tau+1}$ only trade with $\mathcal{S}_{\tau+1}$ and vice versa. First, the buyers in $\mathcal{B}_{\tau+1}$ only trade with the sellers in $\mathcal{S}_{\tau+1}$, because they are not adjacent to the seller types from $\mathcal{S}_{\tau'}$ for any $\tau'\geq\tau+1$; and the seller types with an index lower than $\tau+1$ does not trade with them based on our previous discussion. Second, $\mathcal{S}_{\tau+1}$ only trade with $\mathcal{B}_{\tau+1}$, otherwise we can also obtain $\sum_{i\in\mathcal{S}_{\tau+1}}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_{\tau+1}} \psi_i^s$ following the same argument in (EC.32), which violate Constraint (EC.28c) to Problem (EC.28) given that Problem (EC.30) is a reformulation without loss of optimality. In summary, for any $\tau_1 \neq \tau_2$, $x_{ij}=0$ for $i\in\mathcal{S}_{\tau_1}$ and $j\in B_{\tau_2}$. This allows us to show that for any $i\in\mathcal{S}_\tau$ with $\tau=1,\dots,\overline{\tau}$, we have that if $j: x_{ij}>0$, then we obtain that $j\in\mathcal{B}_\tau$.

Thus, regarding the sellers' incomes, for any $i\in\mathcal{S}_\tau$ with $\tau=1,\dots,\overline{\tau}$ and any $j: x_{ij}>0$, we have that

$$p_i^s - r_i^s = F_s^{-1}\Big(\frac{q_i^s}{s_i}\Big) = F_s^{-1}\Big(\frac{\rho\Big((w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}\Big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}}\Big).$$

Since $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ with $\tau=1,\dots,\overline{\tau}$ in Lemma EC.12, we can next focus on the ranking of $\frac{\rho\Big(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}\Big)}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $\tau=1,\dots,\overline{\tau}$. Recall from Step 1 that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau=1,\dots\overline{\tau}$. Based on Lemma EC.9, for some constant $\widetilde{u}\geq 0$, we have that $\frac{\rho(u)}{u}=1$ for $0<u\leq\widetilde{u}$ and $\frac{\rho(u)}{u}$ strictly decreases in $u$ for $u>\widetilde{u}$. Define $\widetilde{\tau}:=\max\{\tau|u_j<\widetilde{u}$ for $j\in\mathcal{B}_\tau\}$. We observe that (i) for any $\tau\leq\widetilde{\tau}$, we have $\frac{q_i^s}{s_i}=\frac{\rho(u)}{u}=1$ and $p_i^s-r_i^s=F_s^{-1}\big(\frac{\rho(u)}{u}\big)=F_s^{-1}(1)=\overline{v}_{s_i}$ for $i\in S_\tau$; (ii) for any $\tau>\widetilde{\tau}$, we have $\frac{\rho(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}})}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$strictly decreases in $\tau$. Therefore, we can summarize that

$$\begin{aligned}
&\frac{q_{i_1}^s}{s_{i_1}}=\frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1,i_2\in\mathcal{S}_\tau,\tau\in\{1,...,\overline{\tau}\},\\
&\frac{q_i^s}{s_i}=1, && \text{for } i\in\mathcal{S}_\tau,\tau\leq\widetilde{\tau},\\
&\frac{q_{i_1}^s}{s_{i_1}}>\frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1\in\mathcal{S}_{\tau_1},i_2\in\mathcal{S}_{\tau_2},\tau_1,\tau_2\in\{\widetilde{\tau}+1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

and

$$\begin{aligned}
&p_{i_1}^s-r_{i_1}^s=p_{i_1}^s-r_{i_1}^s, && \text{for } i_1,i_2\in\mathcal{S}_\tau,\tau\in\{1,...,\overline{\tau}\},\\
&p_i^s-r_i^s=\overline{v}_{s_i}, && \text{for } i\in\mathcal{S}_\tau,\tau\leq\widetilde{\tau},\\
&p_{i_1}^s-r_{i_1}^s>p_{i_2}^s-r_{i_2}^s, && \text{for } i_1\in\mathcal{S}_{\tau_1},i_2\in\mathcal{S}_{\tau_2},\tau_1,\tau_2\in\{\widetilde{\tau}+1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.
\end{aligned}$$

Summarizing the two steps above, we conclude the claims in this result. ■

**Proof of Corollary 2.** Given the definition of $(\boldsymbol{k}^s,\boldsymbol{k}^b)$ at the beginning of Appendix EC.3.1, for any $\xi_s\in(0,1)$ and $\xi_b\in(0,1)$, we first let $\psi_i^s=(k_i^s)^{\frac{1}{1-\xi_s}}$ and $\psi_j^b=(k_j^b)^{\frac{1}{1-\xi_b}}$ for simplicity of notations. We consider the equivalent reformulation in Problem (EC.28) with decision variables $(\boldsymbol{w}, \boldsymbol{z})$ by Lemma EC.10 and Problem (EC.30) with the decision variable vector $\boldsymbol{w}$ and Lemma EC.11. We let $W_j = (w_j)^{\frac{1}{1-\xi_b}}$ for all $j \in \mathcal{B}$.

Notice that it is without loss of generality to consider a connected graph $G(\mathcal{S} \cup \mathcal{B}, E)$ for the proof arguments. We prove the impact of $\psi^s$ and $\psi^b$ on the service levels in Step 1, and then the impacts on supply/demand and population in Step 2.

