# Optimal Growth of a Two-Sided Platform with Heterogeneous Agents

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We consider the dynamics of a two-sided platform, where the agent population on both sides experiences growth over time with heterogeneous retention and new adoption. The compatibility between buyers and sellers is captured by a bipartite network. The platform strategically sets commissions to optimize its total profit over a horizon of $T$ periods, considering the trade-off between short-term profit and growth as well as the spatial imbalances in supply and demand. We design an asymptotically optimal commission policy with the profit loss upper-bounded by a constant independent of $T$, in contrast with a myopic policy that is shown to be arbitrarily bad even if the platform is a monopoly intermediary in the market. To obtain this policy, we first develop a single-period benchmark problem that captures the optimal steady state of the platform, then delicately control the growth of the agent types with the lowest relative population level compared with the single-period benchmark in each period over time. We further examine the impact of the growth potential and the market network structure on agents' payment/income, and the platform's optimal commissions and optimal profit. To achieve that, we introduce innovative metrics to quantify the long-run growth potential of each agent type. Using these metrics, we first show that for each agent type, higher ratios of their compatible counterparts' long-term growth potential to their own cause lower payments (higher income) at the optimal steady state. In addition, the impact of the relative long-run growth potential on the platform's optimal commissions in a submarket depends on the convexity/concavity of the value distribution function of agents. Finally, we show that a "balanced" network, where the relative long-run growth potential between sellers and buyers for all submarkets is the same as that for the entire market, allows the platform to achieve maximum profitability. The extent of network imbalance determines the lower bound for the achievable long-run average profit.

*Key words*: two-sided market, platform growth, market structure.

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## 1. Introduction

In recent years, the rapid growth of consumer-to-consumer (C2C) platforms such as Airbnb, eBay, and Upwork has transformed buyer-seller interactions. Their success relies on efficiently growing the agent base on both sides, which drives transaction volume and ultimately enhances platform profitability. Existing literature suggests that a pivotal strategy of the platform involves initially subsidizing agents to stimulate their growth and subsequently implementing charges to ensure long-term profitability (Lian and Van Ryzin 2021). Throughout this process, it is crucial to strike a balance between long-term growth and short-term profitability via a tailored commission structure. However, determining which agent type to subsidize or charge higher fees becomes challenging, particularly when considering the heterogeneity in their growth potentials and compatibility with other platform participants.

In general, the growth of an active agent base hinges on two primary factors: retaining previous agents and encouraging adoptions by new agents, the latter of which often relies on word-of-mouth communication between potential adopters and previous agents through online reviews or comments. Different agent types exhibit varying retention rates and word-of-mouth effects. For example, on Airbnb, tourists seeking vacation homes may have lower retention rates compared to regular business travelers due to infrequent revisits (Hamilton et al. 2017). However, they rely more on transaction histories and online reviews from previous guests when selecting properties in unfamiliar destinations (Arndt 1967, Sundaram and Webster 1999). Studies also indicate that the retention rates of drivers on Uber are heterogeneous, influenced by factors such as gender and age (Kooti et al. 2017). To deal with such heterogeneity, Meituan, a Chinese on-demand services app, charges its free users lower delivery fees than its paid members, who exhibit higher retention rates on the platform (Lee 2020). In summary, when encouraging the growth of the supply and demand base, platforms may need to tailor their commission structures based on the distinct retention rates and growth potentials across different agent segments.

Furthermore, based on previous works on the cross-side network effect of a two-sided market (Rochet and Tirole 2003, Eisenmann et al. 2006, Chu and Manchanda 2016), growth on one side of the market has a positive impact on the growth of the other side. However, the value contributed to the opposite side of the network differs across various agent types, as buyers and sellers are horizontally differentiated in terms of their "popularity" and preferences for agents on the other side of the market. This compatibility difference arises from varying tastes, geographical constraints, or skill mismatches (Birge et al. 2021). For instance, on Airbnb, listings located in popular tourist destinations tend to be more popular; on Upwork, freelancers who offer skills that match market demands and have flexible schedules tend to attract more companies. During the platform's phase of growth, the sluggish increase in the number of "marquee users", typically prominent buyers or high-profile suppliers, can impede the growth of their compatible counterparts, who face limited transaction opportunities and therefore lack the motivation to join the platform.

With the intricate interplay of *inter-temporal factors* marked by heterogeneous growth potentials and *spatial factors* characterized by the compatibility between agents, it becomes challenging for the platform to find an optimal commission policy to grow the agent base that maximizes its long-term profits. Furthermore, gaining insight into how both the inter-temporal and spatial factors affect the optimal policy and resulting agent-base sizes is of utmost importance. These are the two primary focal points of our study.

**Results and Contributions.** We consider a two-sided platform that charges commissions to sellers and buyers for facilitating their transactions. The compatibility between the buyers and sellers is captured by a bipartite graph, and the transaction quantities and prices between the agents are determined endogenously in a general equilibrium setting. In each period, the population of each agent type consists of two components: retained agents from the previous period and new adoptions, the number of which depends on the transaction quantity in the previous period. The platform determines the commissions in each period to maximize the total profit in $T$ periods, taking into account the trade-off between immediate revenue and the potential for future expansion. Our main findings are summarized as follows.

First, we formulate the platform's problem as a multi-period pricing optimization model with endogenous transactions between sellers and buyers, which, however, is challenging to solve due to its high-dimensional state space (determined by the sizes of different agent types and planning time interval). To overcome this challenge, we introduce a single-period problem, the solution of which captures the optimal steady state of the system. We show that the gap between $T$ times the optimal objective value of it and that of the original problem is upper bounded by a constant (see Proposition 2), and therefore we see it as a benchmark. We then develop an efficient policy that is shown to be asymptotic optimal (see Theorem 1). The policy focuses only on the scarcest agent base relative to the benchmark problem in each period, and controls its payment/income at the benchmark level to boost its growth. The demand/supply quantities of other types are matched correspondingly to guarantee feasibility. Our result provides managerial insights for platform growth: the key is not to boost the growth of the agent type with the lowest mass in each period. Instead, the platform should first identify the targeted level at which it can maintain and maximize its long-run average profit, and then guarantee the service level (e.g., by offering subsidies or lowering commissions) for the agent types that lag behind this targeted level in each period. In addition, it is perhaps worth pointing that even if the platform serves as a monopoly intermediary in the market, a myopic policy without considering the growth dynamics in the marketplace could be arbitrarily bad (see Proposition 3). This highlights the significance of growth dynamics in the platform's profit optimization problem.

Second, we focus on the platform's optimal steady state characterized by the single-period benchmark problem. We analyze how the growth potential of agent types (inter-temporal factor) and network structure (spacial factor) influence (1) the agents' payments/incomes, (2) the optimal commission, and (3) the platform's profit. We first develop a novel metric to capture the long-run growth potential of each agent type. Regarding (1), we show that the buyer (seller) type with a higher ratio of compatible sellers' (buyers') long-term growth potential to their own long-term growth potential experiences lower payments (higher income) at the optimal steady state (see Proposition 4). Based on this result, we conduct a sensitivity analysis to illustrate the impact of each agent type's long-term growth potential on its or others' income/payment (see Corollary 1). For (2), we show that the optimal commission charged from the submarket first decreases in the relative growth potential between sellers and buyers, and then increases (decreases) in it when the value distribution functions of both sides are convex (concave)(see Proposition 5). Regarding (3), we show that a "balanced" network, where the relative long-run growth potential of sellers and buyers for all submarkets are the same as that for the entire market, leads to maximum platform profitability (see Theorem 2). In contrast, the extent of the "imbalance" of the network in terms of the relative long-run growth potential between the two sides determines the lower bound of the optimal profit the platform can achieve. It is worth noting that the metric of "balances" developed in the previous literature under static settings fails to offer a profit guarantee in our dynamic setting. Our results suggest that the platform should strategically focus its marketing campaigns or loyalty programs on agents who exhibit relatively lower long-term growth potential compared with their compatible agents from the other side of the network.

**Organization of the Paper.** The rest of the paper is organized as follows. After reviewing the relevant literature in Section 2, we introduce the model and computational challenges in Section 3. In section 4, we design a heuristic algorithm with provably good performance. In section 5, we examine the impact of both the structure and intertemporal property on the agents' payments/incomes, the platform's optimal commissions and long-run average profit at the optimal steady state. The concluding remarks are drawn in Section 6.

Throughout the paper, we use "increasing" (and "decreasing") in a weak sense, i.e., meaning "weakly increasing" (and "weakly decreasing") unless otherwise specified. In addition, we use $\mathbb{R}_+$ to denote the non-negative real number set.

## 2. Literature Review

Pricing in two-sided platforms has been extensively studied in the field of Economics and Operations Management. Based on the seminal work by Caillaud and Jullien (2003), Rochet and Tirole (2003, 2006), Armstrong (2006), a growing literature has explored the pricing and matching problems in the context of online platforms (e.g., Hagiu 2009, Cachon et al. 2017, Taylor 2018, Bai et al. 2019, Benjaafar et al. 2019, Hu and Zhou 2020, Benjaafar et al. 2022, Cohen and Zhang 2022). Our work features network effects in a potentially incomplete two-sided market that evolves dynamically. Agents on one side of the market can only trade with a subset of agents on the other, and moreover, the platform's commissions influence the growth of the agent base in the market. Therefore, our work is closely related to the following two streams of literature: (i) the growth of a marketplace and (ii) pricing in a networked market.

Past literature about the growth of a marketplace mainly focused on product diffusion, which provides a model to forecast the growth of the customer base for a new product, see e.g., Bass (1969), Kalish (1985), Norton and Bass (1987). Based on these papers, more recent literature studies how to leverage discounts or investment incentives to influence the growth of new products (e.g., Shen et al. 2014, Ajorlou et al. 2018) and that of two-sided platforms (e.g., Kabra et al. 2016, Lian and Van Ryzin 2021, He and Goh 2022). Specifically, Lian and Van Ryzin (2021) considered a two-sided market in which the platform can subsidize one or both sides to boost their growth. They show that the optimal policy is to employ a subsidy shock to rapidly steer the market towards its optimal long-term size. He and Goh (2022) studied the dynamics of a hybrid workforce comprising on-demand freelancers and traditional employees, both capable of fulfilling customer demands. They investigated how demand should be allocated between employees and freelancers, and under what conditions the system is sustainable in the long run. Our study differs from this stream of work in that agents have heterogeneous compatibility and growth potentials, and the transaction quantities and prices are both formed endogenously in a general equilibrium in each period. This requires us to come up with a customized commission structure for different user types.

Our study is also closely related to the literature on networked markets (e.g., Kranton and Minehart 2001, Bimpikis et al. 2019, Baron et al. 2022, Zheng et al. 2023). In this line of work, the edges of the network capture the trading opportunities between agents, and the impacts of network effects on the market outcomes are analyzed. For example, Chen and Chen (2021) explored duopoly competition within a market involving network-connected buyers, and they show that the symmetry of market shares for two identical sellers in a Nash equilibrium depends on the intensity of network effects and the quality of the product. More closely, some recent studies explore how to improve operational efficiency in a two-sided networked market using central price controls (e.g., Banerjee et al. 2015, Ma et al. 2022, Varma et al. 2023) or non-pricing controls (e.g., Kanoria and Saban 2021). For example, Hu and Zhou (2022) consider a platform that strategically matches buyers and sellers, who are categorized into distinct groups based on varying arrival rates and matching values. They provide sufficient conditions under which the optimal matching policy follows a priority hierarchy among matching pairs, determined by factors such as quality and distance. Our work adopts the framework proposed by Birge et al. (2021), in which a platform determines commission structure to maximize the total profit in a networked market, and the trades and prices are formed endogenously in a competitive equilibrium given the commissions. Differently, we delve into a dynamic setting and demonstrate that utilizing metrics for network connectivity from static settings in prior literature to quantify the impact of network structure is inadequate. We introduce a novel metric that incorporates the intertemporal factor (i.e., the growth potentials of agents).

Some recent literature also explored the expansion of the platform's agent base in a network (e.g., Li et al. 2021, Alizamir et al. 2022). These studies typically assume a uniform retention and growth rate across agents from the same side or all agents in the network, with each agent's payoff determined by an exogenously specified function of the number of participants in the network. In contrast, we account for the heterogeneity of retention and growth rates among various agent types and introduce a novel metric that incorporates both spatial and intertemporal factors to assess the influence of the network structure on the platform's profitability.

## 3. Model

Consider a two-sided market in which a platform charges commissions to buyers and sellers for facilitating transactions. The compatibility between buyers and sellers is captured by a bipartite graph $(\mathcal{B} \cup \mathcal{S}, E)$, where $\mathcal{B} = \{1, 2, ..., N_b\}$ and $\mathcal{S} = \{1, 2, ..., N_s\}$. Here we denote the set of buyer types by $\mathcal{B}$ with $|\mathcal{B}| = N_b$ and that of seller types by $\mathcal{S}$ with $|\mathcal{S}| = N_s$, and let $E$ be the set of edges that captures the potential trading opportunities between them. Specifically, for $i \in \mathcal{S}$ and $j \in \mathcal{B}$, $(i, j) \in E$ if and only if the service or product of type-$i$ sellers can satisfy the demand of type-$j$ buyers. This compatibility difference arises from varying tastes, geographical constraints, or skill mismatch. In each period $t \in \{1, ..., T\}$, the total masses of type-$i$ sellers and type-$j$ buyers are respectively $s_i(t)$ for $i \in \mathcal{S}$ and $b_j(t)$ for $j \in \mathcal{B}$. Specifically, the initial population of each type is finite, i.e., $s_i(1) < \infty$ for $i \in \mathcal{S}$ and $b_j(1) < \infty$ for $j \in \mathcal{B}$.

The buyers/sellers are infinitesimal, and each one of them supplies/demands at most one unit of product/service in one period if they make transactions in the market. For $t \in \{1, ..., T\}$, we use $q_i^s(t)$ and $q_j^b(t)$ respectively to denote the aggregate supply quantities of type-$i$ sellers and the aggregate demand quantities of type-$j$ buyers, where $q_i^s(t) \in [0, s_i(t)]$ for $i \in \mathcal{S}$ and $q_j^b(t) \in [0, b_j(t)]$ for $j \in \mathcal{B}$. Note that given the commission charged by the platform $(\boldsymbol{r}^b(t), \boldsymbol{r}^s(t))$, the supply/demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is endogenously determined in equilibrium, with mechanism details discussed later.

**Population Transition.** A key feature of our model is that the total mass for each agent type evolves dynamically. For any $t \in \{1, ..., T-1\}$, we consider the following population transition equations:

$$s_i(t+1) = \alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t)), \qquad \forall i \in \mathcal{S} \tag{1a}$$

$$b_j(t+1) = \alpha_j^b b_j(t) + \mathcal{G}_j^b(q_j^b(t)), \qquad \forall j \in \mathcal{B}. \tag{1b}$$

In the first term, $\alpha_i^s \in (0, 1)$ and $\alpha_j^b \in (0, 1)$ respectively denote the retention rate of type-$i$ sellers and type-$j$ buyers, which are mainly determined by the agents' habits, their loyalty to the platform, etc. The assumption that the retention rate is exogenous and independent of the commission is commonly seen in the related literature, e.g., Lian and Van Ryzin

(2021), Alizamir et al. (2022), He and Goh (2022). Such population dynamics could coincide with a setting in which infinitesimal sellers and buyers stay in the system for a limited time and then exit with a certain probability. It can be commonly seen in practical examples such as Upwork and TaskRabbit, where the values of freelancers' outside options follow a specific value distribution.

In the second term, $\mathcal{G}_i^s(q_i^s(t))$ and $\mathcal{G}_j^b(q_j^b(t))$ capture the adoption of new agents, which depends on the supply/demand quantities in the last period. Our model is in line with the word-of-mouth effect or the imitation effect (see Bass 1969, Mahajan and Peterson 1985): current agents who transact and obtain a positive surplus can communicate the positive information about the platform to potential new adoptions, who are more likely to imitate their behaviors and be attracted to the platform. For the rest of the paper, $\mathcal{G}_i^s(\cdot)$ and $\mathcal{G}_j^b(\cdot)$ will be referred to as the *growth functions*. Our results can be generalized to the case where new adoptions depend on both supply/demand quantities and population in the previous period (i.e., $\mathcal{G}_i^s(q_i^s(t), s_i(t))$ and $\mathcal{G}_j^b(q_j^b(t), b_j(t))$) under some mild assumptions.

Some previous studies about the growth of two-sided platforms assume that the new adoption rates are homogeneous for agents from one side and depend on the transaction quantity/price/surplus in the last period (see Lian and Van Ryzin 2021, He and Goh 2022). Different from them, we assume that agent types from both sides are heterogeneous in terms of their retention rate and word-of-mouth effect. For example, tourists seeking vacation homes exhibit a lower retention rate but a higher word-of-mouth effect than regular business travelers, as discussed in Section 1. Another difference is that we do not assume a specific expression for the growth functions $\mathcal{G}_i^s(\cdot)$ and $\mathcal{G}_j^b(\cdot)$ for now, but impose the following regularity assumptions.

Assumption 1. *For any $i \in \mathcal{S}$ and any $j \in \mathcal{B}$, $\mathcal{G}_i^s(\cdot)$ and $\mathcal{G}_j^b(\cdot)$ satisfy the following conditions:*

*(i)* $\mathcal{G}_i^s(0) = 0$ *and* $\mathcal{G}_j^b(0) = 0$;

*(ii)* $\mathcal{G}_i^s(q)$ *and* $\mathcal{G}_j^b(q)$ *are continuously differentiable, increasing and strictly concave in* $q \in \mathbb{R}_+$, *and moreover,* $\lim_{q \to \infty} (\mathcal{G}_i^s)'(q) = 0$ *and* $\lim_{q \to \infty} (\mathcal{G}_j^b)'(q) = 0$.

Assumption 1(i) implies that if no transaction occurred in the previous period, then there is no word-of-mouth effect. Assumption 1(ii) requires that the number of new adoptions increases in the transaction quantity, but the marginal effect decreases as the transaction quantity increases. It also requires that the growth rate diminishes to zero as the quantity converges to infinity, which holds because the total amount of potential agents in a market is finite.

We next discuss how the equilibrium supply/demand $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ formed in a networked market given the commission charged by the platform in each period $t \in \{1, ..., T\}$.

**Competitive Equilibrium.** In period $t \in \{1, ..., T\}$, the platform charges commission $r_i^s(t)$ to type-$i$ sellers and $r_j^b(t)$ to type-$j$ buyers if they participate in transactions. The commissions are homogeneous within the same agent type but may vary across different types. When $r_i^s(t) < 0$ or $r_j^b(t) < 0$, the platform subsidizes the sellers or buyers. Given the commissions, type-$i$ sellers offer their products/service at price $p_i(t)$ and receive $p_i(t) - r_i^s(t)$; type-$j$ buyers pay $p_i(t) + r_j^b(t)$ if they trade with the type-$i$ sellers. The market prices $\boldsymbol{p}(t)$ are endogenously formed in equilibrium, given the commission charged by the platform (see Definition 1 later). This is widely observed across various online platforms. For instance, on Airbnb, hosts must compete on their rental offers, and on Upwork, freelancers compete on their hourly rates. We assume that sellers cannot charge different prices to different buyers, aligning with the standard practice of many online platforms, such as Airbnb and Upwork, where seller prices are openly displayed on the webpage. In addition, we assume for a type-$j$ buyer, all compatible sellers (i.e., $i : (i, j) \in E$) provide perfectly substitutable products/services, and the type-$j$ buyer does not have preference over the compatible sellers' products if their prices are the same. Similarly, it is indifferent for a seller to trade with any compatible buyers given that the market price is formed on the seller side. Note that vertical differentiation of sellers can be modeled by adding a quality term for each type of seller in the payoff function of buyers (see Birge et al. 2021). This does not fundamentally change our insights.

We use $F_{b_j} : [0, \overline{v}_{b_j}] \to [0, 1]$ and $F_{s_i} : [0, \overline{v}_{s_i}] \to [0, 1]$ to denote the cumulative distribution function of the (reservation) values respectively for type-$j$ buyers and type-$i$ sellers, in which $\overline{v}_{b_j}$ and $\overline{v}_{s_i}$ are finite for any $j \in \mathcal{B}$ and $i \in \mathcal{S}$. For simplicity, we refer to a seller by "he" and a buyer by "she". Then a type-$i$ seller only engages in trades when the amount he receives from the transaction is weakly higher than his reservation value $v$, i.e., $p_i(t) - r_i^s(t) \geq v$; similarly, a type-$j$ buyer trades when the total payment is weakly lower than her value, i.e., $p_i(t) + r_j^b(t) \leq v$. To simplify our analysis later, we extend the domains of the value distributions to $\mathbb{R}$: let $F_{b_j}(v) = 1$ for $v \geq \overline{v}_{b_j}$ and $F_{b_j}(v) = 0$ for $v \leq 0$ for any

$j \in \mathcal{B}$; similarly, for the seller side, we let $F_{s_i}(v) = 1$ for $v \geq \overline{v}_{s_i}$ and $F_{s_i}(v) = 0$ for $v \leq 0$ for any $i \in \mathcal{S}$. In addition, define $f_{b_j}(v)$ and $f_{s_i}(v)$ respectively as the derivative of $F_{b_j}(v)$ for $v \in [0, \overline{v}_{b_j}]$ and $F_{s_i}(v)$ for $v \in [0, \overline{v}_{s_i}]$ (or the density function of the valuations). We impose the following assumption regarding the value distributions throughout the paper.

ASSUMPTION 2. **(value distribution)** *For any $j \in \mathcal{B}$ and $i \in \mathcal{S}$,*

*(i) $F_{b_j}(v)$ and $F_{s_i}(v)$ are strictly increasing in $v \in [0, \overline{v}_{b_j}]$ and $v \in [0, \overline{v}_{s_i}]$, where $\overline{v}_{b_j} < \infty$ and $\overline{v}_{s_i} < \infty$;*

*(ii) $F_{b_j}(v)$ and $F_{s_i}(v)$ are continuously differentiable respectively in $v \in [0, \overline{v}_{b_j}]$ and $v \in [0, \overline{v}_{s_i}]$, and the density functions are lower bounded by a positive constant.*

Under Assumption 2(i), $F_{b_j}$ is invertible for $v \in [0, \overline{v}_{b_j}]$ and $F_{s_i}$ is invertible for $v \in [0, \overline{v}_{s_i}]$. We define the inverse function $F_{b_j}^{-1} : [0,1] \to [0, \overline{v}_{b_j}]$ and $F_{s_i}^{-1} : [0,1] \to [0, \overline{v}_{s_i}]$ such that $F_{b_j}^{-1}(F_{b_j}(v)) = v$ for $v \in [0, \overline{v}_{b_j}]$ and $F_{s_i}^{-1}(F_{s_i}(v)) = v$ for $v \in [0, \overline{v}_{s_i}]$. Moreover, Assumption 2(ii) ensures that any change in the (reservation) value induces a bounded change in the fraction of the population. Notice that under Assumption 2, $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$ are continuous and differentiable in $x \in [0,1]$, and their density functions are also bounded. We further impose the following Assumption on $F_{b_j}^{-1}(x)$ and $F_{s_i}^{-1}(x)$.

ASSUMPTION 3. **(concavity)** *$F_{b_j}^{-1}(1 - a/b)a$ and $-F_{s_i}^{-1}(a/b)a$ are both strictly concave in $(a,b)$ for $0 \leq a \leq b$.*

Assumptions 2 and 3 hold for many commonly used distributions such as uniform distribution, truncated exponential distribution, and truncated generalized Pareto distribution.

Define $x_{ij}(t)$ as the amount type-$j$ buyers purchase from type-$i$ sellers. Then given the commission vector $(r_i^s(t) : i \in \mathcal{S},\ r_j^b(t) : j \in \mathcal{B})$ by the platform and the population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$, a competitive equilibrium consists of a price vector $\boldsymbol{p}(t)$, a supply/demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ and a flow vector $\boldsymbol{x}(t)$ that satisfy the following equilibrium conditions.

DEFINITION 1. **(competitive equilibrium)** In period $t \in \{1, \ldots, T\}$, given the platform's commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ and the population vector of sellers and buyers $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$, a competitive equilibrium is defined as the price-flow vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ that satisfies the following conditions:

$$q_i^s(t) = s_i(t) F_{s_i}(p_i(t) - r_i^s(t)), \qquad \forall i \in \mathcal{S}, \tag{2a}$$

$$q_j^b(t) = b_j(t) \Big(1 - F_{b_j}\big(\min_{i' : (i',j) \in E} \{p_{i'}(t)\} + r_j^b(t)\big)\Big), \qquad \forall j \in \mathcal{B}, \tag{2b}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}, \tag{2c}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{2d}$$

$$x_{ij}(t) \geq 0, \qquad \forall (i,j) \in E, \tag{2e}$$

$$x_{ij}(t) = 0, \qquad \forall i \notin \arg\min_{i':(i',j)\in E} \{p_{i'}\}, \;\; j \in \mathcal{B}. \tag{2f}$$

In Definition 1, Conditions (2a) and (2b) ensure that the total supply/demand quantities of type-$i$ sellers and type-$j$ buyers equal the mass of agents who can obtain positive utilities from the transaction. Specifically, Condition (2b) assumes that type-$j$ buyers only trade with compatible sellers with the lowest market price to maximize their utilities. Conditions (2c) and (2d) characterize the flow conservation conditions in the networked market. Finally, Condition (2e) requires that the transaction flow is non-negative, and Condition (2f) requires that the buyers only trade with compatible sellers with the lowest prices. Notice that equilibrium concepts similar to Definition 1 have also been adopted in the two-sided market literature by, e.g., Weyl (2010) and Birge et al. (2021). In our setting, the buyers and sellers are myopic, i.e., the demand/supply quantities only depend on the prices and commissions in the current period. This assumption is commonly seen in the literature about dynamic pricing in the monopoly or duopoly competition setting, e.g., Chen and Gallego (2019), Birge et al. (2023).

With Definition 1, the following result characterizes the existence of a competitive equilibrium in each period given any commission profile and the total mass of agents. Moreover, the result shows that the equilibrium is essentially unique.

PROPOSITION 1. **(existence and uniqueness of equilibrium)** *For any* $t \in \{1, ..., T\}$, *given a commission profile* $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_b}$ *and the total mass of agents* $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}_+^{N_s} \times \mathbb{R}_+^{N_b}$,

*(i) a competitive equilibrium* $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ *always exists;*

*(ii) all competitive equilibria share the same supply-demand vector* $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$*, and they share the same prices* $p_i(t)$ *for* $0 < q_i^s(t) < s_i(t)$ *.*

Proposition 1(ii) implies that the supply-demand vector $(\boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ is always unique in an equilibrium. Moreover, given the platform's commissions and the total mass of agents in period $t$, the equilibrium prices are not necessarily unique for type-$i$ seller with $q_i^s(t) = 0$, as any price lower than $r_i^s(t)$ leads to a non-positive income of $p_i(t)-r_i^s(t)$, and therefore no type-$i$ sellers would engage in transactions; similarly, the equilibrium prices are not necessarily unique for type-$i$ sellers with $q_i^s(t)=s_i(t)$, as any equilibrium price higher than $\overline{v}_i^s(t)+r_i^s(t)$ results in income higher than the maximum reservation value $\overline{v}_i^s$, and therefore all type-$i$ sellers are willing to supply their service/goods. Other than these cases, the equilibrium prices are essentially unique in general.

**Platform's Profit Optimization Problem.** Given the mass of agents with different types in the first period (i.e., $(\boldsymbol{s}(1),\boldsymbol{b}(1))$), the platform aims at maximizing its total T-period profit by determining the commission for each type in a period. For simplicity of notation, we let $(\boldsymbol{s},\boldsymbol{b}):=(\boldsymbol{s}(t),\boldsymbol{b}(t))_{t=2}^T$, and $(\boldsymbol{r},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}):=(\boldsymbol{r}(t),\boldsymbol{p}(t),\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t))_{t=1}^T$. The platform's T-period profit maximization problem can be expressed as:

$$\mathcal{R}^*(T)=\max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{r},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}}\sum_{t=1}^{T}\left[\sum_{i\in\mathcal{S}}r_i^s(t)q_i^s(t)+\sum_{j\in\mathcal{B}}r_j^b(t)q_j^b(t)\right] \tag{3a}$$

$$\text{s.t.}\quad (\boldsymbol{s}(t),\boldsymbol{b}(t),\boldsymbol{r}(t),\boldsymbol{p}(t),\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t)) \text{ satisfies (2)},\qquad \forall t\in\{1,...,T\}, \tag{3b}$$

$$s_i(t+1)=\alpha_i^s s_i(t)+\mathcal{G}_i^s(q_i^s(t)),\qquad \forall i\in\mathcal{S},t\in\{1,...,T-1\}, \tag{3c}$$

$$b_j(t+1)=\alpha_j^b b_j(t)+\mathcal{G}_j^b(q_j^b(t)),\qquad \forall j\in\mathcal{B},t\in\{1,...,T-1\}. \tag{3d}$$

The platform's profit consists of commissions from the sellers and buyers who trade in the market during the $T$ periods. Constraint (3b) ensures that given the population vector $(\boldsymbol{s}(t),\boldsymbol{b}(t))$ and commission profile $(\boldsymbol{r}^s(t),\boldsymbol{r}^b(t))$ in period $t$, vector $(\boldsymbol{p}(t),\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t))$ constitutes a competitive equilibrium; Constraints (3c)-(3d) indicate that the dynamics of populations follow the transition equations given in (1). Based on Proposition 1(ii), the equilibrium transaction quantities $(\boldsymbol{q^s}(t),\boldsymbol{q^b}(t))_{t=1}^T$ are unique given any commission $(\boldsymbol{r}^s(t),\boldsymbol{r}^b(t))_{t=1}^T$. Therefore, the maximization problem in (3) is well-defined. In the rest of the paper, we refer to Problem (3) as OPT. Since OPT is non-convex (in $(\boldsymbol{r},\boldsymbol{q})$), we will first reformulate it into a convex optimization problem and then discuss the challenges in solving it.

**Reformulation and Challenges.** Given positive population vector $(\boldsymbol{s}(t),\boldsymbol{b}(t))$ and any feasible trading vector $(\boldsymbol{x}(t),\boldsymbol{q}^s(t),\boldsymbol{q}^b(t))$ in period $t$, we first characterize a feasible commission profile $(\boldsymbol{r}^s(t),\boldsymbol{r}^b(t))$ that can induce this equilibrium trades in the following Lemma.

Lemma 1. **(commissions for feasible transactions)** *For any $t \in \{1, ..., T\}$, given any positive population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ and non-negative trading vector $(\boldsymbol{x}(t), \boldsymbol{q}^{\boldsymbol{s}}(t), \boldsymbol{q}^{\boldsymbol{b}}(t))$ that satisfy (i) the flow conservation conditions in (2c)-(2e) and (ii) $\boldsymbol{q}^s(t) \leq \boldsymbol{s}(t)$ and $\boldsymbol{q}^b(t) \leq \boldsymbol{b}(t)$, a commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ supports $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{x}(t), \boldsymbol{q}^{\boldsymbol{s}}(t), \boldsymbol{q}^{\boldsymbol{b}}(t))$ in a competitive equilibrium if there exists a price vector $\boldsymbol{p}(t) \in \mathbb{R}^{N_s}$ that satisfies the following system of linear inequalities:*

$$p_i(t) - r_i^s(t) \; = \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; > \; 0, \tag{4a}$$

$$p_i(t) - r_i^s(t) \; \leq \; F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big), \qquad \forall i : q_i^s(t) \; = \; 0, \tag{4b}$$

$$p_i(t) + r_j^b(t) \; = \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; > \; 0, \tag{4c}$$

$$p_i(t) + r_j^b(t) \; \geq \; F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big), \qquad \forall (i,j) : x_{ij}(t) \; = \; 0. \tag{4d}$$

Note that the lowest value of a type-$j$ buyer that participates in trading can be expressed as $F_{b_j}^{-1}\big(1 - \frac{q_j^b(t)}{b_j(t)}\big)$; the highest reservation value of a type-$i$ seller that participates in trading can be expressed as $F_{s_i}^{-1}\big(\frac{q_i^s(t)}{s_i(t)}\big)$. We refer to these values as the *marginal value* of the corresponding agent type. The conditions in (4a)-(4b) characterize the relationship between the sellers' income and the marginal value, and the conditions in (4c)-(4d) characterize the relationship between the buyers' payment and the marginal value. In specific, (4a) and (4c) show that the buyers' payments and sellers' incomes per unit equal the marginal value for types with positive transactions. Moreover, we construct a class of commissions that satisfy (4) in the proof of Lemma 1 to demonstrate that the conditions in (4) are non-empty. The commissions that can induce the equilibrium are not necessarily unique, but the buyers' payments and sellers' incomes with positive trades are uniquely determined in any equilibrium.

The relationship between the commissions and the transaction quantities in Lemma 1 shows that the total payment from type-$j$ buyers is upper bounded by $F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)})q_j^b(t)$ and the total income for type-$i$ sellers is lower bounded by $F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)$. As a result, the objective value of OPT is upper bounded by the difference between these two terms, i.e., $\sum_{t=1}^T \Big[\sum_{j \in \mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)q_j^b(t) - \sum_{i \in \mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)q_i^s(t)\Big]$, which is concave in $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{b})$ under Assumption 3. By further relaxing some constraints of OPT, we can obtain a convex optimization problem where the decision variables only consist of population vector $(\boldsymbol{s}, \boldsymbol{b})$, supply/demand profile $(\boldsymbol{q}^s, \boldsymbol{q}^b)$, and flow vector $\boldsymbol{x}$ but not commission $(\boldsymbol{r}^s, \boldsymbol{r}^b)$. We present the formulation in Problem (22) in Appendix A. In addition, we show that the relaxation is tight in Proposition 6 in Appendix A.