Proof of Claim (1): Establish the impact of $\psi^s$ and $\psi^b$ on the service levels. Recall from Step 1 in the proof arguments of Proposition 4 that for any $j \in \mathcal{B}$, when $\frac{W_j}{\psi_j^b}$ becomes larger under the optimal solution $\boldsymbol{W}$ to Problem (EC.30), $\frac{q_j^b}{b_j}$ becomes larger at the optimal solution as well. As a result, we can focus on the impact of $\psi^s$ and $\psi^b$ on $\frac{W_j}{\psi_j^b}$ for $j \in \mathcal{B}_\tau$.

Step (1-i): Establish the impact of $(\psi^s, \psi^b)$ on the service levels of the buyer side. Let $(\boldsymbol{W}, \boldsymbol{z})$ be the optimal solution to (EC.28) given parameters $(\psi^s, \psi^b)$ and let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \ldots \overline{\tau}\}$ be the network components obtained from (10) given this parameter set. We define the index set $\tau_i := \{\tau | i \in S_\tau\}$ and $\tau_j := \{\tau | j \in B_\tau\}$. We consider an alternative vector $(\hat{\psi}^s, \hat{\psi}^b)$ in which we pick any $\widetilde{i} \in \mathcal{S}$, and let $\hat{\psi}_{\widetilde{i}}^s > \psi_{\widetilde{i}}^s$; we also let $\hat{\psi}_i^s := \psi_i^s$ for all $i \neq \widetilde{i}$ and let $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. Then we obtain that the parameter vector $(\hat{\psi}^s, \hat{\psi}^b)$ has only one entry on the seller side that is higher than in $(\psi^s, \psi^b)$. Let $(\hat{W}, \hat{z})$ be the optimal solution to (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, and let $\{(\hat{\mathcal{S}}_\tau, \hat{\mathcal{B}}_\tau) : \tau = 1, \ldots \widetilde{\tau}\}$ be the network components obtained from (10) given this parameter set for some positive integer $\widetilde{\tau}$.

To prove the claim of this step, we want to show that $W_j \le \hat{W}_j$ for all $j \in \mathcal{B}$. This leads to the observation that $\frac{W_j}{\psi_j^b} \le \frac{\hat{W}_j}{\hat{\psi}_j^b}$ given our construction that $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. In this way, we can claim that a higher $\psi_i^s$ leads to weakly higher $\frac{W_j}{\psi_j^b}$ for all $j \in \mathcal{B}$.

Suppose towards a contradiction that there exists a $j_1 \in \mathcal{B}$ such that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution. Based on Constraint (EC.28b), we have that $\sum_{i \in N_E(j_1)} z_{ij_1} = W_{j_1} > \hat{W}_{j_1} = \sum_{i \in N_E(j_1)} \hat{z}_{ij_1}$, which implies that there exists a $i_1 \in N_E(j_1)$ such that $z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0$. Similarly, given $i_1 \in N_E(j_1)$, based on Constraint (EC.28c), we have that $\sum_{j \in N_E(i_1)} z_{i_1 j} = \psi_{i_1}^s \le \hat{\psi}_{i_1}^s = \sum_{j \in N_E(i_1)} \hat{z}_{i_1 j}$ where the inequality follows from the construction of $\hat{\psi}$ above. This implies that there exists $j_2 \in N_E(i_1)$ such that $0 \le z_{i_1 j_2} < \hat{z}_{i_1 j_2}$. Using the same argument as above, there must exist a $i_2 \in N_E(j_2), i_2 \neq i_1$ such that $z_{i_2 j_2} > \hat{z}_{i_2 j_2} \ge 0$ and there exists some $j_3 \in N_E(i_2)$ such that $0 \le z_{i_2 j_3} < \hat{z}_{i_2 j_3}$. In this iteration, given the finiteness of the graph, we have that there exists a finite list $(j_1, i_1, j_2, i_2 \ldots, j_n)$ such that $W_{j_1} > \hat{W}_{j_1}$ and $W_{j_n} \le \hat{W}_{j_n}$. We let $\mathbb{B}_1 = \{j_1\}$, and $\mathbb{S}_1 = \{i | i \in N_E(j_1), z_{i_1 j_1} > \hat{z}_{i_1 j_1} \ge 0\}$. For $t \in \{2, 3 \ldots\}$, we further let

$\mathbb{B}_t = \{j | j \in N_E(i), 0 \leq z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$, and $\mathbb{S}_t = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathbb{B}_{t-1}\}$. We have that $\mathscr{B}_t := \cup_{l\in\{1,\ldots,t\}} \mathbb{B}_l$ and $\mathscr{S}_t := \cup_{l\in\{1,\ldots,t\}} \mathbb{S}_l$ are the sets of all possible buyer types and seller types accessed within the first $2t$ steps in this iteration. Since $\mathscr{B}_{t-1} \subset \mathscr{B}_t \subset \mathcal{B}$ and $|\mathcal{B}|$ is finite, there exists a finite $\bar{t}$ such that $\mathscr{B}_{\bar{t}} = \mathscr{B}_{\bar{t}-1}$, i.e., the set $\mathscr{B}_t$ stops expanding. Under the assumption that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution for $j_1 \in \mathbb{B}_1$, we next show that there exists $j \in \mathscr{B}_{\bar{t}}$ such that $W_j < \hat{W}_j$. We further suppose towards a contradiction that $W_j > \hat{W}_j$ for any $j \in \mathscr{B}_{\bar{t}}$. Consider the set of seller types $\widetilde{S} := \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$. We can show that $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}}$ by definition. Moreover, we would obtain that

$$\begin{aligned}
\sum_{i\in\widetilde{S}} \hat{\psi}_i^s &= \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&\overset{(a)}{=} \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&< \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \\
&\leq \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} = \sum_{i\in\widetilde{S}} \psi_i^s
\end{aligned}$$