Even though the non-convexity challenge of OPT can be circumvented by the reformulation, it is still computationally intractable when $T$ is large due to its high-dimensional state space (i.e., larger than $T \times (2N_s + 2N_b + |E|)$). Moreover, it would be hard to derive structural properties of the optimal policies in this deterministic dynamic program given the cross-side complementarity and same-side substitution. In Section 4 below, we propose a single-period convex problem, which returns the long-run average value of OPT; based on the optimal solution of this benchmark, we design a simple policy with provable performance guarantees.

## 4. Asymptotically Optimal Policy for the Platform

We define an admissible policy as a sequence of functions $\pi =: \{\pi_t : \mathcal{F}_t \to \mathcal{R}^{N_s+N_b}\}_{t=1}^T$ that outputs the commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in each period, where $\mathcal{F}_t$ is the history of population vectors $(\boldsymbol{s}(t'), \boldsymbol{b}(t') : t' = 1, \ldots, t)$ and transaction vectors $(\boldsymbol{x}(t'), \boldsymbol{q}^s(t'), \boldsymbol{q}^b(t') : t' = 1, \ldots, t-1)$ up to the current period. Once the commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ is implemented in each period, the price-transaction vector $(\boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ is formed in the market equilibrium (Proposition 1) and the population profiles $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ will be uniquely updated according to (1). We let $\Pi$ be the set of admissible policies.

For any policy $\pi \in \Pi$, we define $\mathcal{R}^\pi(T)$ as the platform's total profit in $T$ periods, and we evaluate the policy's performance by quantifying its profit loss relative to the optimal objective value $\mathcal{R}^*(T)$ in OPT, which can be formally defined as

$$\mathcal{L}^\pi(T) \;=\; \mathcal{R}^*(T) - \mathcal{R}^\pi(T). \tag{5}$$

We focus on $\mathcal{L}^\pi(T)$ in the asymptotic setting as $T \to \infty$, and devise an admissible policy with a constant upper bound for $\mathcal{L}^\pi(T)$.

Flynn (1978) studies heuristic policies for solving infinite-horizon deterministic dynamic programming problems. He provides the necessary and sufficient conditions for the existence and asymptotic optimality of "steady-state policy", which involves solving a static problem to identify the optimal steady state, moving the system to this state, and maintaining it there. Our algorithm will share a similar spirit. Moreover, Flynn (1975, 1981) also provide examples of constructing feasible rules that move the system from the initial state to the target steady state, mostly by implementing the action at the optimal steady state from the beginning or making straightforward modifications on it. However, we will see that these methods cannot be applied to our setting due to the flow conservation constraints of the competitive equilibrium in the networked market (i.e., Definition 1). To resolve this challenge, we propose a novel method called the Target-Ratio Policy (TRP) that only steers the growth of the scarcest agents in the networked market towards optimality in each period. Interestingly, we establish that such a policy can indeed achieve asymptotic optimality. On the other hand, we will show that the widely-adopted myopic policy, under which the platform completely neglects population growth, could perform arbitrarily badly in general.

**Long-run Average Value Problem (AVG).** Based on the reformulation of OPT, we first develop a corresponding steady-state problem, which serves as a key benchmark for our algorithm. Based on Definition 1, we relax the equilibrium conditions in Constraint (2) as below:

$$q_i^s \le s_i, \sum_{j:(i,j)\in E} x_{ij} = q_i^s, \qquad \forall i \in \mathcal{S}, \tag{6a}$$

$$q_j^b \le b_j, \sum_{j:(i,j)\in E} x_{ij} = q_i^s, \qquad \forall j \in \mathcal{B}, \tag{6b}$$

$$x_{ij} \ge 0, \qquad \forall (i,j) \in E. \tag{6c}$$

Furthermore, we relax the population transition equations in Constraint (1) to inequalities:

$$s_i \le \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s), \qquad \forall i \in \mathcal{S}, \tag{7a}$$

$$b_j \le \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b), \qquad \forall j \in \mathcal{B}. \tag{7b}$$

Define $\widetilde{F}_{b_j}(q_j^b, b_j) := F_{b_j}^{-1}\big(1 - \frac{q_j^b}{b_j}\big) q_j^b$ for $b_j > 0$ and $0 \le q_j^b \le b_j$ and $\widetilde{F}_{b_j}(q_j^b, b_j) := 0$ for $q_j^b = b_j = 0$. Similarly, define $\widetilde{F}_{s_i}(q_i^s, s_i) := F_{s_i}^{-1}\big(\frac{q_i^s}{s_i}\big) q_i^s$ for $s_i > 0$ and $0 \le q_i^s \le s_i$ and $\widetilde{F}_{s_i}(q_i^s, s_i) := 0$ for $q_i^s = s_i = 0$. We show that $\widetilde{F}_{b_j}(q, b)$ and $\widetilde{F}_{s_i}(q, s)$ are both continuous at $(0, 0)$ (see Lemma 6 in Appendix B). Then we consider the following optimization problem which we refer to as AVG:

$$\overline{\mathcal{R}} = \max_{\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \quad \sum_{j \in \mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i \in \mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i), \tag{8a}$$

$$\text{s.t.} \quad (6)-(7). \tag{8b}$$

It is worth noting that the feasible region for Problem (8) is a convex set, and thanks to Assumption 1, the objective function is concave. Before presenting our policy, we characterize the properties of the optimal solution to AVG:

LEMMA 2. **(optimal solution to AVG)** *The optimal solution to Problem* (8) *exists, and moreover,*

*(i) the optimal population* $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ *and the optimal supply-demand vector* $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ *are unique;*

*(ii) constraints* $s_i \leq \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ *and* $b_j \leq \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ *are tight at optimality.*

Lemma 2(i) suggests that for any optimal solution to AVG, the sub-component $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ is always unique. We call $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$ the *optimal steady state* in the rest of the paper (see Flynn 1975, 1992). Lemma 2(ii) implies that it is feasible to preserve the mass of agents at $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ by controlling the supply-demand vector at the level of $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b)$. This property further suggests that the platform could achieve the long-run average optimal profit in AVG.

We establish the following proposition to show that the gap between $T$ times of the optimal objective value $\overline{\mathcal{R}}$ of AVG from (8) and that of OPT from (3) is bounded from above by a constant for any positive integer $T$.

PROPOSITION 2. **(alternative benchmark for OPT)** *There exists a positive constant* $C_1$ *such that for any* $T = 1, 2, \ldots,$

$$|\mathcal{R}^*(T) - T\overline{\mathcal{R}}| \leq C_1.$$

Proposition 2 shows that the difference between $\frac{1}{T}\mathcal{R}^*(T)$ and $\overline{\mathcal{R}}$ converges to zero as $T$ approaches infinity. In addition, in contrast to the high-dimensional problem OPT, AVG is a much more tractable static convex optimization problem. Therefore, we will consider $T\overline{\mathcal{R}}$, instead of $\mathcal{R}^*(T)$, as the benchmark for our algorithm design.

Next, we propose the *Target Ratio Policy (TRP)* that admits fast convergence to the steady-state solutions to AVG and formally establish its asymptotic optimality.

**Target Ratio Policy (TRP).** For simplicity of illustration, we refer to $\frac{s_i(t)}{\bar{s}_i}$ for $i \in \mathcal{S}$ and $\frac{b_j(t)}{\bar{b}_j}$ for $j \in \mathcal{B}$ as the *population ratio* of seller type $i$ and buyer type $j$, respectively. In addition, we notice $\frac{q_j^b(t)}{b_j(t)}$ $\left(\frac{q_i^s(t)}{s_i(t)}\right)$ is the fraction of type-$j$ buyers (type-$i$ sellers) who trade on the platform in period $t$. We refer to this fraction as the *service level* of the corresponding agent type. Recall that the service level also determines the payment/income of agents.

Motivated by Proposition 2, we design our approximation algorithm to steer the mass of each agent type towards the optimal steady-state $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ in the network. A straightforward method is to control the service level of each type at the optimal service levels of AVG, i.e., $\frac{q_i^s(t)}{s_i(t)} \approx \frac{\overline{q}_i^s}{\overline{s}_i}$ for any $i \in \mathcal{S}$ and $\frac{q_j^b(t)}{b_j(t)} \approx \frac{\overline{q}_j^b}{\overline{b}_j}$ for any $j \in \mathcal{B}$ for $t \in \{1, ..., T\}$. This is also equivalent to controlling the income/payment of each agent type at the value in the optimal steady state given by AVG. However, the main challenge is that such a policy is not necessarily feasible for every period in a networked market given the heterogeneous population ratios among different seller and buyer types. For example, consider a simple scenario of one buyer type and one seller type with a positive initial mass vector $(s(1), b(1))$. Given the flow conservation constraint $q^s(t) = q^b(t)$ for all $t \in \{1, ..., T\}$, if we control the supply quantity $q^s(1)$ such that the service level of the supply side is the same as that from AVG (i.e., $\frac{q^s(1)}{s(1)} = \frac{\overline{q}^s}{\overline{s}}$), the service level for buyers in the first period may be significantly different from the optimal service level of AVG in general (i.e., the gap between $\frac{q^b(1)}{b(1)}$ and $\frac{\overline{q}^b}{\overline{b}}$ may be large). In particular, the type with a lower population ratio will limit the transaction quantity of the other type with a higher ratio, which further restricts its growth. To circumvent this challenge, we focus on the type with the lowest population ratio and seek to boost their growth in each period, while guaranteeing the feasibility of the policy in the entire networked market. Towards this direction, we formally define the Target Ratio Policy in Algorithm 1.

One key advantage of TRP is its computational efficiency: it only requires solving the single-period optimization AVG once. In Algorithm 1, TRP first identifies the agent types with strictly positive populations in AVG (for all $j$ such that $\overline{b}_j > 0$ and $i$ such that $\overline{s}_i > 0$). The types with zero population at the optimal steady state either have low growth potential, or are located at less important positions in the network such that the platform should de-prioritize their growth from the very beginning. Among the agent types with positive population masses in AVG, TRP finds the one with the lowest population ratio $m(t)$ in each period $t$ where $m(t) = \min\{\min_{i:\overline{s}_i>0} \frac{s_i(t)}{\overline{s}_i}, \min_{j:\overline{b}_j>0} \frac{b_j(t)}{\overline{b}_j}\}$, and matches its service level to the optimal one from AVG (i.e., $\frac{q_i^s(t)}{s_i(t)} = \frac{\overline{q}_i^s}{\overline{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} = \frac{\overline{q}_j^b}{\overline{b}_j}$). For other types with higher population ratios, their demand/supply quantities are matched correspondingly to guarantee feasibility in the networked market. By Lemma 1, we can always find the commissions to induce our desired transaction quantities in each period by solving a system of linear inequalities.

**Algorithm 1:** Target Ratio Policy

**Input:** Optimal solution to AVG ($\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b, \overline{\boldsymbol{x}}$) and initial mass of agents ($\boldsymbol{s}(1), \boldsymbol{b}(1)$).
**for** $t = 1$ *to* $T$ **do**
  $m(t) \leftarrow \min\left\{ \min_{i:\overline{s}_i>0} \frac{s_i(t)}{\overline{s}_i}, \min_{j:\overline{b}_j>0} \frac{b_j(t)}{\overline{b}_j} \right\}$;
  **for** $i = 1$ *to* $N_s$ **do**
    $q_i^s(t) \leftarrow \overline{q}_i^s m(t)$ ;
  **end**
  **for** $j = 1$ *to* $N_b$ **do**
    $q_j^b(t) \leftarrow \overline{q}_j^b m(t)$;
  **end**
  **for** $(i, j) \in E$ **do**
    $x_{ij}(t) \leftarrow \overline{x}_{ij} m(t)$;
  **end**
  Solve (4) to obtain ($\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)$);
  if there are multiple feasible solution sets for (4), select one arbitrarily;
  population profile ($\boldsymbol{s}(t+1), \boldsymbol{b}(t+1)$) is returned by the system dynamics in (1).
**end**
**Output:** ($\boldsymbol{r}^s, \boldsymbol{r}^b$).

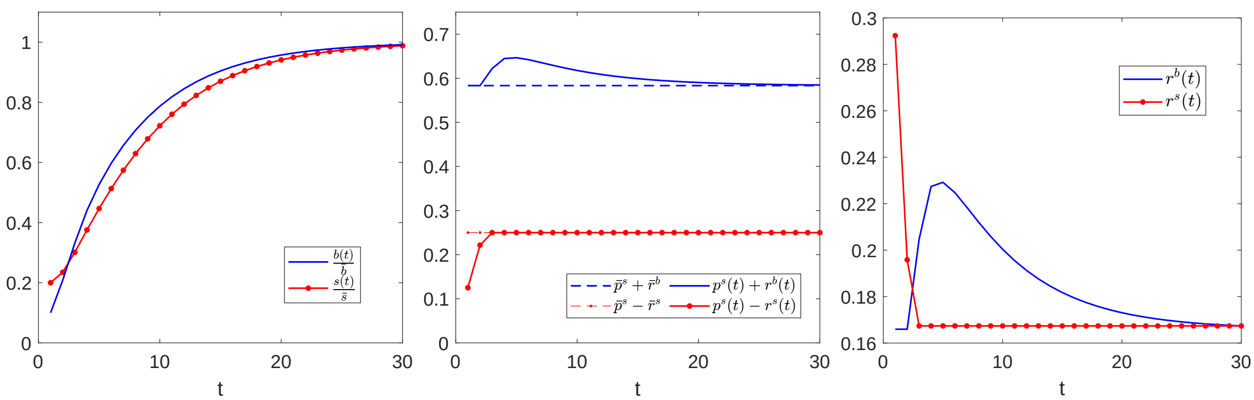
Note that for those agent types with higher initial population ratios, the service level will be lower than that of AVG (i.e., $\frac{q_i^s(t)}{s_i(t)} < \frac{\overline{q}_i^s}{\overline{s}_i}$ or $\frac{q_j^b(t)}{b_j(t)} < \frac{\overline{q}_j^b}{\overline{b}_j}$). Their populations may grow slowly, or even decline at the beginning. As a result, the agent type with the lowest population ratio may change over time in the network. Therefore, the platform may focus on boosting the growth of different types across the planning horizon. A key insight of our study is that perhaps surprisingly, by guaranteeing the growth of the agent types with the *lowest* population ratio in the network in each period, the entire network could eventually converge to the optimal state of AVG. As the main result of this section, we prove a theoretical performance guarantee for TRP.

THEOREM 1. **(performance of TRP)** *There exists a positive constant $C_2$ such that*

$$\mathcal{L}^{TR}(T) \leq C_2,$$

*for any $T = 1, 2 \ldots$.*

Theorem 1 shows that the profit loss of TRP relative to the optimal policy is uniformly bounded (with respect to $T$) by a constant, which further suggests that boosting the growth of the agent type with the lowest population ratio in each period is not only feasible but also asymptotically optimal in the networked market. The proof of Theorem 1 is relegated to Appendix B. In this proof, we show that under TRP, even though the type with the lowest ratio may change over time, the lowest ratio $m(t)$ *monotonically* converges to one at an exponential rate, i.e., $|m(t+1)-1| \leq \gamma |m(t)-1|$ for some $\gamma \in (0,1)$. Therefore, for each type, the transaction quantities $q_i^s(t) = \overline{q}_i^s m(t)$ and $q_j^b(t) = \overline{q}_j^b m(t)$ converge to the optimal levels $\overline{q}_i^s$ and $\overline{q}_j^b$ for any $i \in \mathcal{S}$ and $j \in \mathcal{B}$, which ensures that the population profile $(\boldsymbol{s}(t), \boldsymbol{b}(t))$ also converges to the optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG. By establishing the fast convergence rate, we observe that there exists a constant $C_1'$ such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$. Together with the result in Proposition 2 that $|\mathcal{R}^*(T) - T\overline{\mathcal{R}}| \leq C_1$, we conclude that there exists a constant $C_2 := C_1 + C_1'$ such that $|\mathcal{R}^*(T) - \mathcal{R}^{TR}(T)| \leq C_2$. Note that our result can be generalized into the case where the mass of new adoptions for the seller side is given by $\mathcal{G}_i^s(q_i^s(t), s_i(t))$ for $i \in \mathcal{S}$ where $\mathcal{G}_i^s(q_i^s(t), s_i(t))$ is concave in $(q_i^s(t), s_i(t))$ and the similar form holds for the buyer side.




(a) Population Ratio (b) Payment/income (c) Commission

**Figure 1 TRP. Consider the following parameters:** $\frac{s(1)}{\bar{s}} = 0.2$ **and** $\frac{b(1)}{\bar{b}} = 0.1$ **and** $s(t+1) = 0.5s(t) + 0.5\sqrt{q^s(t)}$ **and** $b(t+1) = 0.5b(t) + 0.7\sqrt{q^b(t)}$.

Figure 1 illustrates the evolution of agent populations, buyers' payment, sellers' income, and platform's commissions in a one-seller-one-buyer network (and so we can drop the subscripts) when the platform applies TRP. The seller side is assumed to have a higher initial population ratio (i.e., $\frac{s(1)}{\bar{s}} > \frac{b(1)}{\bar{b}}$), but a lower growth rate than the buyer side (i.e., $\mathcal{G}^{s\prime}(\cdot) \leq \mathcal{G}^{b\prime}(\cdot)$). Figure 1(a) shows that the population ratios of the seller and the buyer converge to 1. Note that the buyer's population ratio is initially lower but surpasses the seller's afterward. Consequently, in Figure 1(b), under TRP, the platform keeps the buyers' payments at the steady-state level to stimulate their growth in the first few periods and increases their payments above the steady-state level once the buyers' population ratio becomes higher. Conversely, the sellers' incomes start lower than the steady-state value and gradually increase as the seller population ratio decreases. Recall that the equilibrium prices can be any value between the payment of buyers and the incomes of sellers, which is not necessarily unique (see Lemma 1). Without loss of optimality, we set the equilibrium prices at the average of buyer payment and seller income in each period, and observe in Figure 1(c) that the commission charged from the seller side is higher initially, while that charged from the buyer side becomes higher at the later stages. If we consider an alternative scenario with a lower initial population ratio and a higher growth coefficient on the seller side (i.e., $\mathcal{G}^{s\prime}(\cdot)$ is large), we can observe an opposite trend under TRP, as detailed in Appendix B.

**Myopic Policy (MP).** When facing a high-dimensional dynamic program, MP is often used as a heuristic due to its simplicity. Moreover, as a monopoly intermediary, one may wonder if it is without loss of optimality for the platform to implement the myopic policy in the market. Note that Robinson and Lakhani (1975) and Bass and Bultez (1982) both examine the performance of the myopic pricing policy in a product diffusion process, but they draw different results under different diffusion functions. Specifically, Robinson and Lakhani (1975) show that myopic policy results in significant profit loss relative to the optimal policy when the current price could stimulate future demand. In contrast, Bass and Bultez (1982) consider the case that the diffusion process is exogenous, and does not interact with price. They show by a numerical study that there is only a small difference in the discounted profits between the myopic policy and the optimal policy. Here we will examine how MP performs in our model.

Under MP, in each period $t$, the platform determines the commissions $(\boldsymbol{r}^b(t), \boldsymbol{r}^s(t))$ to maximize its profit in the current period (i.e., $\sum_{i\in\mathcal{S}} r_i^s(t)q_i^s(t) + \sum_{j\in\mathcal{B}} r_j^b(t)q_j^b(t)$) subject to the equilibrium constraints in (2), without considering the population dynamics in (1) and its impact on future profit. The formal definition of the myopic policy is given by Definition 2 in Appendix B.

To investigate the performance of MP in our setting, we let $\mathcal{R}^M(t)$ denote the platform's profit under MP in period $t$. Recall that $\overline{\mathcal{R}}$ is the optimal objective value to AVG, which could be achieved under TRP. The following result shows that the performance of MP could be arbitrarily bad.

Proposition 3. **(performance of MP)** *Under MP, for any* $\epsilon > 0$*, there exists a problem instance such that* $\lim_{t\to\infty} \mathcal{R}^M(t) := \overline{\mathcal{R}}^M < \infty$ *and* $\overline{\mathcal{R}}^M < \epsilon\overline{\mathcal{R}}$*, and, hence, there exists* $C_3 > 0$ *such that*

$$\mathcal{L}^{MP}(T) \geq C_3 T.$$

Proposition 3 suggests that ignoring the commissions' impact on population growth could lead to significant profit loss for the platform even if the platform serves as a monopoly intermediary. This result also implies that the profit loss $\mathcal{L}^{MP}(T) = \Omega(T)$. In the proof of Proposition 3, we show that the commissions by the platform under MP at the optimal steady state are higher than those under TRP. Therefore, the platform must sacrifice some short-term margin to achieve long-term profitability.

Since we mentioned that in each period, the service level of the agent type (or equivalently, their payment/income) that lags behind should be controlled at its service level in the optimal static state, we will next focus on the optimal steady state and investigate how service level in the optimal static state is determined by both the network structure $G(\mathcal{S}\cup\mathcal{B}, E)$ and population dynamics from (1).

## 5. Impact of Population Dynamics and Network Structure

In this section, we investigate how the population dynamics and network structure influence the optimal commission and incomes/payments of agents (see Section 5.1) as well as the platform's profit (see Section 5.2) in the optimal steady state. Investigating the impacts of these spatial-temporal factors can provide insights into the platform's revenue management strategy.

Past literature has shown that in a static setting, a network with higher connectivity enables the platform to more efficiently match supply with demand, and the agent types connected to a larger population on the other side would gain higher surplus (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021). However, in our dynamic setting, the steady-state population size of each type is determined endogenously, relying on factors such as the growth potential of agents, the network structure, and the commissions set by the platform. In addition, we will later show in an example in Figure 3 that the metric used in Birge et al. (2021) to quantify the connectivity of the network in the static setting fails to quantify the profitability of the platform in the dynamic setting. Therefore, in this section, we first develop a novel metric to measure both the growth potential and connectivity of the agent types. We then use this metric to characterize the impact of the temporal-spatial factors on the platform profit and agent surplus of different agent types.

To isolate the impact of network structure and better quantify the growth potential of agent types, we make the following additional assumption.

Assumption 4. *(i) The value distributions satisfy that* $F_{s_i}(v) = F_s(v)$ *for any* $i \in \mathcal{S}$ *and* $F_{b_j}(v) = F_b(v)$ *for any* $j \in \mathcal{B}$.

*(ii) The transition equation is given by* $s_i(t+1) = \alpha_i^s s_i(t) + \beta_i^s (q_i^s(t))^{\xi_s}$ *for all* $i \in \mathcal{S}$ *and* $b_j(t+1) = \alpha_j^b b_j(t) + \beta_j^b (q_j^b(t))^{\xi_b}$ *for all* $j \in \mathcal{B}$ *with* $t \in \{1, ..., T-1\}$ *where* $\xi_s \in (0,1)$ *and* $\xi_b \in (0,1)$.

Assumption 4(i) requires that different types of sellers/buyers are homogeneous in their valuations. Assumption 4(ii) specifies the functional form of the growth functions $\mathcal{G}(q)$ in (1), in which the retention rate and growth coefficient are type-specific. In particular, $\beta_i^s$ and $\beta_j^b$ measure the effect of transaction quantities on growth; the exponents $\xi_s$ ($\xi_b$) are homogeneous across different types of sellers (buyers) and capture the elasticity of the new adoptions with respect to the transaction quantities (i.e., $\frac{\partial s_i(t+1)/s_i(t+1)}{\partial q_i^s(t)/q_i^s(t)}$ or $\frac{\partial b_j(t+1)/b_j(t+1)}{\partial q_j^b(t)/q_j^b(t)}$). For the main results of this section, we can generalize the population dynamics in Assumption 4(ii) to $s_i(t+1) = \alpha_i^s s_i(t) + \beta_i^s g_s(s_i(t), q_i^s(t))$ for all $i \in \mathcal{S}$ and $b_j(t+1) = \alpha_j^b b_j(t) + \beta_j^b g_b(b_j(t), q_j^b(t))$ for all $j \in \mathcal{B}$, where the growth functions $g_s(\cdot,\cdot)$ and $g_b(\cdot,\cdot)$ are homogeneous of degree $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$ (a function $g(\cdot,\cdot)$ is homogeneous of degree $\alpha$ means $g(ns, nq) = n^a g(s,q)$ for any $s \geq q \geq 0$, $n > 0$). This more general form can capture the average surplus of the agent (i.e., $\int_0^{\frac{q_i^s(t)}{s_i(t)}} (F_s^{-1})'(y) y dy$), which may contribute to the growth of new adoptions, as modeled by Lian and Van Ryzin (2021).

**Long-run Growth Potential $\psi$ of Agents.** Based on Assumption 4, we develop a metric to measure the growth potential of each agent type. Given type-$i$ sellers' service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ induced by the platform's optimal commissions $(\overline{\boldsymbol{r}}^s, \overline{\boldsymbol{r}}^b)$ at the optimal steady state, by Lemma 2, the population of type-$i$ seller (type-$j$ buyer) converges to $\overline{s}_i$ ($\overline{b}_j$) that satisfies $\overline{s}_i = \alpha_i^s \overline{s}_i + \beta_i^s (\overline{q}_i^s)^{\xi_s}$ ($\overline{b}_j = \alpha_j^b \overline{b}_j + \beta_j^b (\overline{q}_j^b)^{\xi_b}$). Algebraic manipulations suggest that

$$\overline{s}_i = \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}} \left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{\xi_s}{1-\xi_s}}, \qquad \overline{q}_i^s = \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}} \left(\frac{\overline{q}_i^s}{\overline{s}_i}\right)^{\frac{1}{1-\xi_s}} \text{ where } i \in \mathcal{S}, \tag{9a}$$

$$\overline{b}_j = \left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}} \left(\frac{\overline{q}_j^b}{\overline{b}_j}\right)^{\frac{\xi_b}{1-\xi_b}}, \qquad \overline{q}_j^b = \left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}} \left(\frac{\overline{q}_j^b}{\overline{b}_j}\right)^{\frac{1}{1-\xi_b}} \text{ where } j \in \mathcal{B}. \tag{9b}$$

Eqn. (9) reveals that given the service level $\frac{\overline{q}_i^s}{\overline{s}_i}$ for type-$i$ sellers and $\frac{\overline{q}_j^b}{\overline{b}_j}$ for type-$j$ buyers, the population of an agent type and the transaction quantities at the optimal steady state are proportional to the coefficients $\left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}$ for type-$i$ sellers and $\left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}$ for type-$j$ buyers. Based on this, we formally define the *long-run growth potential* as follows:

$$\psi_i^s := \left(\frac{\beta_i^s}{1-\alpha_i^s}\right)^{\frac{1}{1-\xi_s}}, \ i \in \mathcal{S}, \qquad \psi_j^b := \left(\frac{\beta_j^b}{1-\alpha_j^b}\right)^{\frac{1}{1-\xi_b}}, \ j \in \mathcal{B}. \tag{10}$$

We next provide some intuitive explanations for $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$. For simplicity, we omit the superscripts $(s,b)$ and subscripts $(i,j)$ in the population transition parameters. Given the population dynamics in Assumption 4(ii), a fraction $\alpha < 1$ of agents stays in the system after each period, and the impact of the growth coefficient $\beta$ captures the impact of transaction quantities on the population growth. As $\alpha$ and $\beta$ increase, the corresponding value of $\psi$ also increases, and therefore we refer to $\psi_i^s$ for $i \in \mathcal{S}$ and $\psi_j^b$ for $j \in \mathcal{B}$ in (10) as the long-run growth potential that each agent type can achieve.

**Rankings of Relative Growth Potential in the Network.** Based on the long-run growth potential, we introduce a ranking of different types of buyers (sellers). Let $N_E(X)$ denote the set of all neighbors of agent types $X \subseteq \mathcal{B} \cup \mathcal{S}$ in the graph $G(S \cup B, E)$ such that $N_E(X) = \{i \notin X : (i,j) \in E \text{ for } j \in X\}$. Given a network $G(\mathcal{S} \cup \mathcal{B}, E)$ and the long-run growth potential vector $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, we first let $\mathcal{B}^0 = B$, $\mathcal{S}^0 = S$ and $E^0 = E$. For $\tau = 0, 1, \ldots$, we define $\mathcal{B}_\tau$ and $\mathcal{S}_\tau$ iteratively as follows:

$$\mathcal{B}_{\tau+1} = \underset{\widetilde{\mathcal{B}} \subseteq \mathcal{B}^\tau}{\arg\min} \frac{\sum_{i \in N_{E^\tau}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}, \tag{11a}$$

$$\mathcal{S}_{\tau+1} = N_{E^\tau}(\mathcal{B}_{\tau+1}). \tag{11b}$$

where $\mathcal{B}^{\tau+1} = \mathcal{B}^\tau \backslash \mathcal{B}_{\tau+1}$, $\mathcal{S}^{\tau+1} = \mathcal{S}^\tau \backslash \mathcal{S}_{\tau+1}$, $E^\tau = \{(i,j) \in E : i \in \mathcal{S}^\tau \text{ and } j \in \mathcal{B}^\tau\}$ and $N_{E^\tau}(B) = \{i \in \mathcal{S}^\tau : j \in \mathcal{B} \text{ and } (i,j) \in E^\tau\}$[1].

In (11a), for each subset of buyer types $\widetilde{\mathcal{B}}$ of $\mathcal{B}^\tau$, $\frac{\sum_{i \in N_{E^\tau}(\widetilde{\mathcal{B}})} \psi_i^s}{\sum_{j \in \widetilde{\mathcal{B}}} \psi_j^b}$ is the ratio between the total long-term growth potential of its (remaining) compatible sellers and its own. We refer to the ratio as the *relative growth potential* between $N_{E^\tau}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}}$. This metric, similar to those used for comparing two economies in, e.g., Krugman (1989), captures the relative growth potential of sellers and buyers. In (11), we can iteratively identify a subgraph

[1] If multiple sets achieve the minimum, the arg min operator returns the largest one.

such that the relative growth potential of sellers is the lowest. Subsequently, we label it and remove this subgraph from the network, and then the $B^\tau$ and $S^\tau$ are the remaining agent types and $E^\tau$ is the remaining graph after $\tau$ iterations. We repeat the procedure until the remaining subgraph is empty. As a result, the subnetwork with a higher index $\tau$ has a higher relative growth potential of sellers against buyers in the graph. In general, the ranking incorporates both inter-temporal factors captured by the long-term growth potential $\boldsymbol{\psi}$ and spatial factors captured by the graph structure $G(\mathcal{B}\cup\mathcal{S},E)$.

We use the example in Figure 2 below to illustrate the rankings of relative growth potential. This example illustrates the compatibility between freelance coders and clients in need of IT services in terms of their skills and schedule on Upwork. Specifically, clients needing AI Services can only be served by coders with AI skills, and clients requiring immediate delivery of work can only choose coders with flexible working hours. By enumeration, we can obtain the index of each type, and the solid (dotted) line represents the lines between sets with the same (different) index. For a large-scale network, we can obtain the ranking by solving a convex optimization problem. [2] We will next show that the index implies the sellers' incomes and buyers' payments under the platform's optimal commissions.

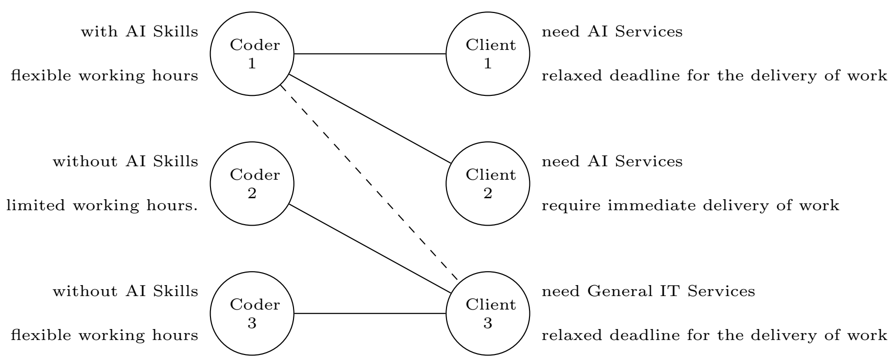



**Figure 2 Compatibility between Freelance Coders and Clients in need of IT Services on Upwork. Suppose that $\psi_i^s=\psi_j^b=1$ for $i=\{1,2,3\}$ and $j=\{1,2,3\}$. Then by enumeration, we know $\{1,2\}=\arg\min_{\widetilde{\mathcal{B}}\subseteq\mathcal{B}}\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})}\psi_i^s}{\sum_{j\in\widetilde{\mathcal{B}}}\psi_j^b}$, which means $\mathcal{B}_1=\{1,2\}$ and $\mathcal{S}_1=\{1\}$. After eliminating $\mathcal{B}_1$ and $\mathcal{S}_1$ from the network $E$, we have $\mathcal{B}^1=\{3\},\mathcal{S}^1=\{2,3\},E^1=\{(2,3),(3,3)\}$. Since there is only one buyer type left, we know $\mathcal{B}_2=\{3\}$ and $\mathcal{S}_2=\{2,3\}$. Finally, all agent types are labeled with an index.**

[2] Notice that AVG in (8) is equivalent to maximizing $\sum_{j\in\mathcal{B}}\left[\psi_j^b h\left(\frac{W_j}{\psi_j^b}\right)\right]$ over a polymatroid $\{\boldsymbol{W}:\sum_{j\in\widetilde{\mathcal{B}}}W_j\leq\sum_{i\in N_E(\widetilde{\mathcal{B}})}\psi_i^s,\forall\widetilde{\mathcal{B}}\subseteq\mathcal{B},\ W_j\geq 0,\forall j\in\mathcal{B}\}$, where $h:[0,\infty)\to[0,\infty)$ is an increasing and concave function (see Lemma 12 in Appendix C.1). Therefore, by solving a convex optimization problem, we can obtain the ranking given in (11). This procedure borrows the algorithmic idea to characterize the lexicographically optimal bases of polymatroids from Fujishige (1980) (see Lemma 9 - Lemma 13 in Appendix C.1 for the connection).