where in Step (a), with $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}} = \cup_{l\in\{1,\ldots,\bar{t}\}} \mathbb{S}_l$, in the iterative construction above, given that $\mathbb{B}_t = \{j | j \in N_E(i), 0 \leq z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$ and that $\mathscr{B}_{\bar{t}} = \cup_{l\in\{1,\ldots,\bar{t}\}} \mathbb{B}_l$, the subset of buyer types $\{j : z_{ij} < \hat{z}_{ij}$ for some $i \in \widetilde{S}\}$ should be a subset of $\mathscr{B}_{\bar{t}}$; based on the definition $\widetilde{S} = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$, we have that $z_{ij} > \hat{z}_{ij}$ for any $i \in \widetilde{S}$ and $j \in \mathscr{B}_{\bar{t}}$, which further implies that $\{j : z_{ij} < \hat{z}_{ij},\ \forall i \in \widetilde{S}\} = \emptyset$ and that $\sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} = 0$. However, the observation that $\sum_{i\in\widetilde{S}} \hat{\psi}_i^s < \sum_{i\in\widetilde{S}} \psi_i^s$ contradicts with the fact that $\sum_{i\in\widetilde{S}} \hat{\psi}_i^s \geq \sum_{i\in\widetilde{S}} \psi_i^s$ by construction of $(\hat{\psi}^s, \hat{\psi}^b)$ above. Therefore, such a contradiction implies that there exists a $j_l \in \mathbb{B}_l \subset \mathscr{B}_{\bar{t}}$ for some $l \in \mathbb{N}_+$ such that $W_{j_l} \leq \hat{W}_{j_l}$. Thus, there must exist a finite path $(j_1, i_1, j_2, i_2 \ldots, j_l)$ for $j_t \in \mathbb{B}_t$ and $i_t \in \mathbb{S}_t$ such that $z_{i_t j_t} > 0$ for $t \in \{1, .., l\}$ and $\hat{z}_{i_{t-1} j_t} > 0$ for $t \in \{2, \ldots, l\}$ under the assumption that $W_{j_l} \leq \hat{W}_{j_l}$. For any $t \in \{1, \ldots, l-1\}$, we let $\tau_{i_t}$ and $\tau_{j_t}$ be the corresponding index for the seller subgroup for $\mathcal{S}_\tau$ and the buyer subgroup $\mathcal{B}_\tau$ by the iterative construction in (10). Since $z_{i_t j_t} > 0$, we know that $\tau_{i_t} = \tau_{j_t}$. With the iterative construction, we have $j_{t+1} \in N_E(i_t)$, which satisfies that $\tau_{i_t} \leq \tau_{j_{t+1}}$ given that $S_{i_t}$ is not adjacent to $\mathcal{B}_l$ with $l < \tau_{i_t}$ with the iterative construction in (10). In summary, $\tau_{j_1} = \tau_{i_1} \leq \tau_{j_2} = \ldots \leq \tau_{j_l}$, which implies that $\frac{W_{j_n}}{\psi_{j_n}^b} \geq \frac{W_{j_1}}{\psi_{j_1}^b}$ based on Lemma EC.12. Therefore, $\frac{\hat{W}_{j_n}}{\hat{\psi}_{j_n}^b} \geq \frac{W_{j_n}}{\psi_{j_n}^b} \geq \frac{W_{j_1}}{\psi_{j_1}^b} > \frac{\hat{W}_{j_1}}{\hat{\psi}_{j_1}^b}$.

We proceed to show that the constructed solution $(\hat{W}, \hat{z})$ cannot be the optimal solution to Problem (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$. We first send a flow $\epsilon$ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$ to construct a new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$: since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\hat{z}_{i_t, j_{t+1}} > 0$ for all $t \in \{1, ..., n-1\}$, we can pick any $\epsilon \in (0, \min\{(\hat{W}_{j_n}\hat{\psi}^b_{j_1} - \hat{W}_{j_1}\hat{\psi}^b_{j_n})/(\hat{\psi}^b_{j_1} + \hat{\psi}^b_{j_n}), \min_{t\in\{1,...,n-1\}}\{\hat{z}_{i_t, j_{t+1}}\}\})$; for $t \in \{1, ..., n-1\}$, let $\widetilde{z}_{i_t j_t} := \hat{z}_{i_t j_t} + \epsilon$, $\widetilde{z}_{i_t j_{t+1}} := \hat{z}_{i_t j_{t+1}} - \epsilon$, $\widetilde{z}_{ij} := \hat{z}_{ij}$ for all $(i,j) \neq (i_t j_{t+1}), (i,j) \neq (i_t j_t)$. Let $\widetilde{W}_{j_1} := \hat{W}_{j_1} + \epsilon$ and $\widetilde{W}_{j_n} := \hat{W}_{j_n} - \epsilon$, $\widetilde{W}_{j'} := \hat{W}_{j'}$ for all $j' \neq j_1, j' \neq j_n$. We next verify the feasibility of this new solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$ in Problem (EC.28). Since $\epsilon \leq \min_{t\in\{1,...,n-1\}}\{\hat{z}_{i_t, j_{t+1}}\}$, we can obtain that $\widetilde{z}_{i_t j_{t+1}} \geq 0$ such that Constraint (EC.28d) is satisfied. In addition, in our construction of the new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$, since we only send a flow $\epsilon$ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$, Constraints (EC.28b) - (EC.28c) are preserved. Thus, $(\widetilde{W}, \widetilde{z})$ is feasible in Problem (EC.28). We define the super-gradient of $h(u)$ as $\partial h(u) = \{z \in \mathbb{R} | h(t) \leq h(u) + z(t-u), \forall t \geq 0\}$. In addition, we define $\partial_- h(u) := \inf\{\partial h(u)\}$ and $\partial_+ h(u) := \sup\{\partial h(u)\}$. Given the strict concavity of $h(u)$ for $u \geq 0$, we have that if $u_2 > u_1 > 0$, then $\partial_+ h(u_2) < \partial_- h(u_1)$, which implies that