### 5.1. Agent Payments/Incomes and Platform Commissions

In this subsection, we analyze the impact of agents' growth potential within a network on the platform's commission decisions, which could provide guidance for the platform's revenue management strategies. Recall that the optimal commission $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at the optimal steady state is not necessarily unique, but any optimal commission profile induces the same (net) payments and incomes for agent types engaged in transactions (see Proposition 1 and Lemma 1). Furthermore, the total commission generated from a transaction (i.e., $r_i^s + r_j^b$ for $(i,j) \in E$), which represents the difference between buyers' payments and sellers' incomes, is inherently unique. Therefore, in this subsection, we will first study the impact of network structure and growth potentials on (net) payments and incomes for agent types and then analyze its impact on the total optimal commission derived from transactions.

**Buyers' Payments and Sellers' Incomes.** In the following proposition, we establish that the ranking of the relative growth potentials of sellers to buyers in the network given in (11) determines the ranking of buyers' payments and sellers' incomes at the optimal steady state given the platform's optimal policy. For simplicity of notation, in any equilibrium from Definition 1 under a platform's optimal commission profile $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$, we denote by $M_j = \min_{i':(i',j)\in E} \{\overline{p}_{i'} + \overline{r}_j^b\}$ the payment of any type-$j$ buyers, and denote by $I_i = \overline{p}_i - \overline{r}_i^s$ the income of any type-$i$ sellers.

Proposition 4. **(ranking of buyers' payments and sellers' incomes)** *In the network $G(\mathcal{S} \cup \mathcal{B}, E)$, under any platform's optimal commission profile $(\overline{\boldsymbol{r}}^{\boldsymbol{s}}, \overline{\boldsymbol{r}}^{\boldsymbol{b}})$ at the steady state,*

*(1) on the buyer side, for any $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ with $\tau_1 \le \tau_2$,*

*(i) the buyers' payments satisfy $M_{j_1} \ge M_{j_2}$;*

*(ii) the buyers' service levels satisfy $\frac{\overline{q}_{j_1}^b}{b_{j_1}} \le \frac{\overline{q}_{j_2}^b}{b_{j_2}}$;*

*(2) on the seller side, for any $i_1 \in \mathcal{S}_{\tau_1}$ and $i_2 \in \mathcal{S}_{\tau_2}$ with $\tau_1 \le \tau_2$,*

*(i) the sellers' incomes satisfy $I_{i_1} \ge I_{i_2}$;*

*(ii) the sellers' service levels satisfy $\frac{\overline{q}_{i_1}^b}{b_{i_1}} \ge \frac{\overline{q}_{i_2}^b}{b_{i_2}}$.*

Proposition 4 suggests that under the platform's optimal commissions at the steady state, when the relative long-run growth potential of sellers to buyers increases (i.e., the ratio $\frac{\sum_{i\in\mathcal{S}_\tau} \psi_i^s}{\sum_{j\in\mathcal{B}_\tau} \psi_j^b}$ in (11) increases as network component index $\tau$ increases), the buyers make less payment, and the sellers receive lower income in equilibrium. By using the example in

Figure 2 to illustrate, the payments on the buyer (i.e., client) side satisfy $M_1 = M_2 > M_3$ given that $\mathcal{B}_1 = \{1,2\}$ and $\mathcal{B}_2 = \{3\}$; the incomes on the seller (i.e., coder) side satisfy that $I_1 > I_2 = I_3$ given that $\mathcal{S}_1 = \{1\}$ and $\mathcal{S}_2 = \{2,3\}$.

The managerial implication from Proposition 4 is that while determining the commission, the platform needs to consider not only the retention rate and growth potentials of the focal agent types but also their trading partners on the other side of the market. Specifically, the platform should incentivize the growth of agents with lower relative growth potential by offering higher commissions to them and extract a higher surplus from those with higher relative growth potential.

From Proposition 4, we see that each agent type's optimal service level depends on their relative growth potentials and their trading partners on the other side of the market. Therefore, any change in the values of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ induces changes in the service level of each agent type, ultimately affecting the equilibrium demand, supply, and population at the optimal steady state. Lastly, we examine the influence of the long-run growth potential $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ to offer guidance for the platform's commission decisions in response to changes in agent types' growth potential in the network.

COROLLARY 1. **(impact of the long-term growth potential)** *Given any* $\xi_s \in (0,1)$ *and* $\xi_b \in (0,1)$*, at the optimal steady state,*

*(1) for the service levels,*

*(i) given* $j \in \mathcal{B}$*,* $\frac{\overline{q}_j^b}{\overline{b}_j}$ *is decreasing in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$ *and increasing in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$*;*

*(ii) given* $i \in \mathcal{S}$*,* $\frac{\overline{q}_i^s}{\overline{s}_i}$ *is decreasing in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$ *and increasing in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$*;*

*(2) for the transaction quantities and populations,*

*(i) given* $j \in \mathcal{B}$*,* $(\overline{q}_j^b, \overline{b}_j)$ *is increasing in* $\psi_j^b \geq 0$*, decreasing in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$ *with* $j' \neq j$*, and increasing in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$*;*

*(ii) given* $i \in \mathcal{S}$*,* $(\overline{q}_i^s, \overline{s}_i)$ *is increasing in* $\psi_i^s \geq 0$*, decreasing in* $\psi_{i'}^s \geq 0$ *for any* $i' \in \mathcal{S}$ *with* $i' \neq i$ *and increasing in* $\psi_{j'}^b \geq 0$ *for any* $j' \in \mathcal{B}$*.*

Note that for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, the vectors $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ are determined by the retention rates $(\boldsymbol{\alpha}^s, \boldsymbol{\alpha}^b)$ and the growth coefficients $(\boldsymbol{\beta}^s, \boldsymbol{\beta}^b)$. Corollary 1(1) suggests that the service level of any agent decreases with the growth potential of all types from the same side but increases with those on the other side of the market. Corollary 1(2) implies that the transaction quantity and population of each type are increasing in their own growth potential and those on the other side of the network, but decreasing in those of other types on the same side.

We discuss the intuition using the buyer side as an example. Both a high long-term growth potential and a high service level contribute to an increase in the population of a buyer type at an optimal steady state. Consequently, when the long-term growth potential of a buyer type is high, the platform can maintain a high population by inducing a relatively lower service level. However, if other buyer types have higher long-term growth potential, their equilibrium demand will rise, resulting in increased prices for the sellers and a reduced service level for our focal buyer type. Conversely, if the corresponding sellers have higher long-term growth potential, their supply will increase, leading to lower prices and benefiting all buyers.

**Platform's Commissions.** We now focus on the total commission charged by the platform from one transaction, viz., the difference between the buyers' payments and the sellers' incomes. Note that under the optimal commission, type-$i$ sellers with $i \in \mathcal{S}_\tau$ only trade with type-$j$ buyers with $j \in \mathcal{B}_\tau$. Therefore, we will examine how the total commission charged from one transaction between sellers in $\mathcal{S}_\tau$ and buyers in $\mathcal{B}_\tau$ depend on the ranking of the relative growth potential of sellers to buyers $\tau$ given in (11). Here, we assume $\xi_s = \xi_b$ to isolate the impact of value distribution.

Proposition 5. **(ranking the platform's commissions)** *Suppose further that $F_s$ and $F_b$ are twice differentiable in their domains and $\xi_s = \xi_b$, there exists a $\widetilde{\tau}$ such that*

*(1) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in $\tau$ for $\tau < \widetilde{\tau}$;*

*(2) $r_i^s + r_j^b$ for $i \in \mathcal{S}_\tau, j \in \mathcal{B}_\tau$ is decreasing in $\tau$ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are concave in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$; whereas it is increasing in $\tau$ for $\tau \geq \widetilde{\tau}$ if $F_s(v)$ and $F_b(v)$ are convex in $v \in [0, \overline{v}_s]$ and $v \in [0, \overline{v}_b]$.*

In Proposition 5(1), when the relative growth potential of sellers to buyers falls below a threshold, the total commission charged from the transaction decreases with the relative growth potential between sellers and buyers. In Proposition 5(2), the concavity of $F_s(v)$ and $F_b(v)$ implies a higher density of agents with lower (reservation) value. In this case, when the relative growth potential of sellers to buyers is higher, the optimal total commission charged by the platform from transactions should be lower. Similarly, the convexity of $F_s(v)$ and $F_b(v)$ implies that the number of agents with higher (reservation) value is higher. In this scenario, the platform charges lower (higher) total commissions for transactions involving agents with moderate (high or low) relative growth potentials of sellers to buyers.

Intuitively, when the relative growth potential between sellers and buyers is below a threshold, the number of sellers is significantly smaller than that of buyers. In such cases, the platform uses its commission to keep the sellers' income at a sufficiently high level to ensure the participation of all sellers in transactions. On the other hand, as the relative growth potential increases, the number of sellers rises, prompting the platform to gradually reduce buyer payments to stimulate demand. As a result, the total commission charged from the transaction, which is the difference between buyer payments and seller incomes, decreases with the relative growth potential between sellers and buyers.

When the relative growth potential between sellers and buyers exceeds the threshold, the number of sellers is sufficient in the market, and the platform no longer needs to provide high subsidies to ensure their full participation. In this case, an increase in the relative growth potential between sellers and buyers suggests that the platform should reduce the service level for sellers and increase the service level for buyers, aimed at achieving a balance between supply and demand. When most agents have a low valuation of the product or service, the platform needs to offer buyers a large price cut to increase their demand, but a slight decrease in sellers' earnings can dampen the supply. As a result, the total commission charged from the transaction decreases with the relative growth potential. Conversely, when most agents highly value the product or service, providing buyers with a modest price reduction is sufficient to encourage their participation, and the platform can substantially reduce sellers' earnings without significantly impacting their supply. As a result, the total commission charged from the transaction increases with the relative growth potential between sellers and buyers.

### 5.2. Optimal Network for the Platform's Profit

The prior studies show that a network that better matches supply with demand achieves a better performance from the system designer's perspective (see Schrijver et al. 2003, Chou et al. 2011, Birge et al. 2021). For example, in a static setting, Chou et al. (2011) show that a graph expander, in which every subset of nodes is linked to a sufficiently large number of neighboring nodes, is optimal for the system. Similarly, Birge et al. (2021) show

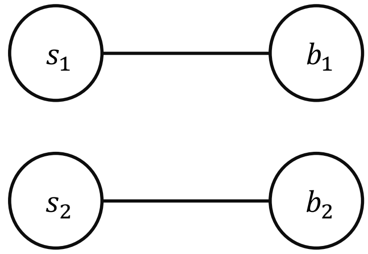



**Figure 3 Bias of Measuring Network Imbalance using Equilibrium Population Ratio: suppose that buyers' and sellers' (reservation) values are uniformly distributed between $[0,1]$ with parameters $\xi_s=\xi_b=0.9$, $\psi_1^s=\psi_2^b=1$, $\psi_1^b=\psi_2^s=3$. The equilibrium population ratio in AVG satisfies $\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})} s_i}{\sum_{j\in\widetilde{\mathcal{B}}} b_j}\geq 0.9\times\frac{\sum_{i\in\mathcal{S}} s_i}{\sum_{j\in\mathcal{B}} b_j}$ for any $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$, but the platform's optimal profit in AVG is only about 85% of that in a complete market; i.e., $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)=0.85\times\overline{\mathcal{R}}(\overline{E},\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$.**

that if the seller-to-buyer population ratio $\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})} s_i}{\sum_{j\in\widetilde{\mathcal{B}}} b_j}$ in each sub-market is sufficiently large, the platform could achieve the maximum optimal profit. One may wonder if the same result would hold in a dynamic setting. Interestingly, we observe from the numerical example in Figure 3 that when the population ratio is endogenously formed as in our model, using the equilibrium population ratio at the optimal steady state as a metric for the network connectivity can overestimate the profit guarantee for the platform relative to the maximum achievable profit given the set of sellers and buyers. Hence, it becomes crucial to incorporate temporal factors, i.e., the long-run growth potential, in (10) into the "connectivity" measure in the network.

To signify the dependence on the network structure $E$ and long-term growth potential $(\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$, we let $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$ denote the platform's optimal steady-state profit from AVG in the network $G(\mathcal{S}\cup\mathcal{B},E)$. We let $\overline{E}$ denote the edge set for the complete graph with the set of seller types $\mathcal{S}$ and that of buyer types $\mathcal{B}$. Since the platform can achieve the maximum optimal profit in a complete graph given that the feasible region for a complete graph is the largest in Problem (3), we use $\overline{\mathcal{R}}(\overline{E},\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$ to benchmark the impact of network structure $E$ on the platform's profit. The following theorem establishes a connection between the temporal-spatial market structure and the platform's optimal profit in network $G(\mathcal{S}\cup\mathcal{B},E)$.

Theorem 2. **($(1-\epsilon)$-optimal network structure)** *For any $\epsilon\in[0,1]$, if $G(\mathcal{B}\cup\mathcal{S},E)$ satisfies that*

$$\frac{\sum_{i\in\mathcal{S}^1}\psi_i^s}{\sum_{j\in\mathcal{B}^1}\psi_j^b}\;\geq\;(1-\epsilon)\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}, \tag{12a}$$

*then*

$$\overline{\mathcal{R}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b) \;\geq\; (1-\epsilon)\overline{\mathcal{R}}(\overline{E}, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b). \tag{12b}$$

In Condition (12a), the right-hand-side expression $\frac{\sum_{i\in\mathcal{S}}\psi_i^s}{\sum_{j\in\mathcal{B}}\psi_j^b}$ represents the relative long-term growth potential of all sellers to all buyers within the entire network $G(\mathcal{B}\cup\mathcal{S}, E)$. Likewise, the left-hand-side term is the relative growth potential of the compatible sellers to a subset of buyers $\mathcal{B}^1$, whose relative long-term growth potential is the lowest (see (11)). Therefore, $\epsilon$ quantifies the degree of imbalance: a positive value of $\epsilon$ indicates that there exists no submarket in which the relative growth potential is $\epsilon$ lower than that of the entire market. The profit guarantee in (12b) implies that the degree of imbalance $\epsilon$ in the network does not cause more than $\epsilon$ optimal profit loss for the platform. When $\epsilon = 0$, the condition in (12a) ensures that the relative growth potential for all submarkets is weakly higher than that for the entire market. In other words, the long-term growth potentials are "balanced" in the network. Note that even though the market $E$ may be incomplete, as long as the market is balanced (i.e., $\epsilon = 0$ in (12a)), the lower bound in (12b) is tight, and the platform's optimal profit achieves the maximum possible optimal profit, i.e., $\overline{\mathcal{R}}(E, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b) = \overline{\mathcal{R}}(\overline{E}, \boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$.

The managerial insight derived from Theorem 2 suggests that the platform should aim to enhance the balance of the network in terms of long-term growth potential to maximize its steady-state optimal profit. Specifically, the platform could prioritize agent types with relatively low long-term growth potential in targeted marketing campaigns to attract new customers and increase customer retention.

**Remark.** A related work by Alizamir et al. (2022) considers a monopoly firm providing service to a network of individual customers with externality. They find that a balanced network with symmetrical mutual interactions among agents results in the lowest profit for the firm. In their setting, they assume a linear impact of agents' consumption on others, and the effects of network externalities go beyond immediate neighbors over time, causing increasing externalities in a network. In contrast, in our setting, increasing the population of one agent type leads to higher transaction quantities on the other side in equilibrium, and the marginal impact on the future population on the other side is decreasing. The marginal decreasing effect of agents' consumption and population on growth can be explained by the fact that the potential market size is usually finite in practice. $\Diamond$

## 6. Conclusion

In this study, we consider a two-sided platform with heterogeneous growth potentials among agent types. The compatibility between buyer and seller types is captured by a bipartite graph, which is not necessarily complete. The platform leverages the commissions to maximize its $T$-period profit. To address the complexity of the platform's profit optimization problem, we consider the long-run average problem as a benchmark and propose a heuristic algorithm with a provable performance guarantee. We show that boosting the growth of the agent type with the lowest population ratio compared with the long-run average benchmark leads to a profit loss bounded by a constant for any positive integer $T$.

Furthermore, we delve into the optimal steady state and explore how the growth potentials of agents and network structure influence the agents' income/payment in the market and the platform's profit. We begin by introducing a set of metrics designed to capture the growth potentials and connectivity of agents. We then show that buyer (seller) types compatible with higher sellers' (buyers') growth potentials experience lower payments (higher income). A sensitivity analysis demonstrates the impact of agent type's long-term growth potential on income/payment. In addition, the commission charged by the platform in a submarket depends on the relative growth potentials from the two sides of the market. Finally, we show that a balanced network, in which sellers with relatively high (low) growth potentials trade with buyers with relatively high (low) growth potentials, results in maximum profitability, while the degree of imbalance in the graph establishes a lower bound for the platform's optimal profit (relative to that under the complete graph).

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# Online Appendix

## A. Proof of Results in Section 3

We first provide some Auxiliary Results used to prove the results in Section 3 in Appendix A.1, then we prove the results in Section 3 in Appendix A.2. Finally, we present the alternative formulation for the platform's profit optimization problem in (3) in Appendix A.3.

### A.1. Auxiliary Results for Section 3

Lemmas 3 - 5 are needed to prove Proposition 1. In Lemma 5, we establish the connection between the equilibrium and the optimal solution to an optimization problem in (14). Before that, we establish some properties for the optimization problem in Lemma 3. We also establish the existence of the optimal solution to this optimization problem in Lemma 4, and show that it is essentially unique. These lemmas enable us to establish the existence and uniqueness of the competitive equilibrium in Definition 1. The proof of Auxiliary Results follows a similar argument as the proof of Proposition EC.1 and Proposition 9 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

For simplicity of notation, we first define that

$$W_{b_j}^t(q_j^b(t)) := \int_0^{q_j^b(t)} F_{b_j}^{-1}\Big(1-\frac{z}{b_j(t)}\Big)dz - r_j^b(t)q_j^b(t), \tag{13a}$$

$$W_{s_i}^t(q_i^s(t)) := -\int_0^{q_i^s(t)} F_{s_i}^{-1}\Big(\frac{z}{s_i(t)}\Big)dz - r_i^s(t)q_i^s(t). \tag{13b}$$

Note that the sum of $W_{b_j}^t(q_j^b(t))$ and $W_{s_i}^t(q_i^s(t))$ can be viewed as the total surplus of buyers and sellers trading in the platform, and is the objective function in Problem (14). Let $W_{b_j}^{t'}(q)$ be the derivative of $W_{b_j}^t(q)$ at $q=q_j^b(t)$ for any $0<q_j^b(t)<b_j(t)$, and abusing some notation, $W_{b_j}^{t'}(0)=\lim_{q_j^b(t)\downarrow 0} W_{b_j}^t(q_j^b(t))$ and $W_{b_j}^{t'}(b_j(t))=\lim_{q_j^b(t)\uparrow b_j(t)} W_{b_j}^t(q_j^b(t))$ given Assumption 2(i). Similarly, we let $W_{s_i}^{t'}(q)$ be the derivative of $W_{s_i}^t(q)$ at $q=q_i^s(t)$ for any $0<q_i^s(t)<s_i(t)$, and we let $W_{s_i}^{t'}(0)=\lim_{q_i^s(t)\downarrow 0} W_{s_i}^t(q_i^s(t))$ and $W_{s_i}^{t'}(s_i(t))=\lim_{q_i^s(t)\uparrow s_i(t)} W_{s_i}^t(q_i^s(t))$ given Assumption 2(i). We consider the following properties of functions $W_{b_j}^t(q_j^b(t))$ and $W_{s_i}^t(q_i^s(t))$.

LEMMA 3. *For any $j\in\mathcal{B}$, $i\in\mathcal{S}$ and $t\in\{1,\dots,T\}$,*

*(i) $W_{b_j}^t(q)$ is continuously differentiable and strictly concave in $q\in(0,b_j(t))$; moreover, both $W_{b_j}^t(q)$ and $W_{b_j}^{t'}(q)$ are right continuous at $q=0$ and left continuous $q=b_j(t)$.*

*(ii) $W_{s_i}^t(q)$ is continuously differentiable and strictly concave in $q\in(0,s_i(t))$; moreover, both $W_{s_i}^t(q)$ and $W_{s_i}^{t'}(q)$ are right continuous at $q=0$ and left continuous $q=s_i(t)$.*

For any $t\in\{1,...,T\}$, we proceed to consider the following optimization problem:

$$W(t) = \max_{\boldsymbol{x}(t),\boldsymbol{q^s}(t),\boldsymbol{q^b}(t)} \sum_{i\in\mathcal{B}}\left(\int_0^{q_j^b(t)} F_{b_j}^{-1}\left(1-\frac{z}{b_j(t)}\right)dz - r_j^b(t)q_j^b(t)\right)$$
$$-\sum_{i\in\mathcal{S}}\left(\int_0^{q_i^s(t)} F_{s_i}^{-1}\left(\frac{z}{s_i(t)}\right)dz + r_i^s(t)q_i^s(t)\right) \tag{14a}$$

$$\text{s.t. } q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j\in\mathcal{B}, \tag{14b}$$

$$\sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S}, \tag{14c}$$

$$q_j^b(t) \le b_j(t), \qquad \forall j \in \mathcal{B}, \tag{14d}$$

$$q_i^s(t) \le s_i(t), \qquad \forall i \in \mathcal{S}, \tag{14e}$$

$$x_{ij}(t) \ge 0, \qquad \forall (i,j) \in E. \tag{14f}$$

From Problem (14), we establish the result below. Before that, we define the notation "$a \le 0 \perp b \ge 0$" as $a \le 0, b \ge 0, ab = 0$.

LEMMA 4. *(i) There exists an optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ to Problem (14).*

*(ii) Given any optimal primal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$, there exists a dual multiplier vector $(\boldsymbol{\theta^b}(t), \boldsymbol{\theta^s}(t), \boldsymbol{\eta}^b(t), \boldsymbol{\eta}^s(t), \boldsymbol{\pi}(t))$ associated with constraints (14b)-(14f) that satisfy the KKT conditions below:*

$$F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) - r_j^b(t) - \theta_j^b(t) - \eta_j^b(t) = 0, \qquad \forall j \in \mathcal{B}, \tag{15a}$$

$$F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) + r_i^s(t) - \theta_i^s(t) + \eta_i^s(t) = 0, \qquad \forall i \in \mathcal{S}, \tag{15b}$$

$$\theta_j^b(t) - \theta_i^s(t) + \pi_{ij}(t) = 0, \qquad \forall (i,j) \in E, \tag{15c}$$

$$q_j^b(t) - b_j(t) \le 0 \perp \eta_j^b(t) \ge 0, \qquad \forall j \in \mathcal{B}, \tag{15d}$$

$$q_i^s(t) - s_i(t) \le 0 \perp \eta_i^s(t) \ge 0, \qquad \forall i \in \mathcal{S}, \tag{15e}$$

$$x_{ij}(t) \ge 0 \perp \pi_{ij}(t) \ge 0, \qquad \forall (i,j) \in E, \tag{15f}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B}, \tag{15g}$$

$$q_i^s(t) = \sum_{j':(i,j')\in E} x_{i,j'}(t), \qquad \forall i \in \mathcal{S}. \tag{15h}$$

*In addition, these KKT conditions in (15) are necessary and sufficient conditions for the optimality of solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$.*

*(iii) All primal optimal solution $(\boldsymbol{x}(t), \boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$ share the same vector $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t))$;*

*(iv) The dual solution $\theta_i^s(t)$ for $i \in \{i' : 0 < q_{i'}^s < s_{i'}\}$ that satisfies (15) is unique.*

The conditions in Lemma 5(i)-(ii) are sufficient and necessary conditions, while those in Lemma 5(iii) are only sufficient conditions for equilibrium, as the prices for type $i \in \{i' : q_{i'}^s(t) = 0 \text{ or } q_{i'}^s(t) = s_{i'}(t)\}$ are not necessarily unique.

LEMMA 5. *In each period $t \in \{1, ..., T\}$, given any commission profile $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$ and population vector $(\boldsymbol{s}(t), \boldsymbol{b}(t)) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{B}|}$,*

*(i) $(\boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t))$ satisfies the equilibrium conditions in Definition 1 if and only if it is an optimal solution to Problem (14);*

*(ii) for $i \in \{i' : 0 < q_{i'}^s(t) < s_{i'}(t)\}$, $p_i(t)$ satisfies the equilibrium conditions in Definition 1 if and only if*

$$p_i(t) = \theta_i^s(t). \tag{16a}$$

*(iii) for* $i \in \{i' : q^s_{i'}(t) = 0$ *or* $q^s_{i'}(t) = s_{i'}(t)\}$, $p_i(t)$ *satisfies the equilibrium conditions in Definition 1 if*

$$p_i(t) = \theta^s_i(t). \tag{16b}$$

Before proceeding, note that functions $F^{-1}_{s_i}(\cdot)$ and $F^{-1}_{b_j}(\cdot)$ have the following properties in an equilibrium:

(1) On the seller side, if $p_i(t) - r^s_i(t) \le 0$, then $q^s_i(t) = 0$ and

$$F^{-1}_{s_i}\left(\frac{q^s_i(t)}{s_i(t)}\right) \ \ge \ p_i(t) - r^s_i(t), \tag{17a}$$

if $0 < p_i(t) - r^s_i(t) < \overline{v}_{s_i}$, then $0 < q^s_i(t) < s_i(t)$ and

$$F^{-1}_{s_i}\left(\frac{q^s_i(t)}{s_i(t)}\right) \ = \ p_i(t) - r^s_i(t), \tag{17b}$$

if $\overline{v}_{s_i} \le p_i(t) - r^s_i(t)$, then $q^s_i(t) = s_i(t)$ and

$$F^{-1}_{s_i}\left(\frac{q^s_i(t)}{s_i(t)}\right) \ \le \ p_i(t) - r^s_i(t). \tag{17c}$$

(2) On the buyer side, if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \le 0$, then $q^b_j(t) = b_j(t)$ and

$$F^{-1}_{b_j}\left(1 - \frac{q^b_j(t)}{b_j(t)}\right) \ \ge \ \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{18a}$$

if $0 < \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} < \overline{v}_{b_j}$, then $0 < q^b_j(t) < b_j(t)$ and

$$F^{-1}_{b_j}\left(1 - \frac{q^b_j(t)}{b_j(t)}\right) \ = \ \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}, \tag{18b}$$

if $\min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\} \ \ge \ \overline{v}_{b_j}$, then $q^b_j(t) = 0$ and

$$F^{-1}_{b_j}\left(1 - \frac{q^b_j(t)}{b_j(t)}\right) \le \min_{i':(i',j)\in E}\{p_{i'}(t) + r^b_j(t)\}. \tag{18c}$$

### A.2. Proof of Results for Section 3

Based on Lemmas 3 - 5, Proposition 1 is proved as below:

**Proof of Proposition 1.** We establish the following two claims of this result.

Claim (i). Lemma 4(i) implies that the optimal primal solution to (14) always exists, and Lemma 5(i) implies that the $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (14). Therefore, the equilibrium transaction vector $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ exists.

Lemma 4(ii) implies that the optimal dual solution to (14) always exists, and Lemma 5(ii) implies that $\boldsymbol{p}$ that satisfies the equality in (16) is the equilibrium price vector. Therefore, there exists a corresponding equilibrium price vector.

Claim (ii). Lemma 4(iii) implies that the optimal primal solution $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ to (14) is unique. Lemma 5(i) implies that the $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is the equilibrium if and only if it is the optimal primal solution to (14). Therefore, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is unique.

Lemma 4(iv) implies that the optimal dual solution $\boldsymbol{\theta}^s$ to Problem (14) is unique for $i \in \{i' : 0 < q^s_{i'} < s_{i'}\}$, and Lemma 5(ii) implies that $p_i(t) = \theta^s_i(t)$ for $i$ that satisfies $0 < q^s_i(t) < s_i(t)$. Therefore, the equilibrium price is unique for $i$ that satisfies $0 < q^s_i(t) < s_i(t)$. ■

**Proof of Lemma 1**. We establish the sufficiency of (4) in Step 1 and construct a feasible commission profile in Step 2 to show that the feasible commission profile always exists.

Step 1: Sufficiency. We show that for any $(\boldsymbol{q}^b(t), \boldsymbol{q}^s(t), \boldsymbol{x}(t))$ that satisfies (2c)-(2e), if vector $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ satisfies the conditions in (4), then it satisfies the conditions in Definition 1.

We first verify the conditions in Definition 1, in which (2c)-(2e) immediately follow from our conditions.

(2a) We consider the following two cases:

When $q_i^s(t) > 0$, $s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) \overset{(a)}{=} s_i(t)F_{s_i}(F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})) = q_i^s(t)$, (a) follows from (4a).

When $q_i^s(t) = 0$, $0 \le s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) \overset{(b)}{\le} s_i(t)F_{s_i}(F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})) = q_i^s(t) = 0$, (b) follows from (4b).This implies that the inequalities are all tight, then $s_i(t)F_{s_i}(p_i(t) - r_i^s(t)) = q_i^s(t)$.

(2b) We consider the following two cases:

When $q_j^b(t) = 0$, then $x_{ij}(t) = 0$ for $\forall i : (i,j) \in E$, then $0 \le b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t))\Big) \overset{(c)}{\le} b_j(t)\Big(1 - F_{b_j}(F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}))\Big) = q_j^b(t) = 0$, where (c) follows from (4d).This implies that the inequalities are all tight, then $b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t))\Big) = q_j^b(t)$.

When $q_j^b(t) > 0$, pick a $i_1$ such that $x_{i_1 j}(t) > 0$ we have $p_{i_1}(t) = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - r_j^b(t)$ based on (4c); if there exists any $i_2$ such that $x_{i_2 j}(t) = 0$, we have $p_{i_2}(t) \ge F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - r_j^b(t)$ based on (4d); then $\min_{i':(i',j)\in E}\{p_{i'}(t)\} = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - r_j^b(t)$ , then $b_j(t)\Big(1 - F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t))\Big) = b_j(t)\Big(1 - F_{b_j}(F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}))\Big) = q_j^b(t)$.

(2f) We consider two cases: When $q_j^b(t) = 0$, then $x_{ij}(t) = 0$ for $\forall i : (i,j) \in E$. When $q_j^b(t) > 0$, we show in proof of (2b) that $p_i(t) \ge \min_{i':(i',j)\in E}\{p_{i'}\} = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - r_j^b(t)$ for $x_{ij}(t) = 0$.

Step 2: construct an instance. In each period, given $(\boldsymbol{q}^b(t), \boldsymbol{q}^s(t), \boldsymbol{x}(t))$ that satisfies (2c)-(2e), consider the following one-period problem:

$$
\begin{aligned}
\widetilde{R}_t = \max_{\boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} & \Big[\sum_{j\in\mathcal{B}} q_j^b + \sum_{i\in\mathcal{S}} q_i^s\Big] \\
\text{s.t. } & q_j^b \le q_j^b(t), && \forall j \in \mathcal{B} && (19a)\\
& q_i^s \le q_i^s(t), && \forall i \in \mathcal{S} && (19b)\\
& \sum_{j':(i,j')\in E} x_{i,j'} = q_i^s, && \forall i \in \mathcal{S} && (19c)\\
& q_j^b = \sum_{i':(i',j)\in E} x_{i',j}, && \forall j \in \mathcal{B} && (19d)\\
& x_{ij} \ge 0, && \forall (i,j) \in E. && (19e)
\end{aligned}
$$

Note that the feasible solution set is not empty, as $q_j^b = q_j^b(t)$ for $\forall j \in \mathcal{B}$, $q_i^s = q_i^s(t)$ for $\forall i \in \mathcal{S}$ and $x_{ij} = x_{ij}(t)$ for $\forall (i,j) \in E$ is a feasible solution. Since the constraints are all linear, the KKT conditions are necessary for the optimal solution in (19). Let $(\omega_i^s(t), \omega_j^b(t), \pi_{ij}(t))$ be the Lagrange multipliers corresponding to the constraint in (19c)-(19e), then we can write down the KKT conditions corresponding to $\boldsymbol{x}$:

$$
\begin{aligned}
& \omega_i^s(t) - \omega_j^b(t) - \pi_{ij}(t) = 0, && \forall (i,j) \in E, && (20a)\\
& x_{ij}(t) \ge 0 \perp \pi_{ij}(t) \ge 0, && \forall i \in \mathcal{S}, \forall (i,j) \in E. && (20b)
\end{aligned}
$$

Then we consider the commission and equilibrium price as follows:

$$p_i(t) = \omega_i^s(t), \qquad \forall i \in \mathcal{S}, \tag{21a}$$

$$r_i^s(t) = \omega_i^s(t) - F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right), \qquad \forall i \in \mathcal{S}, \tag{21b}$$

$$r_j^b(t) = F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) - \omega_j^b(t), \qquad \forall j \in \mathcal{B}. \tag{21c}$$

then the conditions (4a)-(4b) immediately follow. For (4c),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(a)}{=} \omega_j^b(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) = F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (a) follows from (20a) and (20b) that $\pi_{ij}(t) = 0$ when $x_{ij}(t) \geq 0$.

For (4d),

$$p_i(t) + r_j^b(t) = \omega_i^s(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(b)}{=} \omega_j^b(t) + \pi_{ij}(t) + F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) - \omega_j^b(t) \overset{(c)}{\geq} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}).$$

where (b) follows from (20a) and (c) follows from (20b). In summary, (4) holds for our construction in (21). ■

### A.3. Alternative Formulation for the Platform's Profit Optimization Problem

Consider the following convex optimization problem:

$$\mathcal{R}(T) = \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{x},\boldsymbol{q^s},\boldsymbol{q^b}} \sum_{t=1}^{T} \left[ \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big) q_i^s(t) \right] \tag{22a}$$

$$\text{s.t.} \quad q_i^s(t) \leq s_i(t), \qquad \forall i \in \mathcal{S}, t \in \{1,...,T\}, \tag{22b}$$

$$q_j^b(t) \leq b_j(t), \qquad \forall j \in \mathcal{B},\ t \in \{1,...,T\}, \tag{22c}$$

$$\sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i \in \mathcal{S},\ t \in \{1,...,T\}, \tag{22d}$$

$$q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j \in \mathcal{B},\ t \in \{1,...,T\}, \tag{22e}$$

$$x_{ij}(t) \geq 0, \qquad \forall (i,j) \in E,\ t \in \{1,...,T\}, \tag{22f}$$

$$s_i(t+1) \leq \alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t)), \qquad \forall i \in \mathcal{S},\ t \in \{1,...,T-1\}, \tag{22g}$$

$$b_j(t+1) \leq \alpha_j^b b_j(t) + \mathcal{G}_j^b(q_j^b(t)), \qquad \forall j \in \mathcal{B},\ t \in \{1,...,T-1\}. \tag{22h}$$

From Problem (22), we can establish Proposition 6, which enables us to solve a concave maximization problem to obtain the optimal solution $(\boldsymbol{s}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{q^s}, \boldsymbol{q^b})$ to Problem (22), from which we can further establish the optimal commission profile $(\boldsymbol{r}^s, \boldsymbol{r}^b)$ by solving a set of linear inequalities in (4) of Lemma 1.