$$
\begin{aligned}
\hat{\psi}^b_{j_1} h\Big(\frac{\widetilde{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\widetilde{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) &= \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&> \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \epsilon \partial h_-\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) - \epsilon \partial h_+\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&\geq \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big)
\end{aligned}
$$

where the first inequality follows from the concavity of $h(\cdot)$ in $\mathbb{R}_+$; for the second inequality, since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\epsilon < \frac{\hat{W}_{j_n}\hat{\psi}_{j_1} + \hat{W}_{j_1}\hat{\psi}_{j_n}}{\hat{\psi}_{j_1} + \hat{\psi}_{j_n}}$, we have $\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}$, and therefore, $\partial_+ h(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}) < \partial h_-(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}})$. Since other terms in the objective function remain unchanged, $(\widetilde{W}, \widetilde{z})$ leads to a strictly higher objective value than $(\hat{W}, \hat{z})$, which contradicts with the fact that $(\hat{W}, \hat{z})$ be the optimal solution to (EC.28) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$.

In conclusion, we have that $\frac{W_j}{\psi^b_j} \leq \frac{\hat{W}_j}{\hat{\psi}^b_j}$ for all $j \in \mathcal{B}$. This concludes the claim about the impact of $\psi^s_i$. For the impact of $\psi^b_j$, we can apply exactly the same proof-by-contradiction arguments as above to establish that when $\psi^b_{\tilde{j}}$ increases for any $\tilde{j} \in \mathcal{B}$, then we have that the optimal solution $\frac{W_j}{\psi^b_j}$ decreases for any $j \in \mathcal{B}$.

Step (1-ii): Establish the impact of $(\psi^s, \psi^b)$ on the service levels of the seller side. For the impact of $\psi^s$ on the service levels of the seller side, we first recall the construction of

$(\hat{\psi}^s, \hat{\psi}^b)$ based on $(\psi^s, \boldsymbol{\psi}^b)$ in Step (1-i), which satisfies that $\hat{\psi}^s_{\widetilde{i}} > \psi^s_{\widetilde{i}}$, $\hat{\psi}^s_i := \psi^s_i$ for all $i \neq \widetilde{i}$ and $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. Without loss of generality, we suppose that a type-$i$ seller trades with type-$j_1$ buyer where $i \in \mathcal{S}_{l_1}$ and $j_1 \in \mathcal{B}_{l_1}$ given the parameter set $(\psi^s, \psi^b)$; and given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, we suppose that the type-$i$ seller trades with type-$j_2$ buyer for some $j_2 \in \mathcal{B}_{l_2}$. The index satisfies that $l_2 \geq l_1$ given that $\mathcal{S}_{l_1}$ is not connected with $\mathcal{B}_t$ for any $t < l_1$ by the iterative construction of network components in (10). Therefore, we have that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{W_{j_2}}{\psi^b_{j_2}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, where the first inequality follows from Lemma EC.12 given that $l_2 \geq l_1$, and the second inequality follows from the same arguments in Step (1-i). Since type-$i$ sellers have positive trades with type-$j_1$ buyers in the optimal solutions given the parameters $(\psi^s, \psi^b)$, and with type-$j_2$ buyers in the optimal solutions given the parameters $(\hat{\psi}^s, \hat{\psi}^b)$, based on the observation that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, we can establish that

$$\frac{q^s_i}{s_i} \overset{(a)}{=} \left(\frac{\rho(W_{j_1}/\psi^b_{j_1})}{W_{j_1}/\psi_{j_1}}\right)^{1-\xi_s} \overset{(b)}{\geq} \left(\frac{\rho(\hat{W}_{j_2}/\hat{\psi}^b_{j_2})}{\hat{W}_{j_2}/\hat{\psi}_{j_2}}\right)^{1-\xi_s} \overset{(c)}{=} \frac{\hat{q}^s_i}{\hat{s}_i}, \tag{EC.33}$$

where Step (a) and Step (c) follow from the optimality equation in (EC.31) from the proof arguments in Proposition 4; Step (b) follows from the fact that $\frac{\rho(x)}{x}$ monotonically decreases in $x \geq 0$ (see Lemma EC.9). In summary, when $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly lower for all $i \in \mathcal{S}$.

Using the same arguments above, we could establish the impact of $\psi^b$ on the seller side: when $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \in \mathcal{B}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly higher for all $i \in \mathcal{S}$.

Proof of Claim (2): Establish the impact of $\psi^s$ and $\psi^b$ on transaction quantities and populations. Recall from (8) that we have $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$ and $b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{\xi_b}{1-\xi_b}}$ for any $j \in \mathcal{B}$ at the optimal solution to Problem (5) given (7). We establish this claim in the following two substeps.

Step (2-i): Establish the impact of $\psi^b$ on the transaction quantities and populations. For any $j \in \mathcal{B}$, recall from Step (1-i) above that if $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, or if $\psi^s_{\widetilde{i}}$ increases for any $\widetilde{i} \in \mathcal{S}$, then $\frac{q^b_j}{b_j}$ weakly decreases at the optimal solution. Given that $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we can establish that as $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \neq j$, then $q^b_j$ weakly decreases at the optimal solution for any $j \in \mathcal{B}$. From $q^b_j = \psi^b_j(\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we have that $b_j = \psi^b_j(q^b_j)^{\xi_b}$ for any $j \in \mathcal{B}$, which further suggests that $b_j$ weakly decreases at the optimal solution for any $j \in \mathcal{B}$.