Proposition 6. **(tightness of relaxation)** *For any $T \geq 1$, Problem (22) is a tight relaxation of Problem (3): $\mathcal{R}^*(T) = \mathcal{R}(T)$ and any optimal solution $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}})$ to Problem (22) is also optimal to Problem (3).*

**Proof of Proposition 6** We need to prove that the optimal solutions to (3) exist and that they achieve an objective value of $\mathcal{R}^* = \mathcal{R}$. We first show that $\mathcal{R}^* \leq \mathcal{R}$ in step 1, and construct a solution to (3) whose value equals to $\mathcal{R}$ in step 2, which implies that $\mathcal{R}^* = \mathcal{R}$ and the solution is optimal.

Step 1: Establish that $\mathcal{R}^* \le \mathcal{R}$. We show that any feasible solution to (3) is feasible in Problem (22) in Step 1.1, and we further show that it leads to a higher objective value in Problem (22) in Step 1.2.

Step 1.1: Any feasible solution in (3) is feasible in (22). To prove the claim, it is sufficient to verify the constraints (22b)-(22c), as other constraints immediately follow from the constraints in (3).

Based on (2a) and (2b), we have $q_i^s(t) = s_i(t) F_{s_i}(p_i(t) - r_i^s(t)) \le s_i(t)$ as $F_{s_i}(p_i(t) - r_i^s(t)) \in [0,1]$ and $q_j^b(t) = b_j(t)[1 - F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t))] \le b_j(t)$ as $F_{b_j}(\min_{i:(i,j)\in E}\{p_i(t)\} + r_j^b(t)) \in [0,1]$. Therefore, the constraints (22b)-(22c) are satisfied.

Step 1.2: Any feasible solution in (3) results in a higher objective value in (22). We first show that the optimal solution to Problem (3) satisfies the following:

$$\Big(F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i(t)}\Big)\Big) q_i^s(t) \le \Big(p_i(t) - r_i^s(t)\Big) q_i^s(t), \qquad \forall i \in \mathcal{S}, t \in \{1, ..., T\}, \tag{23a}$$

$$\Big(F_{b_j}^{-1}\Big(1 - \frac{q_j^b(t)}{b_j(t)}\Big)\Big) q_j^b(t) \ge \Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\Big) q_j^b(t), \qquad \forall j \in \mathcal{B}, t \in \{1, ..., T\}. \tag{23b}$$

For (23a), when $q_i^s(t) = 0$, (23a) immediately holds; when $q_i^s(t) > 0$, (23a) follows from (17b) and (17c) in the proof of Lemma 5. For (23b), when $q_j^b(t) = 0$, (23b) immediately holds; when $q_j^b(t) > 0$, (23b) follows from (18a) and (18b) in the proof of Lemma 5.

Given (23), the objective function in (3a) satisfies the following:

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} r_j^b(t) q_j^b(t) + \sum_{i\in\mathcal{S}} r_i^s(t) q_i^s(t)\Big] \\
&\overset{(a)}{=} \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} r_j^b(t) \sum_{i':(i',j)\in E} x_{i'j}(t) + \sum_{i\in\mathcal{S}} r_i^s(t) \sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&= \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} \sum_{i':(i',j)\in E} \Big(p_{i'}(t) + r_j^b(t)\Big) x_{i'j}(t) - \sum_{i\in\mathcal{S}} \Big(p_i(t) - r_i^s(t)\Big) \sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&\overset{(b)}{=} \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} \Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\Big) \sum_{i':(i',j)\in E} x_{i'j}(t) - \sum_{i\in\mathcal{S}} \Big(p_i(t) - r_i^s(t)\Big) \sum_{j':(i,j')\in E} x_{ij'}(t)\Big] \\
&= \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} \Big(\min_{i':(i',j)\in E}\{p_{i'}(t)\} + r_j^b(t)\Big) q_j^b(t) - \sum_{i\in\mathcal{S}} \Big(p_i(t) - r_i^s(t)\Big) q_i^s(t)\Big] \\
&\overset{(c)}{\le} \sum_{t=1}^{T} \Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1 - \frac{q_j^b(t)}{b_j(t)}) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)}) q_i^s(t)\Big] = \mathcal{R},
\end{aligned}$$

where (a) follows from (2c)-(2d); (b) follows from (2f) that $x_{ij} = 0$ for $i \notin \underset{i':(i',j)\in E}{argmin}\{p_i + r_i^s\}$; (c) follows from (23).

Step 2: Establish that $\mathcal{R}^* = \mathcal{R}$. Given any feasible solution to (22), we construct a feasible solution for (3) in Step 2.1, and we further obtain an objective value that equals $\mathcal{R}$ in Step 2.2.

Step 2.1: Construct a feasible solution for Problem (3).

In each period, given the solution for Problem (22), we consider the construction from (21) as in the proof of Lemma 1. We need to verify that all the constraints in (3) hold. Notice that we only need to verify that (2a) (2b) (2f) (3c) and (3d) hold, as other constraints exist in (22) and automatically hold.

(2a) from the construction of $p_i(t)$ and $r_i^s(t)$, we can establish that

$$s_i(t)F_{s_i}(p_i(t)-r_i^s(t))=s_i(t)F_{s_i}\left(F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right)\right)=q_i^s(t).$$

(2b) We consider the following two cases:

(i) if $q_j^b>0$, we pick a $i'$ such that $(i',j)\in E$, then there are two further cases: (1) $x_{i'j}>0$, then $p_{i'}(t)\overset{(a)}{=}\omega_{i'}^s(t)\overset{(b)}{=}\omega_j^b(t)+\pi_{i'j}(t)\overset{(c)}{=}\omega_j^b(t)$, where (a) follows form the construction of $p_{i'}(t)$; (b) follows from (20a); (c) follows from (20b) for $x_{i'j}>0$; (2) $x_{i'j}=0$, then $p_{i'}(t)=\omega_{i'}^s(t)=\omega_j^b(t)+\pi_{i'j}(t)\overset{(d)}{\geq}\omega_j^b(t)$, where (d) follows from (20b) for $x_{i'j}=0$. In summary, $\min_{i':(i',j)\in E}\{p_{i'}(t)\}=\omega_j^b(t)$, then

$$b_j(t)[1-F_{b_j}(\min_{i':(i',j)\in E}\{p_{i'}(t)\}+r_j^b(t))]=b_j(t)[1-F_{b_j}(\omega_j^b(t)+r_j^b(t))]\overset{(e)}{=}b_j(t)[1-F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})]=q_j^b(t),$$

where (e) follows from the construction of $r_j^b(t)$;

(ii) if $q_j^b=0$, we have $p_{i'}(t)=\omega_{i'}^s(t)=\omega_j^b(t)+\pi_{i'j}(t)\geq\omega_j^b(t)$, then $0\overset{(f)}{\leq}b_j(t)[1-F_{b_j}(\min\{p_i(t)+r_j^b(t)\})]\leq b_j(t)[1-F_{b_j}(\omega_j^b(t)+r_j^b(t))]\overset{(g)}{=}b_j(t)[1-F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})]=q_j^b(t)=0$,, where (f) follows from $F_{b_j}(\cdot)\leq 1$, (g) follows from the construction of $r_j^b(t)$. This implies that inequality must be tight. Therefore, (2b) holds.

(2f) We have verified in the proof of (2b) that for any $(i,j)\in E$, we have $p_i=\omega_j^b$ for $x_{ij}>0$ and $p_i\geq\omega_j^b$ for $x_{ij}=0$. Therefore, $x_{ij}=0$ for $i\notin\arg\min_{i':(i',j)\in E}p_{i'}$.

(3c) We first prove (22g) holds as equality by contradiction. Suppose that $s_i(t+1)<\alpha_i^s s_i(t)+\mathcal{G}_i^s(q_i^s(t))$ in the optimal solution to (22), then let $s_i'(t+1)=\alpha_i^s s_i(t)+\mathcal{G}_i^s(q_i^s(t))$, we can obtain higher objective value by replacing the $s_i(t+1)$ in the optimal solution with $s_i'(t+1)$ as the objective function in (22a) increases in $s_i(t+1)$; in addition, $s_i(t+2)\leq\alpha_i^s s_i(t+1)+\mathcal{G}_i^s(q_i^s(t+1))<\alpha_i^s s_i'(t+1)+\mathcal{G}_i^s(q_i^s(t+1))$, which implies that the constraint in (22g) still hold. This contradicts to our assumption that $s_i(t+1)<\alpha_i^s s_i(t)+\mathcal{G}_i^s(q_i^s(t))$ in the optimal solution to (22). Therefore, $s_i(t+1)=\alpha_i^s s_i(t)+\mathcal{G}_i^s(q_i^s(t))$ in the optimal solution to (22), and (3c) immediately holds.

(3d) Similarly, suppose that $b_j(t+1)<\alpha_j^b b_j(t)+\mathcal{G}_j^b(q_j^b(t))$ in the optimal solution to (22), then let $b_j'(t+1)=\alpha_j^b b_j(t)+\mathcal{G}_j^b(q_j^b(t))$, we can obtain higher objective value by replacing the $b_j(t+1)$ in the optimal solution with $b_j'(t+1)$. Therefore, $b_j(t+1)=\alpha_j^b b_j(t)+\mathcal{G}_j^b(q_j^b(t))$ in the optimal solution to (22), and (3d) immediately holds.

Step 2.2: Obtain a value that equals $\mathcal{R}$. We can deduce that

$$\begin{aligned}
\mathcal{R}^* &= \sum_{t=1}^T\left[\sum_{i\in\mathcal{S}}r_i^s(t)q_i^s(t)+\sum_{j\in\mathcal{B}}r_j^b(t)q_j^b(t)\right]\\
&\overset{(a)}{=}\sum_{t=1}^T\left[\sum_{i\in\mathcal{S}}(\omega_i^s(t)-F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)}))q_i^s(t)+\sum_{j\in\mathcal{B}}(F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})-\omega_j^b(t))q_j^b(t)\right]\\
&\overset{(b)}{=}\sum_{t=1}^T\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]\\
&\quad+\sum_{t=1}^T\left[\sum_{i\in\mathcal{S}}\omega_i^s(t)\sum_{j':(i,j')\in E}x_{ij'}(t)-\sum_{j\in\mathcal{B}}\omega_j^b(t)\sum_{i':(i',j)\in E}x_{i'j}(t)\right]
\end{aligned}$$

$$=\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]+\sum_{t=1}^{T}\left[\sum_{(i,j)\in E}\Big(\omega_i^s(t)-\omega_j^b(t)\Big)x_{ij}(t)\right]$$

$$\overset{(c)}{=}\sum_{t=1}^{T}\left[\sum_{j\in\mathcal{B}}F_{b_j}^{-1}(1-\frac{q_j^b(t)}{b_j(t)})q_j^b(t)-\sum_{i\in\mathcal{S}}F_{s_i}^{-1}(\frac{q_i^s(t)}{s_i(t)})q_i^s(t)\right]=\mathcal{R},$$

where (a) follows from the construction of $r_i^s(t)$ and $r_j^b(t)$, (b) follows from (22d) and (22e), (c) follows from (20a) and (20b) that when $x_{ij}>0$, $\omega_i^s=\omega_j^b$, while when $x_{ij}=0$, $\omega_i^s\geq\omega_j^b$. ■

## B. Proof of Results in Section 4

We provide and prove some auxiliary results in Appendix B.1 and prove the result in Section 4 in Appendix B.2. Finally, we provide some numerical results for TRP in Appendix B.3.

### B.1. Auxiliary Results for Section 4

Given the definitions of the value functions $\widetilde{F}_{b_j}$ for any $j\in\mathcal{B}$ and $\widetilde{F}_{s_i}$ for any $i\in\mathcal{S}$ from Problem (8), we have the following lemma.

Lemma 6. *$\widetilde{F}_{b_j}(q,b)$ is continuous at $(0,0)$ for $i\in\mathcal{S}$ and $\widetilde{F}_{s_i}(q,s)$ is continuous at $(0,0)$ for $j\in\mathcal{B}$.*

**Proof of Lemma 6.** We need to show that $\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b)=\widetilde{F}_{b_j}(0,0)=0$ and $\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s)=\widetilde{F}_{s_i}(0,0)=0$, which holds because

$$0\;\leq\;\lim_{(q,b)\downarrow(0,0)}\widetilde{F}_{b_j}(q,b)\;=\;\lim_{(q,b)\downarrow(0,0)}F_{b_j}^{-1}\Big(1-\frac{q}{b}\Big)q\;\leq\;\overline{v}_{b_j}\times 0\;=\;0,$$

$$0\;\leq\;\lim_{(q,s)\downarrow(0,0)}\widetilde{F}_{s_i}(q,s)\;=\;\lim_{(q,s)\downarrow(0,0)}F_{s_i}^{-1}\Big(\frac{q}{s}\Big)q\;\leq\;\overline{v}_{s_i}\times 0\;=\;0,$$

where given Assumption 2, all of the inequalities above follow from $F_{b_j}^{-1}(x)\in[0,\overline{v}_{b_j}]$ for $x\in[0,1]$ where $\overline{v}_{b_j}<\infty$ and $F_{s_i}^{-1}(x)\in[0,\overline{v}_{s_i}]$ for $x\in[0,1]$ where $\overline{v}_{s_i}<\infty$. ■

We next develop an auxiliary result about the growth of populations. To simplify the notation, we let $\mathcal{N}:=\{1,...,|\mathcal{S}|,|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$, where the first $|\mathcal{S}|$ nodes represent the types from the seller side and the last $|\mathcal{B}|$ nodes represent the types from the buyer side. In addition, we use $n_i(t)$ and $q_i(t)$ to respectively denote the population and transaction quantity of type $i\in\mathcal{N}$ at time $t\in\{1,\dots,T\}$. We define $\alpha_i:=\alpha_i^s$ for $i\in\{1,...,|\mathcal{S}|\}$ and $\alpha_i:=\alpha_{i-|\mathcal{S}|}^b$ for $i\in\{|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$; similarly, we define $\mathcal{G}_i(\cdot):=\mathcal{G}_i^s(\cdot)$ for $i\in\{1,...,|\mathcal{S}|\}$ and $\mathcal{G}_i(\cdot):=\mathcal{G}_{i-|\mathcal{S}|}^b(\cdot)$ for $i\in\{|\mathcal{S}|+1,...,|\mathcal{S}|+|\mathcal{B}|\}$. In addition, we define $\mathcal{N}^+:=\{i\in\mathcal{N}:\overline{n}_i>0\}$.

Recall that

$$m(t)=\min_{i\in\mathcal{N}^+}\frac{n_i(t)}{\overline{n}_i}. \tag{24}$$

Given the minimum population ratio $m(t)$ in (24), we let $l(t)$ be the agent type with the lowest population ratio at time $t$ or "the lowest node at time $t$" for short:

$$l(t):=\arg\min_{i\in\mathcal{N}^+}\frac{n_i(t)}{\overline{n}_i}. \tag{25}$$

If there is more than one $i$ such that $\frac{n_i(t)}{\overline{n}_i}=m(t)$, we can set $l(t)$ as any node with the minimum population ratio. After the population evolves in period $t$, it is worth noting that the lowest node can change. Let $\tau_0:=0$

and $m(\tau_0)$ be a dummy agent type with the minimum ratio in period 0 with $m(\tau_0) \notin \mathcal{S} \cup \mathcal{B}$. Moreover, we let $X$ be the total number of times that the lowest node changes in Algorithm 1 for some $X \in \{1, \dots, T\}$. we let $\tau_x := \min\{t : t > \tau_{x-1}, l(t) \neq l(\tau_{x-1})\}$ for $t \in \{1, \dots, T\}$, in which $\tau_x$ is the $x^{th}$ time that the lowest node changes for $x \in \{1, \dots, X\}$. For example, for $x \in \{0, 1, ..., X\}$, if node $i$ has the lowest ratio at time $\tau_x - 1$, then $n_{l(\tau_x - 1)}(\tau_x)$ denotes the population ratio of the node $i$ at time $\tau_x$.

Given the lowest node $l(t) \in \mathcal{S} \cup \mathcal{B}$ we let

$$g_t(n) := \alpha_{l(t)} n + \mathcal{G}_{l(t)}\left(n \frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right), \tag{26}$$

where $n \geq 0$. Then $g_t(n)$ is the transition equation for the lowest node in period $t$, as by the population transition specified in Algorithm 1 and the definition of $g_t(\cdot)$, we have that

$$n_{l(t)}(t+1) = \alpha_{l(t)} n_{l(t)}(t) + \mathcal{G}_{l(t)}\left(n_{l(t)}(t) \frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right) = g_t(n_{l(t)}(t)). \tag{27}$$

We have the following observation about function $g_t(\cdot)$.

Lemma 7. *$g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$. Moreover, its derivative satisfies $g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1, ..., T\}$. Moreover, $g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$ and $g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$.*

**Proof of Lemma 7.** We divide the proof arguments into the following components.

Differentiability and monotonicity. From Assumption 1, we have that function $\mathcal{G}_i(n)$ is continuously differentiable, increasing and strictly concave in $n \geq 0$, which directly implies that $g_t(n)$ is differentiable, increasing and strictly concave in $n \geq 0$.

$g_t'(\overline{n}_{l(t)}) < 1$ for all $t \in \{1, ..., T\}$. By Algorithm 1, we have that $\overline{n}_{l(t)} > 0$. Since $g_t(n)$ is continuous in $n \in [0, \overline{n}_{l(t)}]$ and differentiable $(0, \overline{n}_{l(t)})$, by the mean value theorem, there exists a $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$ such that $g_t'(\widetilde{n}_{l(t)}) = \frac{g_t(\overline{n}_{l(t)}) - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(a)}{=} \frac{\overline{n}_{l(t)} - g_t(0)}{\overline{n}_{l(t)} - 0} \overset{(b)}{=} \frac{\overline{n}_{l(t)} - 0}{\overline{n}_{l(t)} - 0} = 1$, where (a) follows from Lemma 2(i) and (b) follows from Assumption 1(i). Since $g_t(n)$ is strictly concave in $n \geq 0$, its derivative strictly decreases in $n \geq 0$, which implies that $g_t'(\overline{n}_{l(t)}) < 1$ given that $\widetilde{n}_{l(t)} \in (0, \overline{n}_{l(t)})$.

$g_t(n) - n < 0$ for $n > \overline{n}_{l(t)}$. we define that $y_t(n) := g_t(n) - n$, and it remains to show that $y_t(n) < 0$ for $n > \overline{n}_{l(t)}$. Since $y_t'(n_{l(t)}) = g_t'(n_{l(t)}) - 1 < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$ and $y_t(\overline{n}_{l(t)}) = 0$ based on Lemma 2(ii), $y_t(n_{l(t)}) < 0$ for $n_{l(t)} > \overline{n}_{l(t)}$.

$g_t(n) - n > 0$ for $0 < n < \overline{n}_{l(t)}$. It remains to show that $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. Note that $y_t(n)$ is concave in $n$. Since $y_t(0) = g_t(0) - 0 = 0$ and $y_t(\overline{n}_{l(t)}) = g_t(\overline{n}_{l(t)}) - \overline{n}_{l(t)} = 0$, we know $y_t((1-a) \times \overline{n}_{l(t)}) > a y_t(0) + (1-a) y_t(\overline{n}_{l(t)}) = 0 + 0 = 0$ for $a \in (0,1)$, therefore $y_t(n) > 0$ for $0 < n < \overline{n}_{l(t)}$. ■

Lastly, we formally define the myopic policy and establish its tractability as a supporting result for our proof arguments for Section 4.

Definition 2. **(myopic policy)** For $t \in \{1, ..., T\}$, given the current population $(\boldsymbol{s}^M(t), \boldsymbol{b}^M(t))$, the myopic policy solves the following optimization problem:

$$\mathcal{R}^{M*}(t) = \max_{\boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)} \sum_{i \in \mathcal{S}} r_i^s(t) q_i^s(t) + \sum_{j \in \mathcal{B}} r_j^b(t) q_j^b(t) \tag{28a}$$

$$\text{s.t.} \quad (\boldsymbol{s^M}(t), \boldsymbol{b^M}(t), \boldsymbol{r}(t), \boldsymbol{p}(t), \boldsymbol{x}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t)) \text{ satisfies } (2), \forall t \in \{1, ..., T\}. \tag{28b}$$

To solve Problem (28), we consider the following optimization problem:

$$\mathcal{R}^M(t) = \max_{\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{x}(t)} \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1-\frac{q_j^b(t)}{b_j^M(t)}\Big)q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s(t)}{s_i^M(t)}\Big)q_i^s(t) \tag{29a}$$

$$\text{s.t.} \quad q_i^s(t) \le s_i^M(t), \sum_{j':(i,j')\in E} x_{i,j'}(t) = q_i^s(t), \qquad \forall i\in\mathcal{S}, t\in\{1,...,T\}, \tag{29b}$$

$$q_j^b(t) \le b_j^M(t),\ q_j^b(t) = \sum_{i':(i',j)\in E} x_{i',j}(t), \qquad \forall j\in\mathcal{B},\ t\in\{1,...,T\}, \tag{29c}$$

$$x_{ij}(t)\ge 0, \qquad \forall (i,j)\in E,\ t\in\{1,...,T\}. \tag{29d}$$

Recalling the observations about Problem (22), we can apply exactly the same arguments as in the proof of Proposition 6 to establish the following result about Problem (29), whose proof will be neglected for avoiding repetition:

Corollary 2. *For any $t\in\{1,...,T\}$, Problem (29) is a tight relaxation of Problem (28), i.e., $\mathcal{R}^{M*}(t) = \mathcal{R}^M(t)$ and any optimal solution $(\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{x}(t))$ to Problem (29) is also optimal to Problem (28).*

### B.2. Proof of Results in Section 4

**Proof of Lemma 2.** Given Assumption 1, we first show that AVG's optimal solution and objective value are finite.

On the seller side, for any $i\in\mathcal{S}$, note that the constraint of AVG requires that $q_i^s \le s_i \le \frac{\mathcal{G}_i^s(q_i^s)}{1-\alpha_i^s}$, or equivalently, $\frac{\mathcal{G}_i^s(q_i^s)}{1-\alpha_i^s} - q_i^s \ge 0$. We consider the following two cases:

(1) when $(\mathcal{G}_i^s)'(0) \le 1-\alpha_i^s$, then for all $q_i^s>0$, we have $\frac{\mathcal{G}_i^s(q_i^s)}{1-\alpha_i^s} - q_i^s \overset{(a)}{\le} \frac{\mathcal{G}_i^s(0)+\mathcal{G}_i^{s\prime}(0)q_i^s}{1-\alpha_i^s} - q_i^s \overset{(b)}{\le} 0$, where (a) follows from the concavity of $\mathcal{G}_i^s(\cdot)$ by Assumption 1, and (b) follows from the condition that $(\mathcal{G}_i^s)'(0)\le 1-\alpha_i^s$ with $\mathcal{G}_i^s(0)=0$. Then, we obtain that any solution with $q_i^s>0$ is not feasible, which implies that a necessary condition for the optimal solution $(\overline{\boldsymbol{q}},\overline{\boldsymbol{s}},\overline{\boldsymbol{b}})$ to AVG in this case is that $\overline{q}_i^s=0$ for some $i\in\mathcal{S}$ with $\mathcal{G}_i^{s\prime}(0)>1-\alpha_i^s$;

(2) when $\mathcal{G}_i^{s\prime}(0)>1-\alpha_i^s$, given that $\lim_{q\to\infty}(\mathcal{G}_i^s)'(q)=0$ and $\mathcal{G}_i^s(q)$ is continuously differentiable in $q\ge 0$ by Assumption 1, there exists a $0<\hat{q}<\infty$ such that $(\mathcal{G}_i^s)'(\hat{q})<1-\alpha_i^s$. We observe that for any solution $\boldsymbol{q}$ where for all $q_i^s > \max\{\hat{q}, \frac{\mathcal{G}_i^s(\hat{q})}{1-\alpha_i^s}/(1-\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s})\}$, we have $\frac{\mathcal{G}_i^s(q_i^s)}{1-\alpha_i^s} - q_i^s \overset{(a)}{\le} \frac{\mathcal{G}_i^s(\hat{q})+\mathcal{G}_i^{s\prime}(\hat{q})(q_i^s-\hat{q})}{1-\alpha_i^s} - q_i^s < \frac{\mathcal{G}_i^s(\hat{q})+\mathcal{G}_i^{s\prime}(\hat{q})q_i^s}{1-\alpha_i^s} - q_i^s = \frac{\mathcal{G}_i^s(\hat{q})}{1-\alpha_i^s} + (\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s}-1)q_i^s \overset{(b)}{<} 0$, where (a) follows from the concavity of $\mathcal{G}_i^s(\cdot)$ in $\mathbb{R}_+\cup\{0\}$, and (b) follows from $\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s}-1<0$ given that $q_i^s > (\frac{\mathcal{G}_i^s(\hat{q})}{1-\alpha_i^s})/(1-\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s})$. In summary, we obtain that any solution $\boldsymbol{q}$ to AVG with $q_i^s > \max\{\hat{q}, (\frac{\mathcal{G}_i^s(\hat{q})}{1-\alpha_i^s})/(1-\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s})\}$ for some $i\in\mathcal{S}$ with $\mathcal{G}_i^{s\prime}(0)>1-\alpha_i^s$ is not feasible. Therefore, any optimal solution $(\overline{\boldsymbol{q}},\overline{\boldsymbol{s}},\overline{\boldsymbol{b}})$ to AVG satisfies that $\overline{q}_i^s \le \max\{\hat{q}, (\frac{\mathcal{G}_i^s(\hat{q})}{1-\alpha_i^s})/(1-\frac{\mathcal{G}_i^{s\prime}(\hat{q})}{1-\alpha_i^s})\}$ if $\mathcal{G}_i^{s\prime}(0)>1-\alpha_i^s$.

In summary, we have that $\overline{q}_i^s$ is finite for all $i\in\mathcal{S}$. Together with the observation that $\overline{q}_i^s \le \overline{s}_i \le \frac{\mathcal{G}_i^s(\overline{q}_i^s)}{1-\alpha_i^s}$, we obtain that $\overline{\boldsymbol{s}}$ is also finite.

On the buyer side, following the same arguments above, we can show that in any optimal solution $(\overline{\boldsymbol{q}},\overline{\boldsymbol{s}},\overline{\boldsymbol{b}})$ to AVG, $(\overline{q}_j^b,\overline{b}_j)$ is finite for all $j\in\mathcal{B}$.

Furthermore, for any $u\in[0,1]$, we have that $F_{s_i}^{-1}(u)\le\overline{v}_{s_i}<\infty$ for any $i\in\mathcal{S}$ and $F_{b_j}^{-1}(u)\le\overline{v}_{b_j}<\infty$ for all $j\in\mathcal{B}$. Therefore, the objective value of AVG is also finite. We have already shown that the feasible set of $(\boldsymbol{q},\boldsymbol{s},\boldsymbol{b})$ is closed and bounded. The constraints in (6a)-(6b) also ensure that the feasible set of $\boldsymbol{x}$ is closed and bounded. In summary, the feasible set characterized by constraint (6a)-(7b) is compact. In addition, the feasible set is not empty, as solution $\mathbf{0}$ is feasible. Furthermore, the objective function in (8a) is continuous in this compact set based on Assumption 2(i). By the extreme value theorem, an optimal solution $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to AVG exists.

We proceed to prove the lemma.

(i). By the extreme value theorem, the optimal solution to (8) exists. Since the objective function is strictly concave and the feasible region is a convex set, the optimal solution to (8) is unique.

(ii). We prove it by contradiction. If there exists a $i \in \mathcal{S}$ such that $\overline{s}_i < \alpha_i^s \overline{s}_i + \mathcal{G}_i^s(\overline{q}_i^s)$, then consider the alternative solution $s_i' := \overline{s}_i + \epsilon$ for a $\epsilon > 0$. We can always find a $\epsilon > 0$ small enough such that $(s_i + \epsilon)(1 - \alpha_i^s) < \mathcal{G}_i^s(q_i^s)$ holds. In addition, $s_i' > \overline{s}_i \geq \overline{q}_i^s$. Furthermore, since the objective function strictly increases in $s_i$, by replacing $\overline{s}_i$ with $s_i'$, we obtain a higher objective value. Therefore, the assumption $\alpha_i^s \overline{s}_i + \mathcal{G}_i^s(\overline{q}^s) - \overline{s}_i > 0$ contradicts the optimality of $(\overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ to Problem (8). The same proof arguments can be applied to show that $b_j \leq \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ is tight for any $j \in \mathcal{B}$. ∎

**Proof of Proposition 2.** By Proposition 6, $\mathcal{R}(T) = \mathcal{R}^*(T)$. So it suffices to show that there exists a constant $C_1$ such that $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \leq C_1$. To prove the result, we establish the following two claims.

Claim 1: $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq -C_1'$. We delay the proof to Step 3 in the proof of Theorem 1 that there exists a constant $C_1'$ and a policy $\pi$ such that $\mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$, which further implies that $\mathcal{R}(T) - T\overline{\mathcal{R}} \geq \mathcal{R}^\pi(T) - T\overline{\mathcal{R}} \geq -C_1'$ given that $\mathcal{R}(T) \geq \mathcal{R}^\pi(T)$.

Claim 2: $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$. Before proving the claim, we first consider the following optimization problem for any $T > 0$:

$$
\begin{aligned}
\widetilde{\mathcal{R}} \; = \; \max_{\boldsymbol{s},\boldsymbol{b},\boldsymbol{q^s},\boldsymbol{q^b},\boldsymbol{x}} \quad & \sum_{j\in\mathcal{B}} \widetilde{F}_{b_j}(q_j^b, b_j) - \sum_{i\in\mathcal{S}} \widetilde{F}_{s_i}(q_i^s, s_i) && && \text{(30a)}\\
\text{s.t.} \quad & q_i^s \leq s_i, && \forall i \in \mathcal{S}, && \text{(30b)}\\
& q_j^b \leq b_j, && \forall j \in \mathcal{B}, && \text{(30c)}\\
& \sum_{j:(i,j)\in E} x_{ij} = q_i^s, && \forall i \in \mathcal{S}, && \text{(30d)}\\
& q_j^b = \sum_{i:(i,j)\in E} x_{ij}, && \forall j \in \mathcal{B}, && \text{(30e)}\\
& x_{ij} \geq 0, && \forall (i,j) \in E, && \text{(30f)}\\
& s_i \leq \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s) + \frac{s_i(1)}{T}, && \forall i \in \mathcal{S}, && \text{(30g)}\\
& b_j \leq \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b) + \frac{b_j(1)}{T}, && \forall j \in \mathcal{B}. && \text{(30h)}
\end{aligned}
$$

Note that the only difference between Problem (30) and Problem (8) is the right-hand side of the constraints (30g)-(30h). Given that $s_i(1) > 0$ for all $i \in \mathcal{S}$ and $b_j(1) > 0$ for all $j \in \mathcal{B}$, Problem (30) could be viewed as a relaxation of Problem (8). We first show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$ and then show that there exists a positive constant $C_1''$ such that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C_1''$ for any $T > 0$. Consequently, we can have $\mathcal{R}(T) - T\overline{\mathcal{R}} \leq C_1''$ for any $T > 0$.

Step 2.1: Show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. For any optimal solution $(\boldsymbol{s}(t), \boldsymbol{b}(t), \boldsymbol{q^s}(t), \boldsymbol{q^b}(t), \boldsymbol{x}(t) : t = 1, \ldots, T)$ to Problem (22), we construct the following alternative solution vector $(\overline{\boldsymbol{s}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{q}}^{\boldsymbol{s}}, \overline{\boldsymbol{q}}^{\boldsymbol{b}}, \overline{\boldsymbol{x}})$ for Problem (30):

$$
\overline{s}_i = \frac{1}{T}\sum_{t=1}^{T} s_i(t) \text{ and } \overline{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t), \qquad \forall i \in \mathcal{S},
$$

$$\bar{b}_j = \frac{1}{T}\sum_{t=1}^{T} b_j(t) \text{ and } \bar{q}_j^b = \frac{1}{T}\sum_{t=1}^{T} q_j^b(t), \qquad \forall j \in \mathcal{B},$$
$$\bar{x}_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{ij}(t), \qquad \forall (i,j) \in E$$

We establish the feasibility of $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}}, \bar{\boldsymbol{x}})$ for Problem (30) in Step 2.1.1 and then show that $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$ in Step 2.1.2.

Step 2.1.1: Feasibility. First, from the constraints in Problem (22), we can easily show (30b) - (30f) hold. In particular, $\bar{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t) \overset{(a)}{\leq} \frac{1}{T}\sum_{t=1}^{T} s_i(t) = \bar{s}_i$. The same argument applies for $\bar{q}_j^b$ and $\bar{b}_j$ on the buyer side. For (30d)-(30e), $\bar{q}_i^s = \frac{1}{T}\sum_{t=1}^{T} q_i^s(t) \overset{(b)}{=} \frac{1}{T}\sum_{j':(i,j')\in E}\sum_{t=1}^{T} x_{ij'}(t) = \sum_{j':(i,j')\in E} \bar{x}_{ij}$. and $\bar{q}_j^b = \frac{1}{T}\sum_{t=1}^{T} q_j^b(t) \overset{(c)}{=} \frac{1}{T}\sum_{i':(i',j)\in E}\sum_{t=1}^{T} x_{i'j}(t) = \sum_{i':(i',j)\in E} \bar{x}_{ij}$. For (30f), $\bar{x}_{ij} = \frac{1}{T}\sum_{t=1}^{T} x_{ij}(t) \overset{(e)}{\geq} 0$.

For constraints in (30g)-(30h), we show that

$$\begin{aligned}
&\bar{s}_i - \Big(\alpha_i^s \bar{s}_i + \mathcal{G}_i^s(\bar{q}_i^s)\Big) - \frac{s_i(1)}{T} \\
\overset{(a)}{=}\ & \frac{1}{T}\sum_{t=1}^{T} s_i(t) - \alpha_i^s \frac{1}{T}\sum_{t=1}^{T} s_i(t) - \mathcal{G}_i^s\Big(\frac{1}{T}\sum_{t=1}^{T} q_i^s(t)\Big) - \frac{s_i(1)}{T} \\
\overset{(b)}{\leq}\ & \frac{1}{T}\sum_{t=1}^{T}\Big[s_i(t) - \Big(\alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t))\Big)\Big] - \frac{s_i(1)}{T} \\
=\ & \frac{1}{T}\sum_{t=1}^{T-1}\Big[s_i(t+1) - \Big(\alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t))\Big)\Big] + \frac{1}{T}(s_i(1) - \alpha_i^s s_i(T) - \mathcal{G}_i^s(q_i^s(T))) - \frac{s_i(1)}{T} \\
\leq\ & 0 + \frac{1}{T}\Big(-\alpha_i^s s_i(T) - \mathcal{G}_i^s(q_i^s(T))\Big) \ \leq\ 0,
\end{aligned}$$

where (a) follows from the construction of $\bar{s}_i$ and $\bar{q}_i^s$ at the beginning of Step 2.1; (b) follows the Assumption 1(ii) that $\mathcal{G}_i^s(\cdot)$ is concave. This proves that Constraint (30g) holds. Following the same argument, we can show that Constraint (30h) holds.

Step 2.1.2: $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$. Given the construction of $\bar{s}_i$ and $\bar{b}_j$, we obtain that $\bar{s}_i > 0$ and $\bar{b}_j > 0$. Given the definitions of $\widetilde{F}_b(\bar{q}_j^b, \bar{b}_j)$ and $\widetilde{F}_s(\bar{q}_i^s, \bar{s}_i)$ in Problem (8), the objective value in (8a) is given by $\sum_{j\in\mathcal{B}} F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{\bar{b}_j})\bar{q}_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}(\frac{\bar{q}_i^s}{\bar{s}_i})\bar{q}_i^s$. This allows us to establish that

$$\begin{aligned}
T\widetilde{\mathcal{R}} &\overset{(a)}{=} T\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{\frac{1}{T}\sum_{t=1}^{T} q_j^b(t)}{\frac{1}{T}\sum_{t=1}^{T} b_j(t)}\Big)\frac{1}{T}\sum_{t=1}^{T} q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{\frac{1}{T}\sum_{t=1}^{T} q_i^s(t)}{\frac{1}{T}\sum_{t=1}^{T} s_i(t)}\Big)\frac{1}{T}\sum_{t=1}^{T} q_i^s(t)\Big] \\
&\overset{(b)}{\geq} T \times \frac{1}{T}\sum_{t=1}^{T}\Big[\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\left(1 - \frac{q_j^b(t)}{b_j(t)}\right) q_j^b(t) - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\left(\frac{q_i^s(t)}{s_i(t)}\right) q_i^s(t)\Big] = \mathcal{R}(T).
\end{aligned}$$

where (a) follows from the construction of $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}}, \bar{\boldsymbol{x}})$ in Step 2-1; (b) follows from the concavity of $F_{b_j}^{-1}(1 - \frac{a}{b})a$ and $-F_{s_i}^{-1}(\frac{a}{b})a$ by Assumption 3.

Summarizing the arguments in these two steps, we have $T\widetilde{\mathcal{R}} \geq \mathcal{R}(T)$.

Step 2.2: Show that $T\widetilde{\mathcal{R}} - T\overline{\mathcal{R}} \leq C_1'' $ for some $C_1'' > 0$. Let $(\boldsymbol{\mu}^s, \boldsymbol{\mu}^b)$ be the dual optimal solution corresponding to the constraint $s_i \leq \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ and $b_j \leq \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ in Problem (8), then $\mu_i^s \geq 0$ for $\forall i \in \mathcal{S}$ and $\mu_j^b \geq 0$ for $\forall j \in \mathcal{B}$ according to duality theory. Note that the only difference between Problem (8) and Problem

(30) is the right-hand side of the constraints in (30g)-(30h). Therefore, based on (5.57) in Boyd et al. (2004), we can establish that

$$\widetilde{\mathcal{R}} \le \overline{\mathcal{R}} + \sum_{i\in\mathcal{S}} \mu_i^s \times \frac{1}{T} s_i(1) + \sum_{j\in\mathcal{B}} \mu_j^b \times \frac{1}{T} b_j(1),$$

which further implies that

$$T(\widetilde{\mathcal{R}} - \overline{\mathcal{R}}) \le T\Big(\sum_{i\in\mathcal{S}} \mu_i^s \times \frac{1}{T} s_i(1) + \sum_{j\in\mathcal{B}} \mu_j^b \times \frac{1}{T} b_j(1)\Big) = \sum_{i\in\mathcal{S}} \mu_i^s s_i(1) + \sum_{j\in\mathcal{B}} \mu_j^b b_j(1).$$

We let $C_1'' := \sum_{i\in\mathcal{S}} \mu_i^s s_i(1) + \sum_{j\in\mathcal{B}} \mu_j^b b_j(1)$, and obtain the desired result.

In summary, $|\mathcal{R}(T) - T\overline{\mathcal{R}}| \le C_1$, where $C_1 = \max\{|C_1'|, |C_1''|\}$. ■

**Proof of Theorem 1.** We divide the proof arguments into the following steps: in Step 1, we show that the solution generated by the TRP is feasible to Problem (22); in Step 2, we show under the TRP, there exists a constant $\gamma \in (0,1)$ such that $|m(t+1)-1| \le \gamma|m(t)-1|$ for $\forall t \in \{1,...,T-1\}$; in Step 3, we show that there exists a constant $C_1'$ such that $T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T) \le C_1'$. Then, together with Proposition 2, we conclude that there exists a constant $C_2 := C_1 + C_1'$ such that $\mathcal{L}^{TR}(T) = \mathcal{R}^*(T) - \mathcal{R}^{TR}(T) = \big(\mathcal{R}^*(T) - T\overline{\mathcal{R}}\big) + \big(T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\big) \le C_2$.

Step 1: Show that the solution generated by the TRP is feasible to Problem (22).

We let $(\overline{\boldsymbol{q}}^s, \overline{\boldsymbol{q}}^b, \overline{\boldsymbol{x}}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{b}})$ be the optimal solution to the AVG in Problem (8). Recall the definition of $m(t)$ in (24), we have that the TRP uses the commissions $(\boldsymbol{r}^s(t), \boldsymbol{r}^b(t))$ in Algorithm 1 to induce the populations and transaction quantities that satisfy $q_i^s(t) = \overline{q}_i^s m(t)$ and $s_i(t+1) = \alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t))$ for $i \in \mathcal{S}$. Similarly, for the buyer side, $q_j^b(t) = \overline{q}_j^b m(t)$ and $b(t+1) = \alpha_j^b b(t) + \mathcal{G}_j^b(q_j^b(t))$ for $j \in \mathcal{B}$.

We first verify that the feasibility of the transaction vector $(\boldsymbol{q}^s(t), \boldsymbol{q}^b(t), \boldsymbol{x}(t))$ to Constraints (22b) - (22h).

(22b)-(22c). $q_i^s(t) \overset{(a)}{=} \overline{q}_i^s m(t) \overset{(b)}{\le} s_i(t)\frac{\overline{q}_i^s}{\overline{s}_i} \overset{(c)}{\le} s_i(t)$, where (a) follows from Algorithm 1; (b) follows directly from the definition of $m(t)$ in (24); (c) follows from Constraint (6a) that $\overline{q}_i^s \le \overline{s}_i$. Similarly, $q_j^b(t) = \overline{q}_j^b m(t) \le b_j(t)\frac{\overline{q}_j^b}{\overline{b}_j} \le b_j(t)$.

(22d)-(22e). $q_i^s(t) = \overline{q}_i^s m(t) \overset{(a)}{=} \sum_{j':(i,j')\in E} \overline{x}_{i,j'} m(t) \overset{(b)}{=} \sum_{j':(i,j')\in E} x_{i,j'}(t)$, where (a) follows from (6a); (b) follows from Algorithm 1. Similarly, $q_j^b(t) = \overline{q}_j^b m(t) = \sum_{i':(i',j)\in E} \overline{x}_{i',j} m(t) = \sum_{i':(i',j)\in E} x_{i',j}(t)$.

(22f) . $x_{i,j} = \overline{x}_{i,j} m(t) \ge 0$ follows from (6c).

(22g)-(22h). Given $s_i(t+1) = \alpha_i^s s_i(t) + \mathcal{G}_i^s(q_i^s(t))$, the inequality is a relaxation, which directly follows. A similar argument holds for the buyer side.

Summarizing the arguments above, the solution generated by the TRP is feasible to Problem (22).

Step 2: Show that there exists a constant $\gamma \in (0,1)$ such that $|m(t+1)-1| \le \gamma|m(t)-1|$ for $t \in \{1,...,T-1\}$.

Recall the definition of $l(t)$ and $g_t(n)$ in (25) and (26), respectively. We discuss three cases: (1) $m(1) > 1$, (2) $m(1) < 1$ and (3) $m(1) = 1$. In each case, we will first show that $m(t)$ gets closer to 1 as $t$ increases, and then we show that the convergence rate can be upper bounded by $\gamma < 1$.

Step 2 - Case 1: $m(1) > 1$.

Step 2 - Case 1 - Step 2.1: Show that $m(1) > m(2) > ... > m(T-1) > m(T) > 1$. To prove the claim of this case, we show that for any $t \in \{1,\ldots,T-1\}$, if $m(t) > 1$, then $m(t) > m(t+1) > 1$. Let $X > 0$ denote the number of times the agent type with the lowest ratio changes. We consider the following two cases for $\forall t \in \{1,\dots,T\}$: (1) the lowest node does not change in the next period, i.e., $\tau_x \le t \le \tau_{x+1}-2$ for $x \in \{0,...,X-1\}$ ; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1}-1$ for $x \in \{0,...,X-1\}$.

(1) For any $\tau_x \le t \le \tau_{x+1}-2$ with $x \in \{0,...,X-1\}$, we show that if $m(t) > 1$, then $m(t) > m(t+1) > 1$. Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds given that $l(t) = l(t+1)$ for $\tau_x \le t \le \tau_{x+1}-2$ and $x \in \{0,\dots,X-1\}$. Then, to show that $m(t) > m(t+1) > 1$, it is equivalent to establish that $n_{l(t)}(t) > n_{l(t)}(t+1) > \overline{n}_{l(t)}$. First, we have

$$n_{l(t)}(t+1) - n_{l(t)}(t) \overset{(b)}{=} g_t(n_{l(t)}(t)) - n_{l(t)}(t) \overset{(c)}{<} 0,$$

where (b) follows from (27); (c) follows directly from Lemma 7. Second, we deduce that

$$n_{l(t)}(t+1) - \overline{n}_{l(t)} \overset{(d)}{=} g_t(n_{l(t)}(t)) - \overline{n}_{l(t)} \overset{(e)}{=} g_t(n_{l(t)}(t)) - g_t(\overline{n}_{l(t)}) \overset{(f)}{>} 0,$$

where (d) follows from (27); (e) follows from Lemma 2(ii); (f) follows from $n_{l(t)}(t) > \overline{n}_{l(t)}$ given that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}} > 1$ and that $g_t(n)$ increases in $n \ge 0$ from Lemma 7.

In summary, for $\tau_x \le t \le \tau_{x+1}-2$, if $m(t) > 1$, then $m(t) > m(t+1) > 1$.

(2) For $t = \tau_x - 1$ with $x \in \{1,\dots,X\}$, we want to show that if $m(\tau_x - 1) > 1$, then $m(\tau_x - 1) > m(\tau_x) > 1$. To prove this, we can deduce that

$$m(\tau_x) = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(a)}{\le} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}} \overset{(b)}{<} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x-1),$$

where (a) follows directly from the definition that $l(\tau_x)$ in (25); (b) follows from $n_{l(\tau_x-1)}(\tau_x) = g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1)) < n_{l(\tau_x-1)}(\tau_x-1)$, where the second inequality follows from $n_{l(\tau_x-1)}(\tau_x-1) > \overline{n}_{l(\tau_x-1)}$ given that $m(\tau_x-1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} > 1$ and Lemma 7. Therefore, $m(\tau_x) < m(\tau_x-1)$.

Next, we show that $m(\tau_x) > 1$. Since

$$\begin{aligned} m(\tau_x) &= \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \overset{(c)}{=} \frac{\alpha_{l(\tau_x)} n_{l(\tau_x)}(\tau_x-1) + \mathcal{G}_{l(\tau_x)}\left(\overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}}\right)}{\overline{n}_{l(\tau_x)}} \\ &\overset{(d)}{\ge} \frac{\alpha_{l(\tau_x)} n_{l(\tau_x)}(\tau_x-1) + \mathcal{G}_{l(\tau_x)}(\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \overset{(e)}{>} \frac{\alpha_{l(\tau_x)} \overline{n}_{l(\tau_x)} + \mathcal{G}_{l(\tau_x)}(\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} = 1, \end{aligned}$$

where (c) follows from Algorithm 1; (d) follows from the condition that $\frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} = m(\tau_x-1) > 1$ given that $\mathcal{G}_{l(\tau_x)}(q)$ increases in $q \ge 0$; (e) follows from $\frac{n_{l(\tau_x)}(\tau_x-1))}{\overline{n}_{l(\tau_x)}} \ge m(\tau_x-1) > 1$. Therefore, $m(\tau_x) > 1$.

Based on the arguments above, if $m(t) > 1$, then $m(t) > m(t+1) > 1$, which holds for any $t \in \{1,...,T-1\}$. Thus, we can conclude that if $m(1) > 1$, then $m(1) > m(2) > ... > m(T-1) > m(T) > 1$.

Step 2 - Case 1 - Step 2.2: Show that there exists a constant $\gamma_1 \in (0,1)$ such that $|m(t+1)-1| \le \gamma_1 |m(t)-1|$ for any $t \in \{1,\dots,T\}$. Again, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x \le t \le \tau_{x+1}-2$ for any $x \in \{0,...,X-1\}$; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1}-1$ for any $x \in \{0,...,X-1\}$. For both cases, we first show that $\big|m(t+1)-1\big| \le g'_t(\overline{n}_{l(t)})\big|m(t)-1\big|$. Then we show that there exists a $\gamma_1 \in (0,1)$ independent from $T$ such that for any positive integer $T$, $\max_{t=1,\dots,T} g'_t(\overline{n}_{l(t)}) \le \gamma_1 < 1$.

(1) For $\tau_x \le t \le \tau_{x+1}-2$, we observe that

$$\begin{aligned}\left|n_{l(t)}(t+1)-\overline{n}_{l(t)}\right| &\overset{(a)}{=} n_{l(t)}(t+1)-\overline{n}_{l(t)} \overset{(b)}{=} g_t(n_{l(t)}(t))-g_t(\overline{n}_{l(t)})\\ &\overset{(c)}{<} (n_{l(t)}(t)-\overline{n}_{l(t)})g_t'(\overline{n}_{l(t)}) \overset{(d)}{=} \left|n_{l(t)}(t)-\overline{n}_{l(t)}\right|g_t'(\overline{n}_{l(t)}),\end{aligned}$$

where (a) follows from $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} \ge m(t+1) > 1$ for any $t \in \{1,...,T-1\}$; (b) follows from (27) and Lemma 2(ii); (c) follows from Lemma 7 given that $g_t(n)$ is strictly concave in $n \ge 0$; (d) follows from $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}} > 1$ for any $t \in \{1,...,T\}$. Therefore, $\left|m(t+1)-1\right| = \left|\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}-1\right| < g_t'(\overline{n}_{l(t)})\left|\frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}-1\right| = g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$.

(2) For $t = \tau_x - 1$,

$$\begin{aligned}\left|m(\tau_x)-1\right| &\overset{(a)}{=} m(\tau_x)-1 = \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}}-1 \overset{(b)}{\le} \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}-1\\ &\overset{(c)}{=} \frac{g_{\tau_x-1}(n_{l(\tau_x-1)}(\tau_x-1))-g_{\tau_x-1}(\overline{n}_{l(\tau_x-1)})}{\overline{n}_{l(\tau_x-1)}} \overset{(d)}{<} \left(\frac{n_{l(\tau_x-1)}(\tau_x-1)-\overline{n}_{l(\tau_x-1)}}{\overline{n}_{l(\tau_x-1)}}\right)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)})\\ &= (m(\tau_x-1)-1)g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}) \overset{(e)}{=} \left|m(\tau_x-1)-1\right|g_{\tau_x-1}'(\overline{n}_{l(\tau_x-1)}),\end{aligned}$$

where (a) follows from $m(t) \ge 1$ for any $t \in \{1,...,T\}$; (b) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \le \frac{n_{l(\tau_x-1)}(\tau_x)}{\overline{n}_{l(\tau_x-1)}}$; (c) follows from $g_t(\cdot)$ in (26) and Lemma 2(ii); (d) follows from the strict concavity of $g_t(\cdot)$ in Lemma 7; (e) follows from $m(\tau_x-1) = \frac{n_{l(\tau_x-1)}(\tau_x-1)}{\overline{n}_{l(\tau_x-1)}} > 1$.

In summary, $\left|m(t+1)-1\right| \le g_t'(\overline{n}_{l(t)})\left|m(t)-1\right|$ for any $t \in \{1,\dots,T\}$.

Given the solution vector $(\overline{\boldsymbol{q}},\overline{\boldsymbol{n}})$, we let $\gamma_1 := \max_{i\in\mathcal{N}^+}\{\alpha_i + \mathcal{G}_i'(\overline{q}_i)\frac{\overline{q}_i}{\overline{n}_i}\}$. We first show that $\gamma_1 \in (0,1)$. Define $y_i(n) := \alpha_i n + \mathcal{G}_i(n\frac{\overline{q}_i}{\overline{n}_i})$. In $[0,\overline{n}_i]$, by the mean value theorem, there exists a $\widetilde{n}_i \in (0,\overline{n}_i)$ such that $y_i'(\widetilde{n}_i) = \frac{y_i(\overline{n}_i)-y_i(0)}{\overline{n}_i-0} \overset{(a)}{=} \frac{\overline{n}_i-y_i(0)}{\overline{n}_i-0} \overset{(b)}{=} \frac{\overline{n}_i-0}{\overline{n}_i-0} = 1$, where (a) follows from $y_i(\overline{n}_i) = \overline{n}_i$ with $y_i(n) := \alpha_i n + \mathcal{G}_i(n\frac{\overline{q}_i}{\overline{n}_i})$ given Lemma 2(ii), and (b) follows from Assumption 1(i). Since $y_i(n)$ is strictly concave in $n \ge 0$, its derivative strictly decreases in $n \ge 0$, which implies that $y_i'(\overline{n}_i) < 1$ given that $\widetilde{n}_i \in (0,\overline{n}_i)$, then $y_i'(\overline{n}_i) = \alpha_i + \mathcal{G}_i'(\overline{q}_i)\frac{\overline{q}_i}{\overline{n}_i} < 1$ for any $i \in \mathcal{N}$. Given the finite network $G(\mathcal{S}\cup\mathcal{B},E)$, $\gamma_1 := \max_{i\in\mathcal{N}^+}\{\alpha_i + \mathcal{G}_i'(\overline{q}_i)\frac{\overline{q}_i}{\overline{n}_i}\} < 1$.

By definition of $g_t(\cdot)$ in (26), we have that

$$\max_{t=1,\dots,T} g_t'(\overline{n}_{l(t)}) = \max_{t=1,\dots,T}\left(\alpha_{l(t)} + \mathcal{G}_{l(t)}'(\overline{q}_{l(t)})\frac{\overline{q}_{l(t)}}{\overline{n}_{l(t)}}\right) \le \max_{i\in\mathcal{N}^+}\left(\alpha_i + \mathcal{G}_i'(\overline{q}_i)\frac{\overline{q}_i}{\overline{n}_i}\right) \le \gamma_1 \ < \ 1,$$

which allows us to conclude the contraction arguments for the case of $m(1) > 1$.

Step 2 - Case 2: $m(1) < 1$.

Step 2 - Case 2 - Step 2.1: Show that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$. Similar to the discussions in Step 2 - Case 1, we consider the following two cases: (1) the lowest node does not change in the next step, i.e., $\tau_x \le t \le \tau_{x+1}-2$ for any $x \in \{0,...,X-1\}$ ; (2) the lowest node changes in next step, i.e., $t = \tau_{x+1}-1$ for any $x \in \{0,...,X-1\}$.

(1) For $\tau_x \le t \le \tau_{x+1}-2$, we want to show that if $m(t) < 1$, then $m(t) < m(t+1) < 1$.

Recall that $m(t) = \frac{n_{l(t)}(t)}{\overline{n}_{l(t)}}$ and $m(t+1) = \frac{n_{l(t+1)}(t+1)}{\overline{n}_{l(t+1)}} \overset{(a)}{=} \frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}}$, where (a) holds as $l(t) = l(t+1)$ for $\tau_x \le t \le \tau_{x+1}-2$. Therefore, $m(t) < 1$ implies that $n_{l(t)}(t) < \overline{n}_{l(t)}$. We observe that $m(t) < m(t+1) < 1$ is then equivalent to $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$, which holds because

$$n_{l(t)}(t+1) - n_{l(t)}(t) \ = \ g_t(n_{l(t)}(t)) - n_{l(t)}(t) \ > \ 0,$$

where the equality follows from (27) and the inequality follows from the condition that $0 < n_{l(t)}(t) < \overline{n}_{l(t)}$ and Lemma 7. In addition,

$$n_{l(t)}(t+1) - \overline{n}_{l(t)} = g_t(n_{l(t)}(t)) - g_t(\overline{n}_{l(t)}) < 0,$$

given that $n_{l(t)}(t) < \overline{n}_{l(t)}$ and that $g_t(n)$ increases in $n \geq 0$ based on Lemma 7. The derivations above allow us to establish that $n_{l(t)}(t) < n_{l(t)}(t+1) < \overline{n}_{l(t)}$.

(2) For $t = \tau_x - 1$, we show that $m(\tau_x - 1) < m(\tau_x) < 1$ if $m(\tau_x - 1) < 1$, then

$$\begin{aligned} m(\tau_x) \ &\overset{(a)}{=} \ \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = \frac{\alpha_{l(\tau_x)} n_{l(\tau_x)}(\tau_x - 1) + \mathcal{G}_{l(\tau_x)}(\overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}})}{\overline{n}_{l(\tau_x)}} \\ &\overset{(b)}{\geq} \frac{\alpha_{l(\tau_x)} \overline{n}_{l(\tau_x)} \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}} + \mathcal{G}_{l(\tau_x)}(\overline{q}_{l(\tau_x)} \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}})}{\overline{n}_{l(\tau_x)}} \\ &\overset{(c)}{>} \frac{\alpha_{l(\tau_x)} \overline{n}_{l(\tau_x)} \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}} + \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}} \mathcal{G}_{l(\tau_x)}(\overline{q}_{l(\tau_x)})}{\overline{n}_{l(\tau_x)}} \\ &= \frac{\overline{n}_{l(\tau_x)} \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}}}{\overline{n}_{l(\tau_x)}} \ \overset{(d)}{=} \ \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}} = m(\tau_x - 1), \end{aligned}$$

where (a) follows the definition of $m(\tau_x)$ in (24) and $l(\tau_x)$ in (25); (b) follows from $\frac{n_{l(\tau_x)}(\tau_x - 1)}{\overline{n}_{l(\tau_x)}} \geq m(\tau_x - 1) = \frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}}$ given the definition of $m(\tau_x - 1)$ in (24); (c) follows from $\mathcal{G}_i(a\overline{q}_i) = \mathcal{G}_i(a\overline{q}_i + (1-a)0) > a\mathcal{G}_i(\overline{q}_i) + (1-a)\mathcal{G}_i(0) = a\mathcal{G}_i(\overline{q}_i)$ for $0 < a < 1$ where the first inequality $\mathcal{G}_i(0) = 0$ and the second inequality follows from the condition that $\mathcal{G}_i(q_i)$ is strictly concave in $q_i$; in addition, (d) follows from $\frac{n_{l(\tau_x - 1)}(\tau_x - 1)}{\overline{n}_{l(\tau_x - 1)}} = m(\tau_x - 1) < 1$ where the inequality holds given the condition that $m(\tau_x - 1) < 1$ in this case. In summary, we have $m(\tau_x) > m(\tau_x - 1)$.

To proceed, we further observe that

$$m(\tau_x) \ = \ \frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} \ \overset{(d)}{\leq} \ \frac{n_{l(\tau_x - 1)}(\tau_x)}{\overline{n}_{l(\tau_x - 1)}} \ \overset{(e)}{<} \ 1,$$

where (d) follows from $\frac{n_{l(\tau_x)}(\tau_x)}{\overline{n}_{l(\tau_x)}} = m(\tau_x) \leq \frac{n_{l(\tau_x - 1)}(\tau_x)}{\overline{n}_{l(\tau_x - 1)}}$ given the definition of $m(\tau_x)$ in (24); (e) follows from Lemma 7 that $n_{l(\tau_x - 1)}(\tau_x) = g_{\tau_x - 1}(n_{l(\tau_x - 1)}(\tau_x - 1)) < \overline{n}_{l(\tau_x - 1)}$ for $n_{l(\tau_x - 1)}(\tau_x - 1) < \overline{n}_{l(\tau_x - 1)}$. Thus, we have that $m(\tau_x) < 1$.

In summary, $m(t) < m(t+1) < 1$ if $m(t) < 1$ for $\forall t \in \{1, ..., T-1\}$. Since $m(t) < 1$, we obtain that $m(1) < m(2) < ... < m(T-1) < m(T) < 1$.

Step 2 - Case 2 - Step 2.2: Show that there exists a constant $\gamma_2 \in (0,1)$ such that $|m(t+1) - 1| \leq \gamma_2 |m(t) - 1|$ for any $t \in \{1, \ldots, T\}$. Following a similar argument in the previous step, we can obtain the desired results.

Step 2 - Case 3: $m(1) = 1$. When $m(1) = 1$, we want to show that $m(t) = 1$ for any $t \in \{1, ..., T\}$. To establish the claim, we show that inductively, if $m(t) = 1$ then $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$. We observe that

$$n_{l(t)}(t+1) \overset{(a)}{=} \alpha_{l(t)} n_{l(t)}(t) + \mathcal{G}_{l(t)}(\overline{q}_{l(t)} m(t)) \overset{(b)}{=} \alpha_{l(t)} \overline{n}_{l(t)} + \mathcal{G}_{l(t)}(\overline{q}_{l(t)}) \overset{(c)}{=} \overline{n}_{l(t)},$$

where (a) follows from the population transition induced by Algorithm 1; (b) holds given that $m(t) = 1$, which further implies that $n_{l(t)}(t) = \overline{n}_{l(t)}$; (c) follows from Lemma 2(ii). Thus, $\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}} = 1$.

In addition, for $i \in \mathcal{N}^+$ with $i \neq l(t)$, we can deduce that

$$n_i(t+1) \;=\; \alpha_i n_i(t) + \mathcal{G}_i(\overline{q}_i m(t)) \;\overset{(d)}{\geq}\; \alpha_i \overline{n}_i + \mathcal{G}_i(\overline{q}_i) \;=\; \overline{n}_i,$$

where (d) follows from $\frac{n_i(t)}{\overline{n}_i} \geq m(t) = 1$ given the definition of $m(t)$ in (24) and the condition that $i \neq l(t)$. The observation above implies that $\frac{n_i(t+1)}{\overline{n}_i} \geq 1$ for $i \in \mathcal{N}^+$ with $i \neq l(t)$. Therefore, we can establish that

$$m(t+1) \;=\; \min\left\{\frac{n_{l(t)}(t+1)}{\overline{n}_{l(t)}},\, \min_{\substack{i \in \mathcal{N}^+,\\ i \neq l(t)}} \left\{\frac{n_i(t+1)}{\overline{n}_i}\right\}\right\} \;=\; 1.$$

Given that $m(1) = 1$, by inductively establishing that $m(t+1) = 1$ for any $t \in \{1, ..., T-1\}$, we have that $m(t) = 1$ for any $t \in \{1, ..., T\}$. Thus, we obtain that $\left|m(t+1) - 1\right| = 0 \leq \gamma_3 \left|m(t) - 1\right| = 0$ for any $\gamma_3 \in (0,1)$.

In summary of the three cases above for $m(t) < 1$, $m(t) > 1$ and $m(t) = 1$, by letting $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$, We have that for some $\gamma \in (0,1)$,

$$\left|m(t+1) - 1\right| \leq \gamma \left|m(t) - 1\right|,$$

for any $t = \{1, ..., T-1\}$.

<u>Step 3: Show that there exists a constant $C_1'$ such that $|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)| \leq C_1'$</u>. We prove this by the following steps. Given $\boldsymbol{q}(t)$ and $\boldsymbol{n}(t)$ induced by TRP, we show in Step 3.1 that there exists a positive constant $C_{q_i}$ such that $\lim\limits_{T \to \infty} \sum_{t=1}^T |q_i(t) - \overline{q}_i| \leq C_{q_i}$; In Step 3.2, we show that the previous two steps induce a positive constant $C_{\frac{q_i}{n_i}}$ that satisfies $\lim\limits_{T \to \infty} \sum_{t=1}^T |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$; In Step 3.3, based on Steps 3.1 - 3.2, we conclude that there exists a constant $C_1'$ such that $\left|T\overline{\mathcal{R}} - \mathcal{R}^{TR}(T)\right| \leq C_1'$.

<u>Step 3.1: Show that there exists constants $C_{q_i}$ such that $\lim\limits_{T \to \infty} \sum_{t=1}^T |q_i(t) - \overline{q}_i| < C_{q_i}$ for any $i \in \mathcal{N}^+$.</u> Notice that

$$\begin{aligned}\lim_{T \to \infty} \sum_{t=1}^T \left|q_i(t) - \overline{q}_i\right| &\overset{(a)}{=} \lim_{T \to \infty} \sum_{t=1}^T \overline{q}_i \left|m(t) - 1\right| \overset{(b)}{\leq} \lim_{T \to \infty} \sum_{t=1}^T \overline{q}_i \left|m(1) - 1\right| \gamma^{t-1} \\ &= \lim_{T \to \infty} \overline{q}_i \left|m(1) - 1\right| \frac{1 - \gamma^T}{1 - \gamma} \overset{(c)}{=} \frac{1}{1-\gamma} \overline{q}_i \left|m(1) - 1\right|,\end{aligned}$$

where (a) follows from $q_i(t) = \overline{q}_i m(t)$ in Algorithm 1; (b) follows from the contraction arguments in Step 2; (c) follows from $\gamma < 1$ in Step 2. Let $C_{q_i} = \frac{\overline{q}_i |m(1) - 1|}{1 - \gamma}$, and then the result follows.

Before proceeding, we provide some supporting results whose proofs will be provided towards the end of this section:

Lemma 8. *For any $i \in \mathcal{N}^+$ with $n_i(1) \geq \overline{n}_i$, there exists a positive constant $C_{n_i}$ such that $\lim\limits_{T \to \infty} \sum_{t=1}^T |n_i(t) - \overline{n}_i| < C_{n_i}$. Moreover, for any $i \in \mathcal{N}^+$ with $n_i(1) < \overline{n}_i$, if $m(1) < 1$, then $n_i(t) < \overline{n}_i$ for $t \in \{1, ..., T\}$.*

<u>Step 3.2:</u> Show that there exists positive constants $C_{\frac{q_i}{n_i}}$ such that $\lim_{T \to \infty} \sum_{t=1}^T |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$. To show the claim for this step, we notice that for any $i \in \mathcal{N}_+$,

$$\left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \;\overset{(a)}{=}\; \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{\overline{q}_i m(t)}{n_i(t)}\right| \;=\; \frac{\overline{q}_i}{\overline{n}_i}\left|1 - \frac{\overline{n}_i m(t)}{n_i(t)}\right| \;\overset{(b)}{\leq}\; \frac{\overline{q}_i}{\overline{n}_i}\left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)}\left|1 - m(t)\right|\right),$$

where (a) follows from the population transition induced by Algorithm 1, and (b) follows directly from the triangle inequality. Therefore,

$$\lim_{T \to \infty} \sum_{t=1}^T \left|\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}\right| \;\leq\; \lim_{T \to \infty} \sum_{t=1}^T \frac{\overline{q}_i}{\overline{n}_i}\left(\left|1 - \frac{\overline{n}_i}{n_i(t)}\right| + \frac{\overline{n}_i}{n_i(t)}\left|1 - m(t)\right|\right)$$

$$= \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{\overline{n}_i}{n_i(t)}\Big|1-\frac{n_i(t)}{\overline{n}_i}\Big| + \sum_{t=1}^{T}\frac{\overline{n}_i}{n_i(t)}\Big|1-m(t)\Big|\right)$$

$$\overset{(c)}{\leq} \lim_{T\to\infty} \frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-\frac{n_i(t)}{\overline{n}_i}\Big| + \sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big|\right), \quad (*) \tag{31}$$

where (c) follow from the definition of $m(t)$ in (24).

Notice that if $m(1)=\min\limits_{i\in\mathcal{N}^+}\frac{n_i(1)}{\overline{n}_i}\geq 1$, then $n(1)\geq\overline{n}_i$ for any $i\in\mathcal{N}^+$. Thus, it is without loss of generality to consider the following three cases for any $i\in\mathcal{N}^+$ to further relax the term in the RHS of (31), which we denote by "(*)".