For any $j \in \mathcal{B}$, it remains to consider the impact of $\psi^b_j$ on $(q^b_j, b_j)$ at the optimal solution for $j \in \mathcal{B}$. We first show that $q^b_j$ increases in $\psi^b_j \geq 0$ for any $j \in \mathcal{B}$. Recall from Constraints

(EC.25d)-(EC.25e) that $\sum_{i\in\mathcal{S}} q_i^s = \sum_{i\in\mathcal{S}}\sum_{j:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}}\sum_{i:(i,j)\in E} x_{ij} = \sum_{j\in\mathcal{B}} q_j^b$, which means that $q_j^b = \sum_{i\in\mathcal{S}} q_i^s - \sum_{j'\neq j, j'\in\mathcal{B}} q_{j'}^b$. Since higher $\psi_j^b$ leads to weakly higher $q_i^s$ for any $i\in\mathcal{S}$ and weakly lower $q_{j'}^b$ for any $j'\in\mathcal{B}$ with $j'\neq j$, we conclude that higher $\psi_j^b$ leads to weakly higher $q_j^b$. Similarly, higher $\psi_i^s$ leads to weakly higher $q_i^s$.

Step (2-ii): Establish the impact of $\psi^s$ on the transaction quantities and populations. By applying the same arguments as in Step (2-i), we can establish that $(q_i, s_i)$ weakly increases in $\psi_i^s$ for all $i\in\mathcal{S}$, and $q_i^s$ and $s_i$ weakly decreases in $\psi_{i'}^s$ for any $i'\neq i$ and weakly increases in $\psi_j^b$ for all $j\in\mathcal{B}$. ■

**Proof of Proposition 5.** Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (EC.25); we let $u_j := (w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}$ for any $j\in\mathcal{B}$ where $(\boldsymbol{w},\boldsymbol{z})$ is the optimal solution to the reformulation into Problem (EC.28) (see Lemma EC.10). Recall that for given $\tau = 1,\dots,\overline{\tau}$ from (10), type-$i$ sellers for $i\in\mathcal{S}_\tau$ trade with type-$j$ buyers for $j\in\mathcal{B}_\tau$. Moreover, for any $i\in\mathcal{S}_\tau$ and $j\in\mathcal{B}_\tau$,

$$r_i^s + r_j^b = F_b^{-1}\Big(1 - \frac{q_j^b}{k_j^b(q_j^b)^{\xi_b}}\Big) - F_s^{-1}\Big(\frac{q_i^s}{k_i^s(q_i^s)^{\xi_s}}\Big) \;=\; F_b^{-1}\Big(1-\rho^{1-\xi_b}(u_j)\Big) - F_s^{-1}\Big(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\Big),$$

where the first equation follows from the conditions in (EC.1a) and (EC.1c) where the expressions of $s_i$ and $b_j$ are given before Problem (EC.25); the second equation follows from the observations in Lemma EC.10(ii) and the definition of $\rho(u)$ in (EC.26). In addition, at the optimal solution, the value of $u_j$ for any $j\in\mathcal{B}_\tau$ increases in $\tau = 1,\dots,\overline{\tau}$ (see Lemma EC.12 and the definition in (10)). For simplicity of notations, we let $r(u) := F_b^{-1}(1-\rho^{1-\xi_b}(u)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u)}{u^{1-\xi_s}})$ for any $u>0$. Recall the definition $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$ before Lemma EC.9.

We prove the two claims of this result.

Claim (1). If $u_j \le \widetilde{u}$, we have $\rho(u_j) = u_j$ (see Lemma EC.9(i)). This implies that $F_b^{-1}(1-\rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}) = F_b^{-1}(1-u_j^{1-\xi_b}) - F_s^{-1}(1)$, which is decreasing in $u_j\in[0,1]$ given that $F_b(\cdot)$ is a strictly increasing function in $[0,\overline{v}^b]$ (see Assumption 2). We let $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j\in\mathcal{B}_\tau\}$. Together with the fact that at the optimal solution, the value of $u_j$ for $j\in\mathcal{B}_\tau$ increases in $\tau = 1,\dots,\overline{\tau}$, we obtain that the value $r(u_j)$ increases in $\tau < \widetilde{\tau}$.

Claim (2). If $u_j \ge \widetilde{u}$, we know that $y_b'(\rho(u_j)) + (y_s)_1'(\rho(u_j), u_j) = 0$. Define $Y(\widetilde{q}_j, u_j) := y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j)$ given the definitions of $y_s$ and $y_b$ before Lemma EC.8: for any $\xi_s\in(0,1)$

and $\xi_b \in (0,1)$, $y_b(q) = F_b^{-1}\left(1-(q)^{1-\xi_b}\right)q$ for $0 \le q \le 1$ and $y_s(q,u) = -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right)q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. We have that

$$\begin{aligned} Y(\widetilde{q}_j, u_j) &= y_b'(\widetilde{q}_j) + (y_s)_1'(\widetilde{q}_j, u_j) \\ &= \left( (\xi_b - 1)\widetilde{q}_j^{1-\xi_b}(F_b^{-1})'\left(1-\widetilde{q}_j^{1-\xi_b}\right) + F_b^{-1}\left(1-\widetilde{q}_j^{1-\xi_b}\right)\right) \\ &\quad + \left( (\xi_s - 1)\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}(F_s^{-1})'\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right) - F_s^{-1}\left(\frac{\widetilde{q}_j}{u_j^{1-\xi_s}}\right)\right). \end{aligned}$$