(1) When $n_i(1)\geq\overline{n}_i$ and $m(1)\geq 1$, we show that

$$(*) \overset{(d)}{\leq} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\Big|1-\frac{n_i(t)}{\overline{n}_i}\Big| + \sum_{t=1}^{T}\Big|1-m(t)\Big|\right) \overset{(e)}{\leq} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1}\right)$$

$$= \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{C_{n_i}}{\overline{n}_i} + \Big|1-m(1)\Big|\frac{1}{1-\gamma}\right),$$

where (d) follows from the result in Step 2 - Case 1- Step 2.1 and Step 2 - Case 3 that if $m(1)>1$, then $m(1)\geq m(2)\geq ...\geq m(T)\geq 1$; (e) follows from Lemma 8 that $\lim\limits_{T\to\infty}\sum_{t=1}^{T}|n_i(t)-\overline{n}_i|\leq C_{n_i}$ given that $n_i(1)\geq\overline{n}_i$, and we also have $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,...,T\}$ by Step 2. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\big(\frac{C_{n_i}}{\overline{n}_i}+|1-m(1)|\frac{1}{1-\gamma}\big)$, we obtain the desired result.

(2) When $n_i(1)<\overline{n}_i$ and $m(1)<1$, we show that

$$(*) \overset{(f)}{\leq} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big| + \sum_{t=1}^{T}\frac{1}{m(t)}\Big|1-m(t)\Big|\right)$$

$$\overset{(g)}{\leq} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1} + \frac{1}{m(1)}\sum_{t=1}^{T}\Big|1-m(1)\Big|\gamma^{t-1}\right) \leq \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\right),$$

where (f) follows from the observation that $m(t)\leq\frac{n_i(t)}{\overline{n}_i}<1$, where the first inequality follows from the definition of $m(t)$ in (24) and the second inequality follows from Lemma 8 that if $n_i(1)<\overline{n}_i$ and $m(1)<1$, then $n_i(t)<\overline{n}_i$ for $t\in\{1,...,T\}$; (g) follows from the observation that $|m(t)-1|\leq\gamma|m(t-1)-1|$ for $\gamma<1$ and $t\in\{2,...,T\}$ by Step 2, and therefore $|m(t)-1|\leq\gamma^{t-1}|m(1)-1|$; in addition, we show in Step 2 - Case 2- Step 2.1 that when $m(1)<1$, we have $m(1)\leq m(t)$ for any $t\in\{1,...,T\}$. Therefore, we can let $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\big(\frac{2|1-m(1)|}{m(1)(1-\gamma)}\big)$, and then obtain the desired result.

(3) When $n_i(1)\geq\overline{n}_i$ and $m(1)<1$, we show that

$$(*) \overset{(h)}{<} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\frac{1}{m(1)}\Big|1-m(t)\Big|\right)$$

$$\overset{(i)}{\leq} \lim_{T\to\infty}\frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \sum_{t=1}^{T}\frac{1}{m(1)}\Big|1-m(1)\Big|\gamma^{t-1}\right) \overset{(j)}{=} \frac{\overline{q}_i}{\overline{n}_i}\left(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i} + \Big|\frac{1}{m(1)}-1\Big|\frac{1}{1-\gamma}\right),$$

where (h) follows from the observation in Step 2 -Case 2- Step 2.1 that $m(1)<m(2)<...<m(T)<1$ when $m(1)<1$ and the result in Lemma 8 that $\lim\limits_{T\to\infty}\sum_{t=1}^{T}|n_i(t)-\overline{n}_i|\leq C_{n_i}$ when $n_i(1)\geq\overline{n}_i$; (i) follows from the results in Step 2 that $|m(t+1)-1|\leq\gamma|m(t)-1|$; (j) follows from the observation in Step 2 that $\gamma<1$. Therefore, by letting $C_{\frac{q_i}{n_i}} := \frac{\overline{q}_i}{\overline{n}_i}\big(\frac{1}{m(1)}\frac{C_{n_i}}{\overline{n}_i}+|\frac{1}{m(1)}-1|\frac{1}{1-\gamma}\big)$, we can establish the desired result.

In summary, we have that for any $i \in \mathcal{N}^+$, there exists a positive constant $C_{\frac{q_i}{n_i}}$ such that

$$\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}_i}{\overline{n}_i}-\frac{q_i(t)}{n_i(t)}\Big| \;\le\; C_{\frac{q_i}{n_i}}.$$

Step 3.3: Show that there exists a constant $C_1'$ such that $|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)|\le C_1'$. Note that for $j\in\mathcal{B}$ with $\overline{b}_j=0$, we have $\widetilde{F}_{b_j}(\overline{q}^b_j,\overline{b}_j)=0$ based on the definition of $\widetilde{F}_{b_j}$ before the formulation of (8). Since $\overline{q}^b_j\le\overline{b}_j=0$, we have $q^b_j(t)=\overline{q}^b_j m(t)=0$ induced by Algorithm 1, which further implies that $F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})q^b_j(t)=0$. Therefore,

$$\lim_{T\to\infty}\sum_{t=1}^{T}\sum_{j\in\mathcal{B}:\,\overline{b}_j=0}\left(\widetilde{F}_{b_j}(\overline{q}^b_j,\overline{b}_j)-F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})q^b_j(t)\right) \;=\; 0.$$

Similarly, we can establish that for any $i\in\mathcal{S}$ with $\overline{s}_i=0$, we have that $\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)=0$, which further implies that $q^s_i(t)=\overline{q}^s_i m(t)=0$. Thus, we have that

$$\lim_{T\to\infty}\quad\sum_{t=1}^{T}\sum_{i\in\mathcal{S}:\overline{s}_i=0}\left(\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)-F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})q^s_i(t)\right) \;=\; 0.$$

Based on the two observations above, with $(\boldsymbol{q}^s(t),\boldsymbol{q}^b(t),\boldsymbol{s}(t),\boldsymbol{b}(t):t=1,\dots T)$ induced by the TRP, we can deduce that

$$\begin{aligned}
&\lim_{T\to\infty}\quad\sum_{t=1}^{T}\Big|T\overline{\mathcal{R}}-\mathcal{R}^{TR}(T)\Big|\\
=&\lim_{T\to\infty}\sum_{t=1}^{T}\Bigg[\sum_{j\in\mathcal{B}}\left(\widetilde{F}_{b_j}(\overline{q}^b_j,\overline{b}_j)-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\right)-\sum_{i\in\mathcal{S}}\left(\widetilde{F}_{s_i}(\overline{q}^s_i,\overline{s}_i)-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)q^s_i(t)\right)\Bigg]\\
=&\lim_{T\to\infty}\sum_{t=1}^{T}\Bigg[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(F^{-1}_{b_j}\Big(1-\frac{\overline{q}^b_j}{\overline{b}_j}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\right)-\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(F^{-1}_{s_i}\Big(\frac{\overline{q}^s_i}{\overline{s}_i}\Big)\overline{q}^s_i-F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})q^s_i(t)\right)\Bigg]\\
\overset{(a)}{\le}&\lim_{T\to\infty}\sum_{t=1}^{T}\Bigg[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\Big|F^{-1}_{b_j}\Big(1-\frac{\overline{q}^b_j}{\overline{b}_j}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)\overline{q}^b_j\Big|+\Big|F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)\overline{q}^b_j-F^{-1}_{b_j}\Big(1-\frac{q^b_j(t)}{b_j(t)}\Big)q^b_j(t)\Big|\right)\\
&+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\Big|F^{-1}_{s_i}\Big(\frac{\overline{q}^s_i}{\overline{s}_i}\Big)\overline{q}^s_i-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)\overline{q}^s_i\Big|+\Big|F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)\overline{q}^s_i-F^{-1}_{s_i}\Big(\frac{q^s_i(t)}{s_i(t)}\Big)q^s_i(t)\Big|\right)\Bigg]\\
\overset{(b)}{\le}&\lim_{T\to\infty}\sum_{t=1}^{T}\Bigg[\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}\Big|\frac{\overline{q}^b_j}{\overline{b}_j}-\frac{q^b_j(t)}{b_j(t)}\Big|+F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})\Big|\overline{q}^b_j-q^b_j(t)\Big|\right)\\
&+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}\Big|\frac{\overline{q}^s_i}{\overline{s}_i}-\frac{q^s_i(t)}{s_i(t)}\Big|+F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})\Big|\overline{q}^s_i-q^s_i(t)\Big|\right)\Bigg]\\
\le&\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}^b_j}{\overline{b}_j}-\frac{q^b_j(t)}{b_j(t)}\Big|+\max_t F^{-1}_{b_j}(1-\frac{q^b_j(t)}{b_j(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\overline{q}^b_j-q^b_j(t)\Big|\right)\\
&+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\frac{\overline{q}^s_i}{\overline{s}_i}-\frac{q^s_i(t)}{s_i(t)}\Big|+\max_t F^{-1}_{s_i}(\frac{q^s_i(t)}{s_i(t)})\lim_{T\to\infty}\sum_{t=1}^{T}\Big|\overline{q}^s_i-q^s_i(t)\Big|\right)\\
\overset{(c)}{\le}&\sum_{j\in\mathcal{B}:\overline{b}_j>0}\left(\overline{q}^b_j\frac{1}{d^b_j}C_{q^b_j/b_j}+\overline{v}^b_jC_{q^b_j}\right)+\sum_{i\in\mathcal{S}:\overline{s}_i>0}\left(\overline{q}^s_i\frac{1}{d^s_i}C_{q^s_i/s_i}+\overline{v}^s_iC_{q^s_i}\right)\;:=\;C_1'.
\end{aligned}$$

where (a) follows from the triangle inequality; (b) follows from Assumption 2(ii) that the derivative of $F_{b_j}$ ($F_{s_i}$) is lower bounded by a positive constant $d^b_j$ ($d^s_i$), and therefore the derivative of $F^{-1}_{b_j}$ ($F^{-1}_{s_i}$) is upper bounded by $\frac{1}{d_j^b}$ ($\frac{1}{d_i^s}$), then $|F_{b_j}^{-1}(x_1) - F_{b_j}^{-1}(x_2)| \leq \frac{1}{d_j^b}|x_1 - x_2|$ for $\forall x_1, x_2$ in the domain, otherwise $\frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|} > \frac{1}{d_j^b}$ implies that there exists a $x_3 \in (x_1, x_2)$ such that $f'(x_3) = \frac{|F_{b_j}^{-1}(x_1)-F_{b_j}^{-1}(x_2)|}{|x_1-x_2|} > \frac{1}{d_j^b}$ by mean value theorem, which contradicts to the fact that the derivative of $F_{b_j}^{-1}$ is upper bounded by $\frac{1}{d_j^b}$; following the same argument, $|F_{s_i}^{-1}(x_1) - F_{s_i}^{-1}(x_2)| \leq \frac{1}{d_i^s}|x_1 - x_2|$ for $\forall x_1, x_2$ in the domain. (c) follows from the results in Step 3.1- Step 3.2 that $\lim_{T\to\infty} \sum_{t=1}^T |q_i(t) - \overline{q}_i| < C_{q_i}$ and $\lim_{T\to\infty} \sum_{t=1}^T |\frac{\overline{q}_i}{\overline{n}_i} - \frac{q_i(t)}{n_i(t)}| \leq C_{\frac{q_i}{n_i}}$ for any $i \in \mathcal{N}^+$; in addition, $F_{b_j}^{-1} \leq \overline{v}_{b_j}$ and $F_{s_i}^{-1} \leq \overline{v}_{s_i}$. Note that we have $\overline{v}_{b_j} < \infty$ for $j \in \mathcal{B}$ and $\overline{v}_{s_i} < \infty$ for $i \in \mathcal{S}$ and $\frac{1}{d_j^b} < \infty$ for $j \in \mathcal{B}$ and $\frac{1}{d_i^s} < \infty$ for $i \in \mathcal{S}$ given Assumption 2(ii).

Together with the observation from Proposition 2, we can conclude that there exists a constant $C_2 := C_1 + C_1'$ such that

$$\mathcal{L}^\pi(T) \;=\; \mathcal{R}^*(T) - \mathcal{R}^\pi(T) \;=\; \big(\mathcal{R}^*(T) - T\overline{\mathcal{R}}\big) + \big(T\overline{\mathcal{R}} - T\mathcal{R}^\pi(T)\big) \;\leq\; C_1 + C_1' \;=\; C_2.$$

This concludes the proof of this result. ■

**Proof of Proposition 3.** We denote by $(\boldsymbol{r^M}(t), \boldsymbol{p^M}(t), \boldsymbol{q}^{s,M}(t), \boldsymbol{q}^{b,M}(t), \boldsymbol{x}^M(t))$ the optimal solution to the optimization problem for the MP in Definition 2. We consider the following problem instance: Consider a simple network in which there is only one buyer type and one seller type with initial population $s(1) = b(1) > 0$. Given the commissions $\boldsymbol{r^M}(t)$ induced by the MP, we let the populations for the next period be $(\boldsymbol{s}^M(t+1), \boldsymbol{b}^M(t+1))$ is updated by $s^M(t+1) = \alpha s^M(t) + \beta(q^{s,M}(t))^\xi$ and $b^M(t+1) = \alpha b^M(t) + \beta(q^{b,M}(t))^\xi$, where we assume $\beta > 0$ and $0 < \xi < 1$ so that the Assumption 1 holds. In addition, we let $F_s(\cdot)$ and $F_b(\cdot)$ be the distribution functions over $[0,1]$ from the uniform distribution.

We establish two claims to complete the proof.

Claim 1: $\lim_{t\to\infty} R^M(t)$ exists . We divide the proof arguments into the following steps. In Step 1.1, we show that if a steady state induced by the MP exists, we characterize the properties of the steady state. In Step 1.2, we show that the populations converge to the steady state under the platform's MP. For simplicity of notations, we let $\mathcal{R}^M(t)$ denote the profit in period $t$ under the MP.

Step 1.1. Characterize the quantity $\overline{q}^M$ and the profit $\overline{R}^M$ in a steady state. We first define a steady state as such that the populations and transaction quantities remain unchanged after the population transition in each period. Given the definition of a steady state, under the platform's myopic policy, the steady-state population vector $(\overline{s}^M, \overline{b}^M, \overline{q}^M)$ should satisfy the following three conditions:

$$\overline{q}^M = \underset{0 \leq q \leq \min\{\overline{s}^M, \overline{b}^M\}}{\arg\max} \left[\left(1 - \frac{q}{\overline{s}^M} - \frac{q}{\overline{b}^M}\right)q\right], \tag{32a}$$

$$\overline{s}^M = \alpha \overline{s}^M + \beta(\overline{q}^M)^\xi, \tag{32b}$$

$$\overline{b}^M = \alpha \overline{b}^M + \beta(\overline{q}^M)^\xi. \tag{32c}$$

Condition (32a) ensures that given the population in each period $(\overline{s}^M, \overline{b}^M)$, the platform's commissions $r$ could induce the equilibrium quantity $\overline{q}^M$ to maximize its profit in the current period (see Corollary 2 for the formulation of optimization problem); (32b) and (32c) ensure that the population vector $(\overline{s}^M, \overline{b}^M)$ remains unchanged after the update in each period.

For Problem (32a), from the first-order-condition $\frac{\partial}{\partial q}\Big[\Big(1-\frac{q}{\overline{s}^M}-\frac{q}{\overline{b}^M}\Big)q\Big]=0$, we can obtain that $\overline{q}^M=\frac{\overline{s}^M\overline{b}^M}{2\overline{s}^M+2\overline{b}^M}$, which falls in the region $(0,\min\{\overline{s}^M,\overline{b}^M\})$. Thus, the optimal solution to (32a) is an interior point. Together with the equations in (32b)-(32c), we obtain that

$$\overline{q}^M = \Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}, \overline{b}^M = k\Big(\frac{k}{4}\Big)^{\frac{\xi}{1-\xi}}, \overline{s}^M = k\Big(\frac{k}{4}\Big)^{\frac{\xi}{1-\xi}}.$$

where we let $k=\frac{\beta}{1-\alpha}$ for simplicity of notations. This allows us to show that the profit induced by the platform's MP satisfies that

$$\overline{\mathcal{R}}^M = \Big(1-\frac{\overline{q}^M}{\overline{s}^M}-\frac{\overline{q}^M}{\overline{b}^M}\Big)\overline{q}^M = \frac{1}{2}\Big(\frac{k}{4}\Big)^{\frac{1}{1-\xi}}.$$

Step 1.2: For the seller side, show that there exists a $\gamma\in(0,1)$ such that $|\overline{s}^M - s^M(t+1)|\le\gamma|(\overline{s}^M - s^M(t))|$.

Next, we establish the convergence of the platform's MP. Without loss of generality, we prove the convergence on the seller side, and notice that the same argument would hold for the buyer side as well.

Since we have $s^M(1)=b^M(1)$ in the problem instance, and in each iteration we have $s^M(t+1)=\alpha s^M(t)+\beta(q^M(t))^\xi$ and $b^M(t+1)=\alpha b^M(t)+\beta(q^M(t))^\xi$, we obtain that $s^M(t)=b^M(t)$ for any $t\in\{1,...,T\}$. Based on this observation, we can obtain that

$$\begin{aligned} q^M(t) &= \underset{0<q<\min\{s^M(t),b^M(t)\}}{\arg\max}\Big\{\Big(1-\frac{q}{s^M(t)}-\frac{q}{b^M(t)}\Big)q\Big\} \\ &= \underset{0<q<s^M(t)}{\arg\max}\Big\{\Big(1-\frac{q}{s^M(t)}-\frac{q}{s^M(t)}\Big)q\Big\} = \frac{s^M(t)}{4}. \end{aligned}$$

From the optimal solution $q^M(t)$ above, we obtain that

$$s^M(t+1)=\alpha s^M(t)+\beta(q^M(t))^\xi=\alpha s^M(t)+\beta\Big(\frac{s^M(t)}{4}\Big)^\xi.$$

Abusing some notations, we let $g_s(s):=\alpha s+\beta(\frac{s}{4})^\xi$ for any $s\ge 0$ such that $g_s(\overline{s}^M)=\overline{s}^M$ based on the condition in (32b). To proceed, we consider the following two cases that $s^M(1)\ge\overline{s}^M$ and $s^M(1)<\overline{s}^M$:

(1) When $s^M(1)\ge\overline{s}^M$, we want to show that $s^M(t)\ge\overline{s}^M$ for $t\in\{1,..,T\}$. By induction, if $s^M(t)\ge\overline{s}^M$, we have $s^M(t+1)=g_s(s^M(t))\ge g_s(\overline{s}^M)=\overline{s}^M$, where the inequality follows from the fact that $g_s(\cdot)$ is an increasing function. Since $s^M(1)\ge\overline{s}^M$, we obtain that $s^M(t)\ge\overline{s}^M$ for $t\in\{1,..,T\}$.

Based on the observation above, we can establish that

$$\Big|s^M(t+1)-\overline{s}^M\Big| = \Big|g_s(s^M(t))-\overline{s}^M\Big| \overset{(a)}{=} g_s(s^M(t))-g_s(\overline{s}^M) \overset{(b)}{\le} \Big|s^M(t)-\overline{s}^M\Big|g_s'(\overline{s}^M), \tag{33}$$

where (a) follows from the observation that $s^M(t)\ge\overline{s}^M$ for $t\in\{1,..,T\}$ in this case; (b) follows from the condion that $g_s$ is concave given that $g_s(s)=\alpha s+\beta(\frac{s}{4})^\xi$ with $a\in(0,1)$. Moreover, we have $g_s'(\overline{s}^M)<1$ given that $g_s(0)=0$ and $g_s(\overline{s}^M)=\overline{s}^M$, and so by the mean value theorem, there exists a $\widetilde{s}\in(0,\overline{s}^M)$ such that $g_s'(\widetilde{s})=\frac{g_s(\overline{s}^M)-g_s(0)}{\overline{s}^M-0}=1$. Since $g_s(\cdot)$ is concave, we have that $g_s'(\overline{s}^M)<g_s'(\widetilde{s})=1$ given that $\overline{s}^M>\widetilde{s}$. By letting $\gamma_1:=g_s'(\overline{s}^M)$, we establish that there exists $\gamma_1\in(0,1)$ such that $|\overline{s}^M-s^M(t+1)|\le\gamma_1|(\overline{s}^M-s^M(t))|$ for $t\in\{1,...,T-1\}$ if $s^M(1)\ge\overline{s}^M$. From the definition of $g_s(\cdot)$ and $\overline{s}^M$, we see that $\gamma_1$ is independent of $T$.

(2) When $s^M(1) < \bar{s}^M$, we want to show that $s^M(t) < \bar{s}^M$ for $t \in \{1,..,T\}$. If $s^M(t) < \bar{s}^M$, we have $s^M(t+1) = g_s(s^M(t)) < g_s(\bar{s}^M) = \bar{s}^M$, where the inequality follows from that $g_s(\cdot)$ is an increasing function given that $s^M(t) < \bar{s}^M$. Since $s^M(1) < \bar{s}^M$, by induction we obtain that $s^M(t) < \bar{s}^M$ for any $t \in \{1,..,T\}$.

Then, we can establish that

$$\frac{\bar{s}^M - g_s(s^M(t))}{\bar{s}^M - s^M(t)} \overset{(c)}{<} \frac{\bar{s}^M - g_s(s^M(1))}{\bar{s}^M - s^M(1)} \overset{(d)}{<} 1,$$

where in Step (c), we establish the following set of observations: (c-i) we first establish that $\frac{\bar{s}^M - g_s(s)}{\bar{s}^M - s}$ decreases in $s \geq 0$ by showing that $\frac{\partial}{\partial s}\left(\frac{\bar{s}^M - g_s(s)}{\bar{s}^M - s}\right) = \frac{(s-\bar{s}^M)g_s'(s) - g_s(s) + \bar{s}^M}{(s-\bar{s}^M)^2} < 0$, with the inequality following as $g_s(s)$ is strictly concave in $s \geq 0$ such that $\bar{s}^M = g_s(\bar{s}^M) < g_s(s) + (\bar{s}^M - s)g_s'(s)$; (c-ii) we then show that $s^M(t) > s^M(1)$ for $t \in \{2,..,T\}$. Note that $g_s(0) = 0$ and $g_s(\bar{s}^M) = \bar{s}^M$. Since $g_s(s) - s$ is strictly concave in $s \geq 0$, by the Jensen's inequality, we obtain that $g_s(a\bar{s}^M) - a\bar{s}^M > a(g_s(\bar{s}^M) - \bar{s}^M) + (1-a)\big(g_s(0) - 0\big) = 0$ for $0 < a < 1$. Therefore, we have $g_s(a\bar{s}^M) > a\bar{s}^M$ for $0 < a < 1$, which further implies that $s^M(t+1) = g_s(s^M(t)) > s^M(t)$ given that $0 < s^M(t) < \bar{s}^M$. Thus, we can obtain that $s^M(t) < s^M(t+1) < \bar{s}^M$ for $t \in \{1,..,T-1\}$. Combining the observations in (c-i) and (c-ii), since $\frac{\bar{s}^M - g_s(s^M(t))}{\bar{s}^M - s^M(t)}$ decreases in $s^M(t)$ and $s^M(t+1) > s^M(t) > s^M(1)$ for $t \in \{2,..,T-1\}$, we have that Step (c) holds. For Step (d), we have $s^M(1) < s^M(2) = g_s(s^M(1)) < g_s(\bar{s}^M) = \bar{s}^M$, where the first inequality follows from $s^M(t+1) = g_s(s^M(t)) > s^M(t)$ for $0 < s^M(t) < \bar{s}^M$ based on previous discussion; the second inequality follows from the condition that $s^M(1) < \bar{s}^M$ in this case and $g_s(.)$ is a increasing function; the last equation follows directly from the observation in (32b). Therefore, we have that $\frac{\bar{s}^M - g_s(s^M(1))}{\bar{s}^M - s^M(1)} < 1$. By letting $\gamma_2 = \frac{\bar{s}^M - g_s(s^M(1))}{\bar{s}^M - s^M(1)}$, we obtain that $\frac{\bar{s}^M - g_s(s^M(t))}{\bar{s}^M - s^M(t)} \leq \gamma_2$, which implies that

$$\left|\bar{s}^M - g_s(s^M(t))\right| \overset{(e)}{=} \bar{s}^M - g_s(s^M(t)) \leq \gamma_2\Big(\bar{s}^M - s^M(t)\Big) \overset{(f)}{=} \gamma_2\left|\bar{s}^M - s^M(t)\right|$$

where (e) and (f) follow from the observations that $s^M(t) < \bar{s}^M$ for $t \in \{1,..,T\}$. In summary, there exists a $\gamma_2 \in (0,1)$ such that $|\bar{s}^M - s^M(t+1)| \leq \gamma_2|(\bar{s}^M - s^M(t))|$ for $t \in \{1,...,T-1\}$ if $s^M(1) < \bar{s}^M$. Again, from the definition of $g_s(\cdot)$, we see that $\gamma_2$ is independent of $T$.

In summary of the two cases above, we let $\gamma := \max\{\gamma_1, \gamma_2\}$, which allows us to obtain the desired result.

Claim 2: For any $\epsilon > 0$, there exists $a \in (0,1)$ for the population transition in this problem instance such that $\overline{\mathcal{R}}^M < \epsilon\overline{\mathcal{R}}$. For the AVG in (8) given the problem instance before Step 1, we have that

$$\begin{aligned}\overline{\mathcal{R}} \;=\; & \max_{s,b,q} \quad \Big(1 - \frac{q}{s} - \frac{q}{b}\Big)q \\ & \text{s.t. } 0 \leq q \leq s, \quad 0 \leq q \leq b, \quad s \leq \alpha s + \beta q^\xi, \quad b \leq \alpha b + \beta q^\xi.\end{aligned}$$

In addition, based on Lemma 2(ii), the inequalities in the last two constraints are both tight. Note that $s = \alpha s + \beta q^\xi$ and $b = \alpha b + \beta q^\xi$ are equivalent to $s = b = kq^\xi$, where $k = \frac{\beta}{1-\alpha}$. By plugging $s = b = kq^\xi$ into the objective function we obtain $\overline{\mathcal{R}} = \max_{0 \leq q \leq kq^\xi}(1 - \frac{q}{kq^\xi} - \frac{q}{kq^\xi})q$. Since $(1 - \frac{q}{kq^\xi} - \frac{q}{kq^\xi})q$ is concave in $q \geq 0$ for $0 < \xi < 1$, from the first-order condition, we have $\bar{q} = \big(\frac{k}{2(2-\xi)}\big)^{\frac{1}{1-\xi}}$, which satisfy $0 < \bar{q} < kq^\xi$. Thus, the optimal commission $\bar{r}$ and the optimal profit $\overline{R}$ for the instance of the AVG in (8) satisfies that

$$\bar{r} \;=\; 1 - \frac{q}{kq^\xi} - \frac{q}{kq^\xi} = \frac{1-\xi}{2-\xi},$$

$$\overline{\mathcal{R}} = \Big(1-\frac{q}{kq^{\xi}}-\frac{q}{kq^{\xi}}\Big)q = \frac{1-\xi}{2-\xi}\Big(\frac{k}{2(2-\xi)}\Big)^{\frac{1}{1-\xi}},$$

which further implies that $\frac{\overline{\mathcal{R}}^M}{\overline{\mathcal{R}}} = (\frac{2-\xi}{2})^{\frac{1}{1-\xi}}\frac{2-\xi}{2(1-\xi)}$. Therefore, we can obtain that

$$\lim_{\xi\to 1}\frac{\overline{\mathcal{R}}^M}{\overline{\mathcal{R}}} = \lim_{\xi\to 1}\Big(\frac{2-\xi}{2}\Big)^{\frac{1}{1-\xi}}\frac{2-\xi}{2(1-\xi)} = 0.$$

■

**Proof of Lemma 8.** We prove the two claims of this result separately. Given that the supporting lemma is located in Step 3 in the proof of Theorem 1, we would borrow some observations from Step 2 in the proof of Theorem 1 in the proof arguments below.

Claim 1. For $i\in\mathcal{N}^+$, when $n_i(1)\ge\overline{n}_i$, we further consider the following two cases: (1) $m(1)\ge 1$; (2) $m(1)<1$.

(1) When $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we first show that if $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, then $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$. Given that $n_i(1)\ge\overline{n}_i$ for any $i\in\mathcal{N}^+$, we assume for induction purpose that $n_i(t)\ge\overline{n}_i$, and then we can establish that

$$n_i(t+1)\overset{(a)}{=}\alpha_i n(t)+\mathcal{G}_i(\overline{q}_i m(t))\overset{(b)}{\ge}\alpha_i n(t)+\mathcal{G}_i(\overline{q}_i)\ge\alpha_i\overline{n}_i+\mathcal{G}_i(\overline{q}_i)\overset{(c)}{=}\overline{n}_i,$$

where (a) follows from the population transitions induced by Algorithm 1; (b) follows from our observations in Step 2 Case 1 in the proof of Theorem 1 that if $m(1)>1$, then we have $m(1)>m(2)>...>m(T)>1$, and in Step 2 Case 3 that if $m(1)=1$, then we have $m(1)=m(2)=...=m(T)=1$ given that function $\mathcal{G}_i(q)$ increases in $q\ge 0$; (c) follows directly from Lemma 2(ii). In summary, if $n_i(t)\ge\overline{n}_i$ and $m(1)\ge 1$, we have $n_i(t+1)\ge\overline{n}_i$ . By induction, with $n_i(1)\ge\overline{n}_i$ and $m(1)\ge 1$, we obtain that $n_i(t)\ge\overline{n}_i$ for any $t\in\{1,...,T\}$.

To proceed, we further notice that for any $t\in\{1,...,T\}$,

$$\begin{aligned}n_i(t)-\overline{n}_i &\overset{(d)}{=} \alpha_i n_i(t-1)+\mathcal{G}_i(\overline{q}_i m(t-1))-\alpha_i\overline{n}_i-\mathcal{G}_i(\overline{q}_i)\\ &\overset{(e)}{\le} \mathcal{G}_i'(\overline{q}_i)\Big(\overline{q}_i m(t-1)-\overline{q}_i\Big)+\alpha_i\Big(n_i(t-1)-\overline{n}_i\Big),\end{aligned}$$

where (d) follows from population transition induced by Algorithm 1 and Lemma 2(ii); (e) follows from the condition that $\mathcal{G}_i(q)$ is concave in $q$ by Assumption 1. Since $n_i(t)\ge\overline{n}_i$, the LHS of the inequality for (e) is nonnegative, and we can take the absolute values and obtain the following inequality:

$$\begin{aligned}\Big|n_i(t)-\overline{n}_i\Big| &\le \Big|\mathcal{G}_i'(\overline{q}_i)\Big(\overline{q}_i m(t-1)-\overline{q}_i\Big)+\alpha_i\Big(n_i(t-1)-\overline{n}_i\Big)\Big|\\ &\le \Big|\mathcal{G}_i'(\overline{q}_i)\Big(\overline{q}_i m(t-1)-\overline{q}_i\Big)\Big|+\Big|\alpha_i\Big(n_i(t-1)-\overline{n}_i\Big)\Big|\\ &= \overline{q}_i\mathcal{G}_i'(\overline{q}_i)\Big|\Big(m(t-1)-1\Big)\Big|+\alpha_i\Big|\Big(n_i(t-1)-\overline{n}_i\Big)\Big|.\end{aligned}\tag{34}$$

From the inequality above, we can establish that

$$\begin{aligned}\sum_{t=2}^{T}\Big|n_i(t)-\overline{n}_i\Big| &\overset{(f)}{=} \sum_{t=2}^{T}\Big[\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\Big|1-m(t-1)\Big|+\alpha_i\Big|n_i(t-1)-\overline{n}_i\Big|\Big]\\ &\overset{(g)}{\le} \sum_{t=2}^{T}\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\gamma^{t-2}\Big|1-m(1)\Big|+\sum_{t=2}^{T}\alpha_i\Big|n_i(t-1)-\overline{n}_i\Big|,\end{aligned}$$

where (f) follows directly from (34); (g) follows from Step 2 in the proof of Theorem 1 that there exists a constant $\gamma \in (0,1)$ such that $|m(t+1)-1| \leq \gamma |m(t)-1|$ for any $t \in \{1,2,\dots\}$. From the inequality above, we subtract a common term $\sum_{t=2}^{T-1} \alpha_i \Big| n_i(t) - \overline{n}_i \Big|$ on both sides of the inequality, and then by moving the term $\Big| n_i(T) - \overline{n}_i \Big|$ from the LHS to the RHS given that both terms have finite values, we can establish that

$$(1-\alpha_i)\sum_{t=2}^{T-1}\Big|n_i(t)-\overline{n}_i\Big| \;\leq\; \sum_{t=2}^{T}\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\gamma^{t-2}\Big|1-m(1)\Big| + \alpha_i\Big|n_i(1)-\overline{n}_i\Big| - \Big|n_i(T)-\overline{n}_i\Big|$$

Then by dividing both sides by $1-\alpha_i$ and adding $\Big|n_i(1)-\overline{n}_i\Big| + \Big|n_i(T)-\overline{n}_i\Big|$, we have

$$\begin{aligned}\sum_{t=1}^{T}\Big|n_i(t)-\overline{n}_i\Big| \;&\leq\; \sum_{t=2}^{T}\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\gamma^{t-2}\Big|1-m(1)\Big| + \frac{1}{1-\alpha_i}\Big|n_i(1)-\overline{n}_i\Big| - \frac{\alpha_i}{1-\alpha_i}\Big|n_i(T)-\overline{n}_i\Big| \\ &\leq \sum_{t=2}^{T}\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\gamma^{t-2}\Big|1-m(1)\Big| + \frac{1}{1-\alpha_i}\Big|n_i(1)-\overline{n}_i\Big|\end{aligned}$$

Therefore, $\lim_{T\to\infty}\sum_{t=1}^{T}\Big|n_i(t)-\overline{n}_i\Big| \leq \frac{1}{1-\alpha_i}\mathcal{G}_i'(\overline{q}_i)\overline{q}_i\Big|1-m(1)\Big|\frac{1}{1-\gamma} + \frac{1}{1-\alpha_i}\Big|n_i(1)-\overline{n}_i\Big|$. In the end, we define the positive constant

$$C_{n_i} := \mathcal{G}_i'(\overline{q}_i)\overline{q}_i\Big|1-m(1)\Big|\frac{1}{1-\gamma} + \Big|n_i(1)-\overline{n}_i\Big|\frac{1}{1-\alpha_i},$$

which allows us to obtain the desired result.