Since $F_s$ and $F_b$ are twice differentiable, we know that $F_s^{-1}$ and $F_b^{-1}$ are continuously differentiable, and therefore $Y(\widetilde{q}_j, u_j)$ is continuously differentiable at $(\widetilde{q}_j, u_j)$ for $0 \le \widetilde{q}_j \le \min\{u_j, 1\}$. By the implicit function theorem, there exists a continuously differentiable function $\rho(u_j)$ such that $\widetilde{q}_j = \rho(u_j)$ given $Y(\widetilde{q}_j, u_j) = 0$. By differentiating $Y(\widetilde{\rho}(u_j), u_j) = 0$ with respect to $u_j$, we obtain

$$\rho'(u_j) = \frac{(\xi_s-1)u_j^{\xi_s-3}\rho(u_j)^{1-2\xi_s}\left((\xi_s-1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s-2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right)}{(\xi_b-1)\rho(u_j)^{-2\xi_b}f_b + (\xi_s-1)u_j^{\xi_s-2}\rho(u_j)^{-2\xi_s}f_s},$$

where

$$\begin{aligned} f_b &:= (\xi_b-2)\rho(u_j)^{\xi_b}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) - (\xi_b-1)\rho(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right), \\ f_s &:= (\xi_s-1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s-2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right). \end{aligned}$$

We proceed to show that $f_s < 0$ and $f_b < 0$ for later use:

$$\begin{aligned} f_b &:= (1-\xi_b)\rho(u_j)^{\xi_b}\left(\frac{(2-\xi_b)}{(\xi_b-1)}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\right) \\ &\overset{(a)}{<} (1-\xi_b)\rho(u_j)^{\xi_b}\left(-2(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)\right) \overset{(b)}{<} 0, \\ f_s &:= (\xi_s-1)u_j\rho^{\xi_s}(u_j)\left(\rho^{1-\xi_s}(u_j)u_j^{\xi_s-1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + \frac{\xi_s-2}{\xi_s-1}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right) \\ &\overset{(c)}{<} (\xi_s-1)u_j\rho^{\xi_s}(u_j)\left(\rho^{1-\xi_s}(u_j)u_j^{\xi_s-1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + 2(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right) \overset{(d)}{<} 0, \end{aligned}$$

where (a) and (c) follow from the facts that $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, which imply that $\frac{2-\xi_b}{\xi_b-1} < -2$ and $\frac{\xi_s-2}{\xi_s-1} > 2$ given that $(F_b^{-1})' > 0$ and $(F_s^{-1})' > 0$ on the domains; (b) and (d) follow from the conditions that $-F_s^{-1}(a/b)a$ and $F_b^{-1}(1-a/b)a$ are concave in $(a,b)$ for

$0 \le a \le b$ and $b > 0$ by Assumption 3, and therefore $\frac{a}{b}(F_s^{-1})''\left(\frac{a}{b}\right) + 2(F_s^{-1})'\left(\frac{a}{b}\right) > 0$ and $\frac{a}{b}(F_b^{-1})''\left(1-\frac{a}{b}\right) - 2(F_b^{-1})'\left(1-\frac{a}{b}\right) < 0$. In summary, we have $f_s < 0$ and $f_b < 0$.

Finally, we want to establish how $r(u_j) = F_b^{-1}(1-\rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}})$ changes in $u_j > 0$. Again, given the continuity of $r(u)$, we define the sup-derivative

$$\partial r(u) = \{z \in \mathbb{R} \mid r(t) \le r(u) + z(t-u),\ \forall t \ge 0\},$$

which implies that

$$\begin{aligned}\partial r(u) &= (\xi_b - 1)\rho(u_j)^{-\xi_b}\rho'(u_j)(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right) \\ &\quad + (\xi_s - 1)u_j^{\xi_s-2}\rho(u_j)^{-\xi_s}\left(u_j\rho'(u_j) - \rho(u_j)\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).\end{aligned}$$

Plugging in the expression of $\rho'(u_j)$, we obtain that

$$\partial r(u) = \frac{(\xi_b - 1)(\xi_s - 1)\rho(u_j)\left(f_1 + f_2 + f_3\right)}{u_j\left((\xi_b - 1)s^{2-\xi_s}\rho(u_j)^{2\xi_s} f_b + (\xi_s - 1)\rho(u_j)^{2\xi_b} f_s\right)},$$

where

$$\begin{aligned}f_1 &= (\xi_b - 1)u_j\rho(u_j)^{\xi_s+1}(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right), \\ f_2 &= (\xi_s - 1)u_j^{\xi_s}\rho(u_j)^{\xi_b+1}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right), \\ f_3 &= -u_j(\xi_b - \xi_s)\rho(u_j)^{\xi_b+\xi_s}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).\end{aligned}$$

Based on the observation above, we discuss the two cases of this claim:

(i) if $F_s(v)$ and $F_b(v)$ are convex in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$, we have $(F_b^{-1})''(v) < 0$ and $(F_s^{-1})''(v) < 0$ in their domains. Given $(F_b^{-1})'(v) > 0$ and $(F_s^{-1})'(v) > 0$, $\rho(u_j) < 1$ (see (EC.26)) and $\xi_s, \xi_b \in (0,1)$, we know that $f_1 > 0$ and $f_2 > 0$. Since $\xi_s = \xi_b$, $f_3 = 0$. Therefore, the numerator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. Since $f_s < 0$ and $f_b < 0$, the denominator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. In summary, $\frac{\partial r(u_j)}{\partial u_j} > 0$ for $u_j \ge \widetilde{u}$;

(ii) if $F_s(v)$ and $F_b(v)$ are concave in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$ respectively, we have $(F_b^{-1})''(v) > 0$ and $(F_s^{-1})''(v) > 0$, then $f_1 < 0$ and $f_2 < 0$. Therefore, $\frac{\partial r(u_j)}{\partial u_j} < 0$ for $u_j \ge \widetilde{u}$.