(2) Given that $m(1) < 1$ and that $n_i(1) \geq \overline{n}_i$, we consider two cases. In the first case, we consider the scenario where there exists a threshold $\widetilde{t} \in \{2,...,T\}$ such that $n_i(t) \geq \overline{n}_i$ for $t < \widetilde{t}$ and $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. In the second case, we consider the scenario where $n_i(t) \geq \overline{n}_i$ for all $t \in \{1,\dots,T\}$.

In the first case, given $\widetilde{t} \in \{2,\dots,T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we want to show that $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$. We prove the claim by induction. Given that $n_i(\widetilde{t}) < \overline{n}_i$, for any $t \geq \widetilde{t}$, suppose towards an induction purpose that $n_i(t) < \overline{n}_i$, and we can establish that

$$n_i(t+1) \overset{(a)}{=} \alpha_i n(t) + \mathcal{G}_i(\overline{q}_i m(t)) \overset{(b)}{<} \alpha_i n(t) + \mathcal{G}_i(\overline{q}_i) \;<\; \alpha_i \overline{n}_i + \mathcal{G}_i(\overline{q}_i) \overset{(c)}{=} \overline{n}_i, \tag{35}$$

where (a) follows from the population transition rule induced by Algorithm 1 such that $q_i(t) = \overline{q}_i m(t)$; (b) follows from the condition that $\mathcal{G}_i(q)$ strictly increases in $q \geq 0$ and from the observation in Step 2.1 from the proofs of Theorem 1 that if $m(1) < 1$, then $m(1) < m(2) < ... < m(T) < 1$; (c) follows directly from Lemma 2(ii). Therefore, we obtain that if there exists a $\widetilde{t} \in \{2,\dots,T\}$ such that $n_i(\widetilde{t}) < \overline{n}_i$, we have $n_i(t) < \overline{n}_i$ for $t \geq \widetilde{t}$.

We then show that $\widetilde{t}$ is independent of $T$. Given the definition of $\widetilde{t}$ as the first time that $n_i(t) < \overline{n}_i$, it is equivalent to show that the value of $n_i(t)$ for $0 \leq t \leq \widetilde{t}$ is independent of $T$. This is true as given $n_i(1)$ and $m(1)$, for $t \in \{1,...,\widetilde{t}-1\}$, $n_i(t+1) = \alpha_i n(t) + \mathcal{G}_i(\overline{q}_i m(t))$, where $m(t) = \min_{i' \in \mathcal{N}^+}\{\frac{n_{i'}(t)}{\overline{n}_{i'}}\}$ is independent of $T$ for $1 \leq t \leq \widetilde{t}-1$.

The observations above allow us to deduce that in the first case,

$$\lim_{T\to\infty}\sum_{t=1}^{T}\Big|n_i(t)-\overline{n}_i\Big| \;=\; \sum_{t=1}^{\widetilde{t}-1}\Big|n_i(t)-\overline{n}_i\Big| + \lim_{T\to\infty}\sum_{t=\widetilde{t}}^{T}\Big|n_i(t)-\overline{n}_i\Big|$$

$$\stackrel{(d)}{=} \sum_{t=1}^{\tilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\tilde{t}}^{T} \Big| m(t) - 1 \Big| \stackrel{(e)}{\le} \sum_{t=1}^{\tilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \lim_{T\to\infty} \sum_{t=\tilde{t}}^{T} \Big| m(\tilde{t}) - 1 \Big| \gamma^{t-\tilde{t}}$$
$$= \sum_{t=1}^{\tilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(\tilde{t}) - 1 \Big| \frac{1}{1-\gamma} \stackrel{(f)}{\le} \sum_{t=1}^{\tilde{t}-1} \Big| n_i(t) - \overline{n}_i \Big| + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma}$$
$$\stackrel{(g)}{\le} \mathcal{G}_i'(\overline{q}_i)\overline{q}_i \Big| 1 - m(1) \Big| \frac{1}{1-\gamma} + \Big| n_i(1) - \overline{n}_i \Big| \frac{1}{1-\alpha_i} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma},$$

where (d) follows from the definition of $m(t)$, (e) follows from Step 2, and (f) follows from $m(1) < m(2) < ... < m(T) < 1$ if $m(1) < 1$ in Step 2.1; (g) follows from the (34).

In the second case, if $n_i(t) \ge \overline{n}_i$ for all $t \in \{1, ..., T\}$, we can apply the same upper bound $\sum_{t=1}^{T} |n_i(t) - \overline{n}_i| \le \lim_{T\to\infty} \sum_{t=1}^{T} |n_i(t) - \overline{n}_i| \le \mathcal{G}_i'(\overline{q}_i)\overline{q}_i \Big| 1 - m(1) \Big| \frac{1}{1-\gamma} + \Big| n_i(1) - \overline{n}_i \Big| \frac{1}{1-\alpha_i}$ in as in Case (1) above under Claim 1.

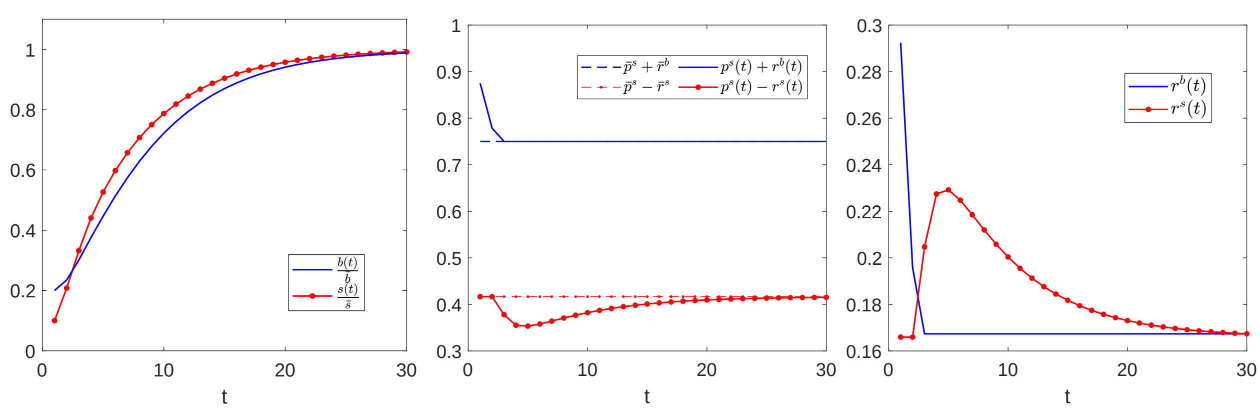
Then let $C_{n_i} = \Big| n_i(1) - \overline{n}_i \Big| + \mathcal{G}_i'(\overline{q}_i)\overline{q}_i \Big| 1 - m(1) \Big| \frac{1}{1-\gamma} + \Big| n_i(1) - \overline{n}_i \Big| \frac{\alpha_i}{1-\alpha_i} + \overline{n}_i \Big| m(1) - 1 \Big| \frac{1}{1-\gamma}$, we obtain the desired result.

Claim 2. To establish the second claim of this result, when $n_i(1) \le \overline{n}_i$ and $m(1) < 1$, by applying the same induction arguments as in (35) from the previous claim, we can establish that $n_i(t) \le \overline{n}_i$ for any $t \in \{1, \ldots, T\}$.

Summarizing the arguments above, we complete the proofs of the two claims in this result. ■

### B.3. Additional Numerical Results

We first provide additional numerical examples for the TRP in Section 4, and then provide the additional analytical results in Section 5 and their proof.




(a) Population ratio (b) Payment/Receipt (c) Commission

**Figure 4** **TRP**

## C. Proof of Results in Section 5

In this section, we develop some auxiliary results that are needed for the proofs of results in Section 5 in C.1. We then respectively prove the results from Section 5.1 in C.2 and those from 5.2 in C.3.

### C.1. Auxiliary Results for Section 5.

In this section, we first develop a simper formulation for Problem (8) in (41). To do that, we first characterize the properties of Problem (8) in Lemma 9 and Lemma 10. Next, we reformulate it in Lemma 11, and will further simplify its formulation into (41) in Lemma 12. We then show the connection between the optimal solution to (41) $\boldsymbol{w}^*$ and $(\mathcal{S}_\tau, \mathcal{B}_\tau)$ constructed in (11) in Lemma 13. The proof of the auxiliary results follows a similar argument to the proof of Lemma 1, Lemma 2 and Proposition 10 in Birge et al. (2021). Therefore, we omit the detail of the proof of auxiliary results for simplicity.

To develop an equivalent reformulation in $(\boldsymbol{q}, \boldsymbol{x})$ for **AVG**, recall from Lemma 2(ii) that the relaxed population dynamics constraints $s_i \le \alpha_i^s s_i + \mathcal{G}_i^s(q_i^s)$ and $b_j \le \alpha_j^b b_j + \mathcal{G}_j^b(q_j^b)$ with the optimal solutions to **AVG** are tight. Together with Assumption 4(ii), on the seller side, we have $s_i = \frac{\beta_i^s (q_i^s)^{\xi_s}}{1-\alpha_i^s}$ for any $i \in \mathcal{S}$. We further let $k_i^s := \frac{\beta_i^s}{1-\alpha_i^s}$, which allows us to obtain that $s_i = k_i^s (q_i^s)^{\xi_s}$ for any $i \in \mathcal{S}$. Similarly, on the buyer side, we have $b_j = k_j^b (q_j^b)^{\xi_b}$ for any $j \in \mathcal{B}$, where $k_j^b = \frac{\beta_j^b}{1-\alpha_j^b}$. Plugging the expressions of $s_i = k_i^s (q_i^s)^{\xi_s}$ and $b_j = k_j^b (q_j^b)^{\xi_b}$ into **AVG**, we obtain the following reformulation of **AVG**:

$$
\begin{aligned}
\overline{\mathcal{R}} \;=\; & \max_{\boldsymbol{q^s}, \boldsymbol{q^b}, \boldsymbol{x}} \Big[ \sum_{j\in\mathcal{B}} \widetilde{F}_b(q_j^b, k_j^b (q_j^b)^{\xi_b}) - \sum_{i\in\mathcal{S}} \widetilde{F}_s(q_i^s, k_i^s (q_i^s)^{\xi_s}) \Big] && (36a)\\
& \text{s.t. } q_i^s \le k_i^s (q_i^s)^{\xi_s}, && \forall i \in \mathcal{S}, \quad (36b)\\
& q_j^b \le k_j^b (q_j^b)^{\xi_b}, && \forall j \in \mathcal{B}, \quad (36c)\\
& \sum_{j:(i,j)\in E} x_{ij} = q_i^s, && \forall i \in \mathcal{S}, \quad (36d)\\
& q_j^b = \sum_{i:(i,j)\in E} x_{ij}, && \forall j \in \mathcal{B}, \quad (36e)\\
& x_{ij} \ge 0, && \forall (i,j) \in E. \quad (36f)
\end{aligned}
$$

where $\widetilde{F}_b(\cdot)$ and $\widetilde{F}_s(\cdot)$ are defined before Problem (8).

For $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, define $y_b(q) := F_b^{-1}(1-(q)^{1-\xi_b})\, q$ for $0 \le q \le 1$. Define $y_s(q,u) := -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \le q \le u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. For simplicity of notations, we let $y_b'(q) := \frac{dy_b(q)}{dq}$ for $0<q<1$ and $y_s^{(1,0)}(q,u) := \frac{\partial y_s(q,u)}{\partial q}$ for $0<q<u$. Furthermore, we let $y_b'(0) := \lim_{q\downarrow 0} y_b'(q)$, $y_b'(1) := \lim_{q\uparrow 1} y_b'(q)$; for $u>0$, we let $y_s^{(1,0)}(0,u) := \lim_{q\to 0} y_s^{(1,0)}(q,u)$, $y_s^{(1,0)}(u,u) := \lim_{q\to u} y_s^{(1,0)}(q,u)$; for $q>0$, we let $y_s^{(0,1)}(q,q) := \lim_{u\to q} y_s^{(0,1)}(q,u)$. We show in the following lemma that all of the limiting values are finite.

LEMMA 9. *(i) $y_b(q)$ is continuously differentiable and strictly concave in $q\in[0,1]$;*

*(ii) $y_s(q,u)$ is continuous and strictly concave in $(q,u) \in \{(q',u') : 0 \le q' \le u'\}$; moreover, $y_s(q,u)$ is continuously differentiable in $(q,u) \in \{(q',u') : 0 \le q' \le u', u'>0\}$;*

*(iii) for any $0<\xi_s<1$, $-(1-\xi_s)\left[F_s^{-1}\right]'(x)x - F_s^{-1}(x)$ strictly decreases in $x\in[0,1]$.*

Before the next auxiliary result, we define

$$
\begin{aligned}
\rho(u) \;&:=\; \operatorname*{arg\,max}_{0\le q\le \min\{1,u\}} \Big( y_b(q) + y_s(q,u) \Big), && \text{for } u \ge 0, \quad (37)\\
h(u) \;&=\; \max_{0\le q\le \min\{1,u\}} \Big( y_b(q) + y_s(q,u) \Big), && \text{for } u \ge 0. \quad (38)
\end{aligned}
$$

Given the definition of $\rho(u)$ and $h(u)$ above, we proceed to consider the following auxiliary result about $(\rho(u), h(u))$ for $u \geq 0$. Notice that $-y_s^{(1,0)}(u,u) = (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s > 0$, which is a constant. To support our proof arguments below, when $u > 0$, if $y_b'(0) > (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := (y_b')^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$; if $y_b'(0) \leq (1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s$, we let $\widetilde{u} := 0$.

Lemma 10. *(i) $\rho(u)$ is a well-defined and strictly increasing in $u \geq 0$; moreover, given $\widetilde{u} \geq 0$ defined before the lemma statement, $\frac{\rho(u)}{u} = 1$ for $u \in (0, \widetilde{u}]$ and $\frac{\rho(u)}{u}$ strictly decreases in $u \geq \widetilde{u}$;*

*(ii) $h(u)$ is continuous, strictly increasing and strictly concave in $u \geq 0$.*

We next develop an alternative optimization for Problem (36). Consider the following optimization problem:

$$\overline{\mathcal{V}} = \max_{\boldsymbol{w},\boldsymbol{z}} \sum_{j\in\mathcal{B}} \Big[ (k_j^b)^{\frac{1}{1-\xi_b}} h\Big( \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} \Big) \Big] \tag{39a}$$

$$\text{s.t.} \quad (w_j)^{\frac{1}{1-\xi_b}} = \sum_{i:(i,j)\in E} z_{ij}, \qquad j \in \mathcal{B} \tag{39b}$$

$$\sum_{j:(i,j)\in E} z_{ij} = (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad i \in \mathcal{S}, \tag{39c}$$

$$z_{ij} \geq 0, \qquad \forall (i,j) \in E. \tag{39d}$$

where

$$h(u) = \max_{0\leq \widetilde{q}_j \leq \min\{1,u\}} F_b^{-1}(1-(\widetilde{q}_j)^{1-\xi_b})\widetilde{q}_j - F_s^{-1}\Big( \frac{(\widetilde{q}_j)^{1-\xi_s}}{u^{1-\xi_s}} \Big) \widetilde{q}_j \text{ for any } u > 0 \tag{40}$$

and $h(0) = 0$. We consider the following result:

Lemma 11. *We have the following equivalence properties between Problem (39) and Problem (40):*

*(i) let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (36), and construct $(\boldsymbol{w}, \boldsymbol{z})$ such that $w_j = (\frac{q_j^b}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}})^{1-\xi_b}$ for any $i : x_{ij} > 0$ and $z_{ij} = \frac{x_{ij}}{q_i^s}(k_i^s)^{\frac{1}{1-\xi_s}}$, $\widetilde{q}_j = \frac{q_j^b}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then $(\boldsymbol{w}, \boldsymbol{z})$ is the optimal solution to Problem (39) and $\widetilde{q}_j$ is the optimal solution to Problem (40) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$;*

*(ii) let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (39) and $\widetilde{q}_j$ is the optimal solution to Problem (40) with $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, then construct $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ such that $x_{ij} = \frac{z_{ij}(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j}{(w_j)^{\frac{1}{1-\xi_b}}}$ and $q_i^s = \frac{(k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j(k_i^s)^{\frac{1}{1-\xi_s}}}{w_j^{\frac{1}{1-\xi_b}}}$ for $j : z_{ij} > 0$, $q_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}\widetilde{q}_j$, then $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is the optimal solution to (36);*

*(iii) Problem (36) and Problem (39) share the same optimal objective value, i.e., $\overline{\mathcal{R}} = \overline{\mathcal{V}}$.*

We can further simplify the formulation in (39) in the following Lemma 12.

Lemma 12. *Problem (39) and the following problem share the same optimal solution vector $\boldsymbol{w}$,*

$$\overline{\mathcal{Y}} = \max_{\boldsymbol{w}} \sum_{j\in\mathcal{B}} \Big[ (k_j^b)^{\frac{1}{1-\xi_b}} h\Big( \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} \Big) \Big] \tag{41a}$$

$$s.t. \quad \sum_{j\in\widetilde{\mathcal{B}}} (w_j)^{\frac{1}{1-\xi_b}} \leq \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, \qquad \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \tag{41b}$$

$$w_j \geq 0, \qquad \forall j \in \mathcal{B}, \tag{41c}$$

*and moreover, $\overline{\mathcal{Y}} = \overline{\mathcal{V}}$ where $\overline{\mathcal{V}}$ is the optimal objective value for Problem (39).*

The next lemma establishes the connection between the optimal solution $\boldsymbol{w}^*$ to Problem (41) and the network components $G(\mathcal{S}_\tau \cup \mathcal{B}_\tau, E_\tau)$ constructed in (11). Given the finiteness of the network $G(\mathcal{S}\cup\mathcal{B}, E)$, the iteration in (11) yields a maximum index $\overline{\tau}$.

LEMMA 13. *For any* $\tau \in \{1,...,\overline{\tau}\}$ *and any* $j' \in \mathcal{B}_\tau$, *we have* $\frac{(w^*_{j'})^{\frac{1}{1-\xi_b}}}{(k^b_{j'})^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in\mathcal{S}_\tau}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$.

### C.2. Proof of Results in Section 5.1.

**Proof of Proposition 4.** Recall that we have established the connection for the optimal solution and the optimal objective value of Problem (36) with those of Problem (39) and Problem (41) in Lemma 11 and Lemma 12. Therefore, we focus on characterizing the properties of optimization problems in (39) and (41) instead of (36) in this proof. We have already shown that (39) and (41) share the same optimal solution $\boldsymbol{w}^*$ in Lemma 12. To prove the claim, we consider the buyer side in Step 1 and the seller side in Step 2.

Step 1: Establish the ranking of buyers' service levels and payments. Based on Lemma 11(ii), we let $(\boldsymbol{w}, \boldsymbol{z})$ be the optimal solution to Problem (39) and $\widetilde{q}_j$ is the optimal solution to Problem (40) with the parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}$. We know the optimal solution to Problem (36) satisfies

$$\frac{q^b_j}{b_j} \overset{(a)}{=} \frac{(q^b_j)^{1-\xi_b}}{k^b_j} \overset{(b)}{=} (\widetilde{q}_j)^{1-\xi_b} \overset{(c)}{=} \rho^{1-\xi_b}\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}\Big),$$

where Step (a) follows from the observation that $b_j = k^b_j (q^b_j)^{\xi_b}$ in Problem (36); Step (b) follows from the solution property of $\widetilde{q}_j$ in Problem (40) by Lemma 11(ii); Step (c) follows from the definition of the optimal solution $\rho$ to Problem (37). Therefore, the ranking of service levels $(\frac{q^b_j}{b_j})_{j\in\mathcal{B}}$ is the same as that of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

For buyers' payments, we know that

$$\min_{i':(i',j)\in E}\{p^s_{i'}\} + r^b_j = F_b^{-1}\Big(1 - \frac{q^b_j}{b_j}\Big) = F_b^{-1}\Big(1 - \rho\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}\Big)\Big).$$

Therefore, the ranking of buyers' payments $(\min_{i':(i',j)\in E}\{p^s_{i'}\} + r^b_j)_{j\in\mathcal{B}}$ is the opposite of $\Big(\rho\big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}}\big)\Big)_{j\in\mathcal{B}}$.

By Lemma 10(i), we have that $\rho(u)$ strictly increases in $u > 0$. From Lemma 13, we know that $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k^b_j)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for $j\in\mathcal{B}_\tau$ and $\tau = 1,\dots\overline{\tau}$. Furthermore, the definition in (11) implies that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k^b_j)^{\frac{1}{1-\xi_b}}}$ strictly increases in $\tau = 1,\dots,\overline{\tau}$. Therefore, we have

$$\frac{q^b_{j_1}}{b_{j_1}} = \frac{q^b_{j_2}}{b_{j_2}}, \qquad \text{for } j_1, j_2 \in \mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},$$
$$\frac{q^b_{j_1}}{b_{j_1}} < \frac{q^b_{j_2}}{b_{j_2}}, \qquad \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.$$

and

$$\min_{i':(i',j_1)\in E}\{p^s_{i'}\} + r^b_{j_1} = \min_{i':(i',j_2)\in E}\{p^s_{i'}\} + r^b_{j_2}, \qquad \text{for } j_1, j_2\in\mathcal{B}_\tau, \tau\in\{1,...,\overline{\tau}\},$$
$$\min_{i':(i',j_1)\in E}\{p^s_{i'}\} + r^b_{j_1} > \min_{i':(i',j_2)\in E}\{p^s_{i'}\} + r^b_{j_2}, \qquad \text{for } j_1\in\mathcal{B}_{\tau_1}, j_2\in\mathcal{B}_{\tau_2}, \tau_1,\tau_2\in\{1,...,\overline{\tau}\} \text{ and } \tau_1<\tau_2.$$

Step 2: Establish the ranking of sellers' service levels and incomes. To establish the ranking of sellers' service levels, given the optimal solution $\boldsymbol{w}$ to Problem (41) and the optimal solution $\widetilde{q}_j$ to Problem (40) with parameter $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that for any $i \in \mathcal{S}$ and $j : x_{ij} > 0$,

$$\frac{q_i^s}{s_i} \overset{(a)}{=} \frac{(q_i^s)^{1-\xi_s}}{k_i^s} \overset{(b)}{=} \left( \frac{\rho\big((w_j)^{\frac{1}{1-\xi_b}} / (k_j^b)^{\frac{1}{1-\xi_b}}\big)}{(w_j)^{\frac{1}{1-\xi_b}} / (k_j^b)^{\frac{1}{1-\xi_b}}} \right)^{1-\xi_s}, \tag{42}$$

where (a) follows from our discussion before Problem (36) that $s_i = k_i^s (q_i^s)^{\xi_s}$; (b) follows from Lemma 11(ii) for $j : x_{ij} > 0$.

We next show that for any $\tau_1 \neq \tau_2$, we have $x_{ij} = 0$ with $i \in \mathcal{S}_{\tau_1}$ and $j \in B_{\tau_2}$. Based on Lemma 11(ii), it is equivalent to show the optimal solution to Problem (39) satisfies that for any $\tau_1 \neq \tau_2$, $z_{ij} = 0$ for $i \in \mathcal{S}_{\tau_1}$ and $j \in B_{\tau_2}$. We show it by induction. Again, to simplify the notation in Problem (39), we let $W_j := (w_j)^{\frac{1}{1-\xi_b}}$ and $\psi_j^b := (k_j^b)^{\frac{1}{1-\xi_b}}$ for any $j \in \mathcal{B}$ and let $\psi_i^s := (k_i^s)^{\frac{1}{1-\xi_s}}$ for any $i \in \mathcal{S}$. We first consider $\tau = 1$. The buyers in $\mathcal{B}_1$ can only trade with the sellers in $\mathcal{S}_1$ given that they are not connected to any other seller types. It remains to show that the sellers in $\mathcal{S}_1$ only trade with the buyers in $\mathcal{B}_1$ at the platform's optimal commissions. Suppose towards a contradiction that there exist $\tau_1 \neq 1$ such that $z_{ij} > 0$ for some $i \in \mathcal{S}_1$ and $j \in B_{\tau_1}$, then

$$\begin{aligned}\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} &= \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\in\mathcal{B}_1} z_{ij} + \sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E, j\notin\mathcal{B}_1} z_{ij} \\ &\overset{(a)}{>} \sum_{j\in\mathcal{B}_1}\sum_{i:(i,j)\in E, i\in\mathcal{S}_1} z_{ij} \overset{(b)}{=} \sum_{j\in\mathcal{B}_1} W_j \overset{(c)}{=} \sum_{j\in\mathcal{B}_1} \psi_j^b \frac{\sum_{i\in\mathcal{S}_1}\psi_i^s}{\sum_{j\in\mathcal{B}_1}\psi_j^b} = \sum_{i\in\mathcal{S}_1}\psi_i^s\end{aligned} \tag{43}$$

where (a) follows from the assumption that $z_{ij} > 0$ for some $i \in \mathcal{S}_1$ and some $j \in B_{\tau_1}$ with $\tau_1 \neq 1$; (b) follows from (39b); (c) follows from the observation in Lemma 13. In summary, $\sum_{i\in\mathcal{S}_1}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_1}\psi_i^s$, which violate Constraint (39c). In summary, we have that $z_{ij} = 0$ for all $i \in \mathcal{S}_1$ and $j \in B_{\tau_1}$ if $\tau_1 \neq 1$. Assuming that $\mathcal{B}_\tau$ only trade with $\mathcal{S}_\tau$ and vice versa, we proceed to show that $\mathcal{B}_{\tau+1}$ only trade with $\mathcal{S}_{\tau+1}$ and vice versa. First, the buyers in $\mathcal{B}_{\tau+1}$ only trade with the sellers in $\mathcal{S}_{\tau+1}$, because they are not adjacent to the seller types from $\mathcal{S}_{\tau'}$ for any $\tau' \geq \tau+1$; and the seller types with an index lower than $\tau+1$ does not trade with them based on our previous discussion. Second, $\mathcal{S}_{\tau+1}$ only trade with $\mathcal{B}_{\tau+1}$, otherwise we can also obtain $\sum_{i\in\mathcal{S}_{\tau+1}}\sum_{j:(i,j)\in E} z_{ij} > \sum_{i\in\mathcal{S}_{\tau+1}}\psi_i^s$ following the same argument in (43), which violate Constraint (39c) to Problem (39) given that Problem (41) is a reformulation without loss of optimality. In summary, for any $\tau_1 \neq \tau_2$, $x_{ij} = 0$ for $i \in \mathcal{S}_{\tau_1}$ and $j \in B_{\tau_2}$. This allows us to show that for any $i \in \mathcal{S}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$, we have that if $j : x_{ij} > 0$, then we obtain that $j \in \mathcal{B}_\tau$.

Thus, regarding the sellers' incomes, for any $i \in \mathcal{S}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$ and any $j : x_{ij} > 0$, we have that

$$p_i^s - r_i^s = F_s^{-1}\Big(\frac{q_i^s}{s_i}\Big) = F_s^{-1}\Big(\frac{\rho\big((w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}\big)}{(w_j)^{\frac{1}{1-\xi_b}}/(k_j^b)^{\frac{1}{1-\xi_b}}}\Big).$$

Since $\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $j \in \mathcal{B}_\tau$ with $\tau = 1, \ldots, \overline{\tau}$ in Lemma 13, we can next focus on the ranking of $\frac{\rho\left(\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}\right)}{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}/\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$ for $\tau = 1, \ldots, \overline{\tau}$. Recall from Step 1 that $\frac{\sum_{i\in N_{E^{\tau-1}}(\mathcal{B}_\tau)}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}_\tau}(k_j^b)^{\frac{1}{1-\xi_b}}}$

strictly increases in $\tau = 1, \dots \overline{\tau}$. Based on Lemma 10, for some constant $\widetilde{u} \geq 0$, we have that $\frac{\rho(u)}{u} = 1$ for $0 < u \leq \widetilde{u}$ and $\frac{\rho(u)}{u}$ strictly decreases in $u$ for $u > \widetilde{u}$. Define $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j \in \mathcal{B}_\tau\}$. We observe that (i) for any $\tau \leq \widetilde{\tau}$, we have $\frac{q_i^s}{s_i} = \frac{\rho(u)}{u} = 1$ and $p_i^s - r_i^s = F_s^{-1}\left(\frac{\rho(u)}{u}\right) = F_s^{-1}(1) = \overline{v}_{s_i}$ for $i \in S_\tau$; (ii) for any $\tau > \widetilde{\tau}$, we have $\frac{\rho(\sum_{i \in N_{E^{\tau-1}}(\mathcal{B}_\tau)} (k_i^s)^{\frac{1}{1-\xi_s}} / \sum_{j \in \mathcal{B}_\tau} (k_j^b)^{\frac{1}{1-\xi_b}})}{\sum_{i \in N_{E^{\tau-1}}(\mathcal{B}_\tau)} (k_i^s)^{\frac{1}{1-\xi_s}} / \sum_{j \in \mathcal{B}_\tau} (k_j^b)^{\frac{1}{1-\xi_b}}}$ strictly decreases in $\tau$. Therefore, we can summarize that

$$
\begin{aligned}
&\frac{q_{i_1}^s}{s_{i_1}} = \frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1, i_2 \in \mathcal{S}_\tau, \tau \in \{1, \dots, \overline{\tau}\}, \\
&\frac{q_i^s}{s_i} = 1, && \text{for } i \in \mathcal{S}_\tau, \tau \leq \widetilde{\tau}, \\
&\frac{q_{i_1}^s}{s_{i_1}} > \frac{q_{i_2}^s}{s_{i_2}}, && \text{for } i_1 \in \mathcal{S}_{\tau_1}, i_2 \in \mathcal{S}_{\tau_2}, \tau_1, \tau_2 \in \{\widetilde{\tau}+1, \dots, \overline{\tau}\} \text{ and } \tau_1 < \tau_2.
\end{aligned}
$$

and

$$
\begin{aligned}
&p_{i_1}^s - r_{i_1}^s = p_{i_1}^s - r_{i_1}^s, && \text{for } i_1, i_2 \in \mathcal{S}_\tau, \tau \in \{1, \dots, \overline{\tau}\}, \\
&p_i^s - r_i^s = \overline{v}_{s_i}, && \text{for } i \in \mathcal{S}_\tau, \tau \leq \widetilde{\tau}, \\
&p_{i_1}^s - r_{i_1}^s > p_{i_2}^s - r_{i_2}^s, && \text{for } i_1 \in \mathcal{S}_{\tau_1}, i_2 \in \mathcal{S}_{\tau_2}, \tau_1, \tau_2 \in \{\widetilde{\tau}+1, \dots, \overline{\tau}\} \text{ and } \tau_1 < \tau_2.
\end{aligned}
$$

Summarizing the two steps above, we conclude the claims in this result. ■

**Proof of Corollary 1.** Given the definition of $(\boldsymbol{k}^s, \boldsymbol{k}^b)$ at the beginning of Appendix C.1, for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, we first let $\psi_i^s = (k_i^s)^{\frac{1}{1-\xi_s}}$ and $\psi_j^b = (k_j^b)^{\frac{1}{1-\xi_b}}$ for simplicity of notations. We consider the equivalent reformulation in Problem (39) with decision variables $(\boldsymbol{w}, \boldsymbol{z})$ by Lemma 11 and Problem (41) with the decision variable vector $\boldsymbol{w}$ and Lemma 12. We let $W_j = (w_j)^{\frac{1}{1-\xi_b}}$ for all $j \in \mathcal{B}$.

Notice that it is without loss of generality to consider a connected graph $G(\mathcal{S} \cup \mathcal{B}, E)$ for the proof arguments. We prove the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on the service levels in Step 1, and then the impacts on supply/demand and population in Step 2.

Proof of Claim (1): Establish the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on the service levels. Recall from Step 1 in the proof arguments of Proposition 4 that for any $j \in \mathcal{B}$, when $\frac{W_j}{\psi_j^b}$ becomes larger under the optimal solution $\boldsymbol{W}$ to Problem (41), $\frac{q_j^b}{b_j}$ becomes larger at the optimal solution as well. As a result, we can focus on the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on $\frac{W_j}{\psi_j^b}$ for $j \in \mathcal{B}_\tau$.

Step (1-i): Establish the impact of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ on the service levels of the buyer side. Let $(\boldsymbol{W}, \boldsymbol{z})$ be the optimal solution to (39) given parameters $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ and let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \dots \overline{\tau}\}$ be the network components obtained from (11) given this parameter set. We define the index set $\tau_i := \{\tau | i \in S_\tau\}$ and $\tau_j := \{\tau | j \in B_\tau\}$. We consider an alternative vector $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$ in which we pick any $\widetilde{i} \in \mathcal{S}$, and let $\hat{\psi}_{\widetilde{i}}^s > \psi_{\widetilde{i}}^s$; we also let $\hat{\psi}_i^s := \psi_i^s$ for all $i \neq \widetilde{i}$ and let $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. Then we obtain that the parameter vector $(\hat{\psi}^s, \hat{\psi}^b)$ has only one entry on the seller side that is higher than in $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$. Let $(\hat{W}, \hat{z})$ be the optimal solution to (39) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, and let $\{(\hat{\mathcal{S}}_\tau, \hat{\mathcal{B}}_\tau) : \tau = 1, \dots \widetilde{\tau}\}$ be the network components obtained from (11) given this parameter set for some positive integer $\widetilde{\tau}$.

To prove the claim of this step, we want to show that $W_j \leq \hat{W}_j$ for all $j \in \mathcal{B}$. This leads to the observation that $\frac{W_j}{\psi_j^b} \leq \frac{\hat{W}_j}{\hat{\psi}_j^b}$ given our construction that $\hat{\psi}_j^b := \psi_j^b$ for all $j \in \mathcal{B}$. In this way, we can claim that a higher $\psi_i^s$ leads to weakly higher $\frac{W_j}{\psi_j^b}$ for all $j \in \mathcal{B}$.