■

### EC.3.3. Proof of Results in Section 5.1.

**Proof of Theorem 2.** Recall that $\overline{\mathcal{R}}(E,\psi^s,\psi^b), \overline{\mathcal{V}}(E,\psi^s,\psi^b), \overline{\mathcal{Y}}(E,\psi^s,\psi^b)$ are respectively the optimal objective value to (EC.25), (EC.28) and (EC.30). To simplify the notations, we use $\overline{\mathcal{R}}(E)$, $\overline{\mathcal{V}}(E)$, $\overline{\mathcal{Y}}(E)$ to denote $\overline{\mathcal{R}}(E,\psi^s,\psi^b)$, $\overline{\mathcal{V}}(E,\psi^s,\psi^b)$, $\overline{\mathcal{Y}}(E,\psi^s,\psi^b)$. From Lemma EC.10 and EC.11, we have that $\overline{\mathcal{R}}(E)=\overline{\mathcal{V}}(E)=\overline{\mathcal{Y}}(E)$. Therefore, to prove the claim in this result, it is equivalent to focus on Problem (EC.30) and show that $\overline{\mathcal{Y}}(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.

We next consider Problem (EC.34) below with an additional constraint $F_b^{-1}(1-q_j^{1-\xi_b})-F_s^{-1}(\frac{q_j^{1-\xi_s}}{u_j^{1-\xi_s}})\geq r$ for some $r\in\mathbb{R}$ in comparison with Problem (EC.30). We then show that even the problem with this constraint can obtain the objective value weakly higher than $(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$, from which we can conclude that $\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. Given the edge set $\overline{E}$ of the complete graph, for any edge set $E\subset\overline{E}$, we define this auxiliary problem below

$$\mathcal{Y}^h(E) = \max_{\boldsymbol{w},r} \sum_{j\in\mathcal{B}}\left[(k_j^b)^{\frac{1}{1-\xi_b}}h\left(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}},r\right)\right] \tag{EC.34a}$$

$$\text{s.t.}\quad \sum_{j\in\widetilde{\mathcal{B}}}(w_j)^{\frac{1}{1-\xi_b}}\leq\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}},\qquad \forall\widetilde{\mathcal{B}}\subseteq\mathcal{B}, \tag{EC.34b}$$

$$w_j\geq 0,\qquad \forall j\in\mathcal{B} \tag{EC.34c}$$

$$r\leq\overline{v}_b, \tag{EC.34d}$$

where for any $u>0$,

$$h(u,r)=\max_{\substack{0\leq\widetilde{q}\leq\min\{1,u\},\\ F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right)\geq r}}\left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right)\right)\widetilde{q}. \tag{EC.34e}$$

Step 1: Show that $\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E)$. Note that the only difference between (EC.34) and (EC.30) is that one more constraint $F_b^{-1}(1-(\widetilde{q}_j)^{1-\xi_b})-F_s^{-1}(\frac{\widetilde{q}_j^{1-\xi_s}}{u^{1-\xi_s}})\geq r$ for any $(i,j)\in E$ is added to Problem (EC.34). With $r\leq\overline{v}_b$, we have that the constraint for the maximization problem in $(h,r)$ is non-empty given that solution $\widetilde{q}=0$ is feasible. Therefore, the solution to Problem (EC.34) is also feasible in Problem (EC.30), and two problems share the same objective functions. Thus, we have that

$$\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E).$$

Step 2: Show that $\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, we first reformulate the optimization problems for $\mathcal{Y}^h(E)$ and $\overline{\mathcal{Y}}(\overline{E})$.

Step 2.1: Reformulate the problem for $\mathcal{Y}^h(E)$. With $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j \in \mathcal{B}$, we define

$$\hat{q}_j(r, u_j) := \max\left\{\widetilde{q}: \ r \le F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\right), 0 \le \widetilde{q} \le \min\{1, u_j\}\right\}. \quad \text{(EC.35)}$$

Note that since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\right)$ strictly decreases in $\widetilde{q} \in [0, \min\{1, u_j\}]$, we know $\hat{q}_j(r, u_j)$ is unique given $(r, u_j)$. Given that $r$ is a lower bound of $F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\left(\frac{(\widetilde{q})^{1-\xi_s}}{(u_j)^{1-\xi_s}}\right)$ and $\hat{q}_j(r, u_j)$ is suboptimal to Problem (EC.34e), the optimal objective value $\mathcal{Y}^h(E)$ from Problem (EC.34e) is weakly higher than the optimal objective value of following optimization problem

$$\begin{aligned}
\max_{\boldsymbol{w}, r} \quad & \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r \hat{q}_j\left(r, \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\right) \\
\text{s.t.} \quad & \sum_{j \in \widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} \le \sum_{i \in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, && \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \\
& w_j \ge 0, && \forall j \in \mathcal{B}, \\
& r \le \overline{v}_b.
\end{aligned}$$

For any $r \in (-\infty, \overline{v}^b]$ and $\epsilon \in (0,1)$, we observe that $(w_j)^{\frac{1}{1-\xi_b}} = (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}}$ is feasible in the optimization problem above given that $w_j \ge 0$ for any $j \in \mathcal{B}$ and for any $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$,

$$\sum_{\in \widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} = \sum_{j \in \widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}} (1-\epsilon) \frac{\sum_{i' \in \mathcal{S}} (k_{i'}^s)^{\frac{1}{1-\xi_s}}}{\sum_{j' \in \mathcal{B}} (k_{j'}^b)^{\frac{1}{1-\xi_b}}} \le \sum_{i \in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}},$$

where the inequality follows directly from the condition in the theorem statement. By letting $\overline{u} := \frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that

$$\mathcal{Y}^h(E) \ \ge \ \max_{r \le \overline{v}_b} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r \hat{q}_j\left(r, (1-\epsilon)\frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}}\right) \ = \ \max_{r \le \overline{v}_b} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r \hat{q}_j\big(r, (1-\epsilon)\overline{u}\big).$$

Step 2.2: Reformulate the problem for $\overline{\mathcal{Y}}(\overline{E})$. We first show that given the graph set to the complete graph $G(\mathcal{S} \cup \mathcal{B}, \overline{E})$, the optimal solution to Problem (EC.30) satisfies $(w_{j'}^*)^{\frac{1}{1-\xi_b}} = (k_{j'}^b)^{\frac{1}{1-\xi_b}} \frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j' \in \mathcal{B}$. Given the definition of $(\mathcal{S}_\tau, \mathcal{B}_\tau)$ in (10), in a complete graph, we have that $\mathcal{B}_1 = \mathcal{B}$, as for any $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$, we have that

$$\frac{\sum_{i \in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(a)}{=} \frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}}} \overset{(b)}{\ge} \frac{\sum_{i \in \mathcal{S}} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i \in N_E(\mathcal{B})} (k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}}},$$

where Step (a) follows from the fact that network $G(\mathcal{S}\cup\mathcal{B},\overline{E})$ is complete; Step (b) follows from the condition that $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$. By Lemma EC.12, we have $\frac{(w^*_{j'})^{\frac{1}{1-\xi_b}}}{(k^b_{j'})^{\frac{1}{1-\xi_b}}}=\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Therefore, we can obtain that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right).$$

Similar to Step 2.1, given definition of $h(.)$ in (EC.29), we could reformulate $h(.)$ by defining that

$$\overline{q}:=\underset{\widetilde{q}\in[0,\min\{1,\overline{u}\}]}{\arg\max}\left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right)\right)\widetilde{q}, \tag{EC.36}$$

where we recall that we have set $\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}=\overline{u}$ in Step 2.1 above. By letting $\overline{r}:=F_b^{-1}(1-(\overline{q})^{1-\xi_b})-F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}})$, given definition of $h(.)$ in (EC.29), we have that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}\overline{r}\overline{q}.$$

Step 2.3: Establish that $\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, for any $j\in\mathcal{B}$, we want to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$.

By the definition of $\hat{q}_j(r,u)$ in (EC.35), we have that for any $j\in\mathcal{B}$,

$$\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}):=\max\left\{\widetilde{q}:\overline{r}\leq F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right),0\leq\widetilde{q}\leq\min\left\{1,(1-\epsilon)\overline{u}\right\}\right\}.$$

For simplicity of notations, we use $\hat{q}_j$ to denote $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})$. Since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b})-F_s^{-1}(\frac{(\widetilde{q})^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ decreases in $\widetilde{q}\in[0,\min\{1,(1-\epsilon)\overline{u}\}]$, we have that either $\overline{r}=F_b^{-1}(1-(\hat{q}_j)^{1-\xi_b})-F_s^{-1}(\frac{(\hat{q}_j)^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ or $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}$.

For any $j\in\mathcal{B}$, to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$, we consider the following two cases:

(1) if $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}$, then $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}\geq(1-\epsilon)\min\{1,\overline{u}\}=(1-\epsilon)\overline{q}$, where the last equality follows from the constraint in Problem (EC.36);

(2) if $\overline{r}=F_b^{-1}(1-\hat{q}_j^{1-\xi_b})-F_s^{-1}(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$, then based on the definition that $\overline{r}=F_b^{-1}(1-\overline{q}^{1-\xi_b})-F_s^{-1}(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}})$ in Step 2.2, we have that

$$F_b^{-1}(1-\overline{q}^{1-\xi_b})-F_s^{-1}\left(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right)=F_b^{-1}(1-\hat{q}_j^{1-\xi_b})-F_s^{-1}\left(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right).$$

Note that $F_b^{-1}(1-q^{1-\xi_b}) - F_s^{-1}(\frac{q^{1-\xi_s}}{u^{1-\xi_s}})$ strictly increases in $u \geq q \geq 0$ and strictly decreases in $q \in [0, \min\{1, u\}]$. With the equation above, given that $0 < (1-\epsilon)\overline{u} \leq \overline{u}$, we have that $\overline{q} \geq \hat{q}_j$, which further implies that $\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} \leq \frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}$. This allows us to establish that $\hat{q}_j^{1-\xi_s} \geq ((1-\epsilon)\overline{u})^{1-\xi_s} \frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} = (\overline{q})^{1-\xi_s}(1-\epsilon)^{1-\xi_s}$. Therefore, we have $\hat{q}_j \geq (1-\epsilon)\overline{q}$.

Summarizing the two cases above, we can establish that

$$\mathcal{Y}^h(E) \overset{(a)}{\geq} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} \overline{r} \hat{q}_j(\overline{r}, (1-\epsilon)\overline{u}) \overset{(b)}{=} \sum_{j \in \mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} \overline{r}(1-\epsilon)\overline{q} \overset{(c)}{=} (1-\epsilon)\overline{\mathcal{Y}}(\overline{E}),$$

where (a) follows from Step 2.1 and $\overline{r} = F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}}) \leq F_b^{-1}(1) = \overline{v}_b$; (b) follows from the observation that $\hat{q}_j(\overline{r}, (1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$ for any $j \in \mathcal{B}$; (c) follows directly from the reformulation in Step 2.2. ■