Suppose towards a contradiction that there exists a $j_1 \in \mathcal{B}$ such that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution. Based on Constraint (39b), we have that $\sum_{i\in N_E(j_1)} z_{ij_1} = W_{j_1} > \hat{W}_{j_1} = \sum_{i\in N_E(j_1)} \hat{z}_{ij_1}$, which implies that there exists a $i_1 \in N_E(j_1)$ such that $z_{i_1j_1} > \hat{z}_{i_1j_1} \geq 0$. Similarly, given $i_1 \in N_E(j_1)$, based on Constraint (39c), we have that $\sum_{j\in N_E(i_1)} z_{i_1j} = \psi^s_{i_1} \leq \hat{\psi}^s_{i_1} = \sum_{j\in N_E(i_1)} \hat{z}_{i_1j}$ where the inequality follows from the construction of $\hat{\boldsymbol{\psi}}$ above. This implies that there exists $j_2 \in N_E(i_1)$ such that $0 \leq z_{i_1j_2} < \hat{z}_{i_1j_2}$. Using the same argument as above, there must exist a $i_2 \in N_E(j_2), i_2 \neq i_1$ such that $z_{i_2j_2} > \hat{z}_{i_2j_2} \geq 0$ and there exists some $j_3 \in N_E(i_2)$ such that $0 \leq z_{i_2j_3} < \hat{z}_{i_2j_3}$. In this iteration, given the finiteness of the graph, we have that there exists a finite list $(j_1, i_1, j_2, i_2 ..., j_n)$ such that $W_{j_1} > \hat{W}_{j_1}$ and $W_{j_n} \leq \hat{W}_{j_n}$. We let $\mathbb{B}_1 = \{j_1\}$, and $\mathbb{S}_1 = \{i | i \in N_E(j_1), z_{i_1j_1} > \hat{z}_{i_1j_1} \geq 0\}$. For $t \in \{2, 3 \dots\}$, we further let $\mathbb{B}_t = \{j | j \in N_E(i), 0 \leq z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$, and $\mathbb{S}_t = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathbb{B}_{t-1}\}$. We have that $\mathscr{B}_t := \cup_{l\in\{1,\dots,t\}} \mathbb{B}_l$ and $\mathscr{S}_t := \cup_{l\in\{1,\dots,t\}} \mathbb{S}_l$ are the sets of all possible buyer types and seller types accessed within the first $2t$ steps in this iteration. Since $\mathscr{B}_{t-1} \subset \mathscr{B}_t \subset \mathcal{B}$ and $|\mathcal{B}|$ is finite, there exists a finite $\bar{t}$ such that $\mathscr{B}_{\bar{t}} = \mathscr{B}_{\bar{t}-1}$, i.e., the set $\mathscr{B}_t$ stops expanding. Under the assumption that $W_{j_1} > \hat{W}_{j_1}$ at the optimal solution for $j_1 \in \mathbb{B}_1$, we next show that there exists $j \in \mathscr{B}_{\bar{t}}$ such that $W_j < \hat{W}_j$. We further suppose towards a contradiction that $W_j > \hat{W}_j$ for any $j \in \mathscr{B}_{\bar{t}}$. Consider the set of seller types $\widetilde{S} := \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$. We can show that $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}}$ by definition. Moreover, we would obtain that

$$
\begin{aligned}
\sum_{i\in\widetilde{S}} \hat{\psi}^s_i &= \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&\overset{(a)}{=} \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} \hat{z}_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} \hat{z}_{ij} \\
&< \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} \\
&\leq \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}>\hat{z}_{ij}} z_{ij} + \sum_{i\in\widetilde{S}} \sum_{j:z_{ij}=\hat{z}_{ij}} z_{ij} = \sum_{i\in\widetilde{S}} \psi^s_i
\end{aligned}
$$

where in Step (a), with $\widetilde{S} \subseteq \mathscr{S}_{\bar{t}} = \cup_{l\in\{1,\dots,\bar{t}\}} \mathbb{S}_l$, in the iterative construction above, given that $\mathbb{B}_t = \{j | j \in N_E(i), 0 \leq z_{ij} < \hat{z}_{ij},\ \forall i \in \mathbb{S}_{t-1}\}$ and that $\mathscr{B}_{\bar{t}} = \cup_{l\in\{1,\dots,\bar{t}\}} \mathbb{B}_l$, the subset of buyer types $\{j : z_{ij} < \hat{z}_{ij}$ for some $i \in \widetilde{S}\}$ should be a subset of $\mathscr{B}_{\bar{t}}$; based on the definition $\widetilde{S} = \{i | i \in N_E(j), z_{ij} > \hat{z}_{ij} \geq 0,\ \forall j \in \mathscr{B}_{\bar{t}}\}$, we have that $z_{ij} > \hat{z}_{ij}$ for any $i \in \widetilde{S}$ and $j \in \mathscr{B}_{\bar{t}}$, which further implies that $\{j : z_{ij} < \hat{z}_{ij},\ \forall i \in \widetilde{S}\} = \emptyset$ and that $\sum_{i\in\widetilde{S}} \sum_{j:z_{ij}<\hat{z}_{ij}} \hat{z}_{ij} = 0$. However, the observation that $\sum_{i\in\widetilde{S}} \hat{\psi}^s_i < \sum_{i\in\widetilde{S}} \psi^s_i$ contradicts with the fact that $\sum_{i\in\widetilde{S}} \hat{\psi}^s_i \geq \sum_{i\in\widetilde{S}} \psi^s_i$ by construction of $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$ above. Therefore, such a contradiction implies that there exists a $j_l \in \mathbb{B}_l \subset \mathscr{B}_{\bar{t}}$ for some $l \in \mathbb{N}_+$ such that $W_{j_l} \leq \hat{W}_{j_l}$. Thus, there must exist a finite path $(j_1, i_1, j_2, i_2 ..., j_l)$ for $j_t \in \mathbb{B}_t$ and $i_t \in \mathbb{S}_t$ such that $z_{i_tj_t} > 0$ for $t \in \{1, .., l\}$ and $\hat{z}_{i_{t-1}j_t} > 0$ for $t \in \{2, \dots, l\}$ under the assumption that $W_{j_l} \leq \hat{W}_{j_l}$. For any $t \in \{1, \dots, l-1\}$, we let $\tau_{i_t}$ and $\tau_{j_t}$ be the corresponding index for the seller subgroup for $\mathcal{S}_\tau$ and the buyer subgroup $\mathcal{B}_\tau$ by the iterative construction in (11). Since $z_{i_tj_t} > 0$, we know that $\tau_{i_t} = \tau_{j_t}$. With the iterative construction, we have $j_{t+1} \in N_E(i_t)$, which satisfies that $\tau_{i_t} \leq \tau_{j_{t+1}}$ given that $S_{i_t}$ is not adjacent to $\mathcal{B}_l$ with $l < \tau_{i_t}$ with the iterative construction in (11). In summary, $\tau_{j_1} = \tau_{i_1} \leq \tau_{j_2} = ... \leq \tau_{j_l}$, which implies that $\frac{W_{j_n}}{\psi^b_{j_n}} \geq \frac{W_{j_1}}{\psi^b_{j_1}}$ based on Lemma 13. Therefore, $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} \geq \frac{W_{j_n}}{\psi^b_{j_n}} \geq \frac{W_{j_1}}{\psi^b_{j_1}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$.

We proceed to show that the constructed solution $(\hat{W}, \hat{z})$ cannot be the optimal solution to Problem (39) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$. We first send a flow $\epsilon$ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$ to construct a new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$: since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\hat{z}_{i_t,j_{t+1}} > 0$ for all $t \in \{1,...,n-1\}$, we can pick any $\epsilon \in (0, \min\{(\hat{W}_{j_n}\hat{\psi}^b_{j_1} - \hat{W}_{j_1}\hat{\psi}^b_{j_n})/(\hat{\psi}^b_{j_1} + \hat{\psi}^b_{j_n}), \min_{t\in\{1,\dots,n-1\}}\{\hat{z}_{i_t,j_{t+1}}\}\})$; for $t \in \{1,...,n-1\}$, let $\widetilde{z}_{i_t j_t} := \hat{z}_{i_t j_t} + \epsilon$, $\widetilde{z}_{i_t j_{t+1}} := \hat{z}_{i_t j_{t+1}} - \epsilon$, $\widetilde{z}_{ij} := \hat{z}_{ij}$ for all $(i,j) \neq (i_t j_{t+1}), (i,j) \neq (i_t j_t)$. Let $\widetilde{W}_{j_1} := \hat{W}_{j_1} + \epsilon$ and $\widetilde{W}_{j_n} := \hat{W}_{j_n} - \epsilon$, $\widetilde{W}_{j'} := \hat{W}_{j'}$ for all $j' \neq j_1, j' \neq j_n$. We next verify the feasibility of this new solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$ in Problem (39). Since $\epsilon \leq \min_{t\in\{1,\dots,n-1\}}\{\hat{z}_{i_t,j_{t+1}}\}$, we can obtain that $\widetilde{z}_{i_t j_{t+1}} \geq 0$ such that Constraint (39d) is satisfied. In addition, in our construction of the new feasible solution $(\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{z}})$, since we only send a flow $\epsilon$ along $j_n \to i_{n-1} \to j_{n-1} \to ... \to i_1 \to j_1$, Constraints (39b) - (39c) are preserved. Thus, $(\widetilde{W}, \widetilde{z})$ is feasible in Problem (39). We define the super-gradient of $h(u)$ as $\partial h(u) = \{z \in \mathbb{R} | h(t) \leq h(u) + z(t-u), \forall t \geq 0\}$. In addition, we define $\partial_- h(u) := \inf\{\partial h(u)\}$ and $\partial_+ h(u) := \sup\{\partial h(u)\}$. Given the strict concavity of $h(u)$ for $u \geq 0$, we have that if $u_2 > u_1 > 0$, then $\partial_+ h(u_2) < \partial_- h(u_1)$, which implies that

$$
\begin{aligned}
\hat{\psi}^b_{j_1} h\Big(\frac{\widetilde{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\widetilde{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) &= \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&> \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \epsilon \partial h_-\Big(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big) - \epsilon \partial h_+\Big(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}\Big) \\
&\geq \hat{\psi}^b_{j_1} h\Big(\frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}\Big) + \hat{\psi}^b_{j_n} h\Big(\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}}\Big)
\end{aligned}
$$

where the first inequality follows from the concavity of $h(\cdot)$ in $\mathbb{R}_+$; for the second inequality, since $\frac{\hat{W}_{j_n}}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1}}{\hat{\psi}^b_{j_1}}$ and $\epsilon < \frac{\hat{W}_{j_n}\hat{\psi}_{j_1} + \hat{W}_{j_1}\hat{\psi}_{j_n}}{\hat{\psi}_{j_1} + \hat{\psi}_{j_n}}$, we have $\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}} > \frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}}$, and therefore, $\partial_+ h(\frac{\hat{W}_{j_n} - \epsilon}{\hat{\psi}^b_{j_n}}) < \partial h_-(\frac{\hat{W}_{j_1} + \epsilon}{\hat{\psi}^b_{j_1}})$. Since other terms in the objective function remain unchanged, $(\widetilde{W}, \widetilde{z})$ leads to a strictly higher objective value than $(\hat{W}, \hat{z})$, which contradicts with the fact that $(\hat{W}, \hat{z})$ be the optimal solution to (39) given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$.

In conclusion, we have that $\frac{W_j}{\psi^b_j} \leq \frac{\hat{W}_j}{\hat{\psi}^b_j}$ for all $j \in \mathcal{B}$. This concludes the claim about the impact of $\psi^s_i$.

For the impact of $\psi^b_j$, we can apply exactly the same proof-by-contradiction arguments as above to establish that when $\psi^b_{\widetilde{j}}$ increases for any $\widetilde{j} \in \mathcal{B}$, then we have that the optimal solution $\frac{W_j}{\psi^b_j}$ decreases for any $j \in \mathcal{B}$.

Step (1-ii): Establish the impact of $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ on the service levels of the seller side. For the impact of $\boldsymbol{\psi}^s$ on the service levels of the seller side, we first recall the construction of $(\hat{\psi}^s, \hat{\psi}^b)$ based on $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$ in Step (1-i), which satisfies that $\hat{\psi}^s_{\widetilde{i}} > \psi^s_{\widetilde{i}}$, $\hat{\psi}^s_i := \psi^s_i$ for all $i \neq \widetilde{i}$ and $\hat{\psi}^b_j := \psi^b_j$ for all $j \in \mathcal{B}$. Without loss of generality, we suppose that a type-$i$ seller trades with type-$j_1$ buyer where $i \in \mathcal{S}_{l_1}$ and $j_1 \in \mathcal{B}_{l_1}$ given the parameter set $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$; and given the parameter set $(\hat{\psi}^s, \hat{\psi}^b)$, we suppose that the type-$i$ seller trades with type-$j_2$ buyer for some $j_2 \in \mathcal{B}_{l_2}$. The index satisfies that $l_2 \geq l_1$ given that $\mathcal{S}_{l_1}$ is not connected with $\mathcal{B}_t$ for any $t < l_1$ by the iterative construction of network components in (11). Therefore, we have that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{W_{j_2}}{\psi^b_{j_2}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, where the first inequality follows from Lemma 13 given that $l_2 \geq l_1$, and the second inequality follows from the same arguments in Step (1-i). Since type-$i$ sellers have positive trades with type-$j_1$ buyers in the optimal solutions given the parameters $(\boldsymbol{\psi}^s, \boldsymbol{\psi}^b)$, and with type-$j_2$ buyers in the optimal solutions given the parameters $(\hat{\boldsymbol{\psi}}^s, \hat{\boldsymbol{\psi}}^b)$, based on the observation that $\frac{W_{j_1}}{\psi^b_{j_1}} \leq \frac{\hat{W}_{j_2}}{\hat{\psi}^b_{j_2}}$, we can establish that

$$
\frac{q^s_i}{s_i} \overset{(a)}{=} \Big(\frac{\rho(W_{j_1}/\psi^b_{j_1})}{W_{j_1}/\psi_{j_1}}\Big)^{1-\xi_s} \overset{(b)}{\geq} \Big(\frac{\rho(\hat{W}_{j_2}/\hat{\psi}^b_{j_2})}{\hat{W}_{j_2}/\hat{\psi}_{j_2}}\Big)^{1-\xi_s} \overset{(c)}{=} \frac{\hat{q}^s_i}{\hat{s}_i}, \tag{44}
$$

where Step (a) and Step (c) follow from the optimality equation in (42) from the proof arguments in Proposition 4; Step (b) follows from the fact that $\frac{\rho(x)}{x}$ monotonically decreases in $x \geq 0$ (see Lemma 10). In summary, when $\psi^s_{\tilde{i}}$ increases for any $\tilde{i} \in \mathcal{S}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly lower for all $i \in \mathcal{S}$.

Using the same arguments above, we could establish the impact of $\boldsymbol{\psi}^b$ on the seller side: when $\psi^b_{\tilde{j}}$ increases for any $\tilde{j} \in \mathcal{B}$, we have that $\frac{q^s_i}{s_i}$ becomes weakly higher for all $i \in \mathcal{S}$.

Proof of Claim (2): Establish the impact of $\boldsymbol{\psi}^s$ and $\boldsymbol{\psi}^b$ on transaction quantities and populations. Recall from (9) that we have $q^b_j = \psi^b_j (\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$ and $b_j = \psi^b_j (\frac{q^b_j}{b_j})^{\frac{\xi_b}{1-\xi_b}}$ for any $j \in \mathcal{B}$ at the optimal solution to Problem (8) given Assumption 4. We establish this claim in the following two substeps.

Step (2-i): Establish the impact of $\boldsymbol{\psi}^b$ on the transaction quantities and populations. For any $j \in \mathcal{B}$, recall from Step (1-i) above that if $\psi^b_{\tilde{j}}$ increases for any $\tilde{j} \neq j$, or if $\psi^s_{\tilde{i}}$ increases for any $\tilde{i} \in \mathcal{S}$, then $\frac{q^b_j}{b_j}$ weakly decreases at the optimal solution. Given that $q^b_j = \psi^b_j (\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we can establish that as $\psi^b_{\tilde{j}}$ increases for any $\tilde{j} \neq j$, then $q^b_j$ weakly decreases at the optimal solution for any $j \in \mathcal{B}$. From $q^b_j = \psi^b_j (\frac{q^b_j}{b_j})^{\frac{1}{1-\xi_b}}$, we have that $b_j = \psi^b_j (q^b_j)^{\xi_b}$ for any $j \in \mathcal{B}$, which further suggests that $b_j$ weakly decreases at the optimal solution for any $j \in \mathcal{B}$.

For any $j \in \mathcal{B}$, it remains to consider the impact of $\psi^b_j$ on $(q^b_j, b_j)$ at the optimal solution for $j \in \mathcal{B}$. We first show that $q^b_j$ increases in $\psi^b_j \geq 0$ for any $j \in \mathcal{B}$. Recall from Constraints (36d)-(36e) that $\sum_{i \in \mathcal{S}} q^s_i = \sum_{i \in \mathcal{S}} \sum_{j:(i,j)\in E} x_{ij} = \sum_{j \in \mathcal{B}} \sum_{i:(i,j)\in E} x_{ij} = \sum_{j \in \mathcal{B}} q^b_j$, which means that $q^b_j = \sum_{i \in \mathcal{S}} q^s_i - \sum_{j' \neq j, j' \in \mathcal{B}} q^b_{j'}$. Since higher $\psi^b_j$ leads to weakly higher $q^s_i$ for any $i \in \mathcal{S}$ and weakly lower $q^b_{j'}$ for any $j' \in \mathcal{B}$ with $j' \neq j$, we conclude that higher $\psi^b_j$ leads to weakly higher $q^b_j$. Similarly, higher $\psi^s_i$ leads to weakly higher $q^s_i$.

Step (2-ii): Establish the impact of $\boldsymbol{\psi}^s$ on the transaction quantities and populations. By applying the same arguments as in Step (2-i), we can establish that $(q_i, s_i)$ weakly increases in $\psi^s_i$ for all $i \in \mathcal{S}$, and $q^s_i$ and $s_i$ weakly decreases in $\psi^s_{i'}$ for any $i' \neq i$ and weakly increases in $\psi^b_j$ for all $j \in \mathcal{B}$. ∎

**Proof of Proposition 5.** Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the optimal solution to Problem (36); we let $u_j := (w_j)^{\frac{1}{1-\xi_b}} / (k^b_j)^{\frac{1}{1-\xi_b}}$ for any $j \in \mathcal{B}$ where $(\boldsymbol{w}, \boldsymbol{z})$ is the optimal solution to the reformulation into Problem (39) (see Lemma 11). Recall that for given $\tau = 1, \ldots, \overline{\tau}$ from (11), type-$i$ sellers for $i \in \mathcal{S}_\tau$ trade with type-$j$ buyers for $j \in \mathcal{B}_\tau$. Moreover, for any $i \in \mathcal{S}_\tau$ and $j \in \mathcal{B}_\tau$,

$$r^s_i + r^b_j = F_b^{-1}\Big(1 - \frac{q^b_j}{k^b_j (q^b_j)^{\xi_b}}\Big) - F_s^{-1}\Big(\frac{q^s_i}{k^s_i (q^s_i)^{\xi_s}}\Big) \;=\; F_b^{-1}\Big(1 - \rho^{1-\xi_b}(u_j)\Big) - F_s^{-1}\Big(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\Big),$$

where the first equation follows from the conditions in (4a) and (4c) where the expressions of $s_i$ and $b_j$ are given before Problem (36); the second equation follows from the observations in Lemma 11(ii) and the definition of $\rho(u)$ in (37). In addition, at the optimal solution, the value of $u_j$ for any $j \in \mathcal{B}_\tau$ increases in $\tau = 1, \ldots, \overline{\tau}$ (see Lemma 13 and the definition in (11)). For simplicity of notations, we let $r(u) := F_b^{-1}(1 - \rho^{1-\xi_b}(u)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u)}{u^{1-\xi_s}})$ for any $u > 0$. Recall the definition $\tilde{u} := (y'_b)^{-1}((1-\xi_s)[F_s^{-1}]'(1) + \overline{v}_s)$ before Lemma 10.

We prove the two claims of this result.

Claim (1). If $u_j \leq \tilde{u}$, we have $\rho(u_j) = u_j$ (see Lemma 10(i)). This implies that $F_b^{-1}(1 - \rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}) = F_b^{-1}(1 - u_j^{1-\xi_b}) - F_s^{-1}(1)$, which is decreasing in $u_j \in [0,1]$ given that $F_b(\cdot)$ is a strictly increasing function in $[0, \overline{v}^b]$ (see Assumption 2). We let $\widetilde{\tau} := \max\{\tau | u_j < \widetilde{u} \text{ for } j \in \mathcal{B}_\tau\}$. Together with the fact that at the optimal solution, the value of $u_j$ for $j \in \mathcal{B}_\tau$ increases in $\tau = 1, \ldots, \overline{\tau}$, we obtain that the value $r(u_j)$ increases in $\tau < \widetilde{\tau}$.

Claim (2). If $u_j \geq \widetilde{u}$, we know that $y_b'(\rho(u_j)) + y_s^{(1,0)}(\rho(u_j), u_j) = 0$. Define $Y(\widetilde{q}_j, u_j) := y_b'(\widetilde{q}_j) + y_s^{(1,0)}(\widetilde{q}_j, u_j)$ given the definitions of $y_s$ and $y_b$ before Lemma 9: for any $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, $y_b(q) = F_b^{-1}\left(1 - (q)^{1-\xi_b}\right) q$ for $0 \leq q \leq 1$ and $y_s(q, u) = -F_s^{-1}\left(\frac{(q)^{1-\xi_s}}{u^{1-\xi_s}}\right) q$ for $0 \leq q \leq u$ and $u > 0$, $y_s(0,0) := \lim_{(q,u)\to(0,0)} y_s(q,u)$. We have that

$$\begin{aligned} Y(\widetilde{q}_j, u_j) &= y_b'(\widetilde{q}_j) + y_s^{(1,0)}(\widetilde{q}_j, u_j) \\ &= \left( (\xi_b - 1)\widetilde{q}_j^{1-\xi_b} (F_b^{-1})' \left(1 - \widetilde{q}_j^{1-\xi_b}\right) + F_b^{-1}\left(1 - \widetilde{q}_j^{1-\xi_b}\right) \right) + \left( (\xi_s - 1) \frac{\widetilde{q}_j}{u_j^{1-\xi_s}} (F_s^{-1})' \left( \frac{\widetilde{q}_j}{u_j^{1-\xi_s}} \right) - F_s^{-1}\left( \frac{\widetilde{q}_j}{u_j^{1-\xi_s}} \right) \right) \end{aligned}$$

Since $F_s$ and $F_b$ are twice differentiable, we know that $F_s^{-1}$ and $F_b^{-1}$ are continuously differentiable, and therefore $Y(\widetilde{q}_j, u_j)$ is continuously differentiable at $(\widetilde{q}_j, u_j)$ for $0 \leq \widetilde{q}_j \leq \min\{u_j, 1\}$. By the implicit function theorem, there exists a continuously differentiable function $\rho(u_j)$ such that $\widetilde{q}_j = \rho(u_j)$ given $Y(\widetilde{q}_j, u_j) = 0$. By differentiating $Y(\widetilde{\rho}(u_j), u_j) = 0$ with respect to $u_j$, we obtain

$$\rho'(u_j) = \frac{(\xi_s - 1)u_j^{\xi_s - 3}\rho(u_j)^{1-2\xi_s}\left( (\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s - 2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)\right)}{(\xi_b - 1)\rho(u_j)^{-2\xi_b} f_b + (\xi_s - 1)u_j^{\xi_s - 2}\rho(u_j)^{-2\xi_s} f_s}$$

where

$$\begin{aligned} f_b &:= (\xi_b - 2)\rho(u_j)^{\xi_b}(F_b^{-1})'\left(1 - \rho(u_j)^{1-\xi_b}\right) - (\xi_b - 1)\rho(u_j)(F_b^{-1})''\left(1 - \rho(u_j)^{1-\xi_b}\right), \\ f_s &:= (\xi_s - 1)\rho(u_j)u_j^{\xi_s}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + (\xi_s - 2)u_j\rho(u_j)^{\xi_s}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right). \end{aligned}$$

We proceed to show that $f_s < 0$ and $f_b < 0$ for later use:

$$\begin{aligned} f_b &:= (1 - \xi_b)\rho(u_j)^{\xi_b}\left( \frac{(2 - \xi_b)}{(\xi_b - 1)}(F_b^{-1})'\left(1 - \rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1 - \rho(u_j)^{1-\xi_b}\right) \right) \\ &\overset{(a)}{<} (1 - \xi_b)\rho(u_j)^{\xi_b}\left( -2(F_b^{-1})'\left(1 - \rho(u_j)^{1-\xi_b}\right) + \rho^{1-\xi_b}(u_j)(F_b^{-1})''\left(1 - \rho(u_j)^{1-\xi_b}\right) \right) \overset{(b)}{<} 0, \\ f_s &:= (\xi_s - 1)u_j\rho^{\xi_s}(u_j)\left( \rho^{1-\xi_s}(u_j)u_j^{\xi_s - 1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + \frac{\xi_s - 2}{\xi_s - 1}(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) \right) \\ &\overset{(c)}{<} (\xi_s - 1)u_j\rho^{\xi_s}(u_j)\left( \rho^{1-\xi_s}(u_j)u_j^{\xi_s - 1}(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) + 2(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right) \right) \overset{(d)}{<} 0, \end{aligned}$$

where (a) and (c) follow from the facts that $\xi_s \in (0,1)$ and $\xi_b \in (0,1)$, which imply that $\frac{2-\xi_b}{\xi_b - 1} < -2$ and $\frac{\xi_s - 2}{\xi_s - 1} > 2$ given that $(F_b^{-1})' > 0$ and $(F_s^{-1})' > 0$ on the domains; (b) and (d) follow from the conditions that $-F_s^{-1}(a/b)a$ and $F_b^{-1}(1 - a/b)a$ are concave in $(a,b)$ for $0 \leq a \leq b$ and $b > 0$ by Assumption 3, and therefore $\frac{a}{b}(F_s^{-1})''\left(\frac{a}{b}\right) + 2(F_s^{-1})'\left(\frac{a}{b}\right) > 0$ and $\frac{a}{b}(F_b^{-1})''\left(1 - \frac{a}{b}\right) - 2(F_b^{-1})'\left(1 - \frac{a}{b}\right) < 0$. In summary, we have $f_s < 0$ and $f_b < 0$.

Finally, we want to establish how $r(u_j) = F_b^{-1}(1 - \rho^{1-\xi_b}(u_j)) - F_s^{-1}(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}})$ changes in $u_j > 0$. Again, given the continuity of $r(u)$, we define the sup-derivative

$$\partial r(u) = \{z \in \mathbb{R} \mid r(t) \leq r(u) + z(t - u), \ \forall t \geq 0\},$$

which implies that

$$\partial r(u) \;=\; (\xi_b-1)\rho(u_j)^{-\xi_b}\rho'(u_j)(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)+(\xi_s-1)u_j^{\xi_s-2}\rho(u_j)^{-\xi_s}\left(u_j\rho'(u_j)-\rho(u_j)\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).$$

Plugging in the expression of $\rho'(u_j)$, we obtain that

$$\partial r(u)=\frac{(\xi_b-1)(\xi_s-1)\rho(u_j)\,(f_1+f_2+f_3)}{u_j\left((\xi_b-1)s^{2-\xi_s}\rho(u_j)^{2\xi_s}f_b+(\xi_s-1)\rho(u_j)^{2\xi_b}f_s\right)},$$

where

$$f_1=(\xi_b-1)u_j\rho(u_j)^{\xi_s+1}(F_b^{-1})''\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)$$

$$f_2=(\xi_s-1)u_j^{\xi_s}\rho(u_j)^{\xi_b+1}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})''\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right)$$

$$f_3=-u_j(\xi_b-\xi_s)\rho(u_j)^{\xi_b+\xi_s}(F_b^{-1})'\left(1-\rho(u_j)^{1-\xi_b}\right)(F_s^{-1})'\left(\frac{\rho^{1-\xi_s}(u_j)}{u_j^{1-\xi_s}}\right).$$

Based on the observation above, we discuss the two cases of this claim:

(i) if $F_s(v)$ and $F_b(v)$ are convex in $v\in[0,\overline{v}_s]$ and $v\in[0,\overline{v}_b]$, we have $(F_b^{-1})''(v)<0$ and $(F_s^{-1})''(v)<0$ in their domains. Given $(F_b^{-1})'(v)>0$ and $(F_s^{-1})'(v)>0$, $\rho(u_j)<1$ (see (37)) and $\xi_s,\xi_b\in(0,1)$, we know that $f_1>0$ and $f_2>0$. Since $\xi_s=\xi_b$, $f_3=0$. Therefore, the numerator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. Since $f_s<0$ and $f_b<0$, the denominator of $\frac{\partial r(u_j)}{\partial u_j}$ is positive. In summary, $\frac{\partial r(u_j)}{\partial u_j}>0$ for $u_j\geq\widetilde{u}$;

(ii) if $F_s(v)$ and $F_b(v)$ are concave in $v\in[0,\overline{v}_s]$ and $v\in[0,\overline{v}_b]$ respectively, we have $(F_b^{-1})''(v)>0$ and $(F_s^{-1})''(v)>0$, then $f_1<0$ and $f_2<0$. Therefore, $\frac{\partial r(u_j)}{\partial u_j}<0$ for $u_j\geq\widetilde{u}$.

■

### C.3. Proof of Results in Section 5.2.

**Proof of Theorem 2.** Recall that $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b),\overline{\mathcal{V}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b),\overline{\mathcal{Y}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$ are respectively the optimal objective value to (36), (39) and (41). To simplify the notations, we use $\overline{\mathcal{R}}(E)$, $\overline{\mathcal{V}}(E)$, $\overline{\mathcal{Y}}(E)$ to denote $\overline{\mathcal{R}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$, $\overline{\mathcal{V}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$, $\overline{\mathcal{Y}}(E,\boldsymbol{\psi}^s,\boldsymbol{\psi}^b)$. From Lemma 11 and 12, we have that $\overline{\mathcal{R}}(E)=\overline{\mathcal{V}}(E)=\overline{\mathcal{Y}}(E)$. Therefore, to prove the claim in this result, it is equivalent to focus on Problem (41) and show that $\overline{\mathcal{Y}}(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$.

We next consider Problem (45) below with an additional constraint $F_b^{-1}(1-q_j^{1-\xi_b})-F_s^{-1}\big(\frac{q_j^{1-\xi_s}}{u_j^{1-\xi_s}}\big)\geq r$ for some $r\in\mathbb{R}$ in comparison with Problem (41). We then show that even the problem with this constraint can obtain the objective value weakly higher than $(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$, from which we can conclude that $\overline{\mathcal{Y}}(E)\geq\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. Given the edge set $\overline{E}$ of the complete graph, for any edge set $E\subset\overline{E}$, we define this auxiliary problem below

$$\mathcal{Y}^h(E) \;=\; \max_{\boldsymbol{w},r}\sum_{j\in\mathcal{B}}\left[(k_j^b)^{\frac{1}{1-\xi_b}}h\Big(\frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}},r\Big)\right] \tag{45a}$$

$$\text{s.t.}\quad \sum_{j\in\widetilde{\mathcal{B}}}(w_j)^{\frac{1}{1-\xi_b}}\leq\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k_i^s)^{\frac{1}{1-\xi_s}},\qquad\forall\widetilde{\mathcal{B}}\subseteq\mathcal{B}, \tag{45b}$$

$$w_j\geq0,\qquad\forall j\in\mathcal{B} \tag{45c}$$

$$r \le \overline{v}_b, \tag{45d}$$

where for any $u > 0$,

$$h(u,r) = \max_{\substack{0 \le \widetilde{q} \le \min\{1,u\}, \\ F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right) \ge r}} \left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u^{1-\xi_s}}\right)\right)\widetilde{q}. \tag{45e}$$

Step 1: Show that $\overline{\mathcal{Y}}(E) \ge \mathcal{Y}^h(E)$. Note that the only difference between (45) and (41) is that one more constraint $F_b^{-1}\big(1-(\widetilde{q}_j)^{1-\xi_b}\big) - F_s^{-1}\big(\frac{\widetilde{q}_j^{1-\xi_s}}{u^{1-\xi_s}}\big) \ge r$ for any $(i,j) \in E$ is added to Problem (45). With $r \le \overline{v}_b$, we have that the constraint for the maximization problem in $(h,r)$ is non-empty given that solution $\widetilde{q} = 0$ is feasible. Therefore, the solution to Problem (45) is also feasible in Problem (41), and two problems share the same objective functions. Thus, we have that

$$\overline{\mathcal{Y}}(E) \ge \mathcal{Y}^h(E).$$

Step 2: Show that $\mathcal{Y}^h(E) \ge (1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, we first reformulate the optimization problems for $\mathcal{Y}^h(E)$ and $\overline{\mathcal{Y}}(\overline{E})$.

Step 2.1: Reformulate the problem for $\mathcal{Y}^h(E)$. With $u_j = \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}$ for any $j \in \mathcal{B}$, we define

$$\hat{q}_j(r,u_j) := \max\left\{\widetilde{q}:\ r \le F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\right), 0 \le \widetilde{q} \le \min\{1,u_j\}\right\}. \tag{46}$$

Note that since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{u_j^{1-\xi_s}}\right)$ strictly decreases in $\widetilde{q} \in [0, \min\{1,u_j\}]$, we know $\hat{q}_j(r,u_j)$ is unique given $(r,u_j)$. Given that $r$ is a lower bound of $F_b^{-1}(1-(\widetilde{q})^{1-\xi_b}) - F_s^{-1}\left(\frac{(\widetilde{q})^{1-\xi_s}}{(u_j)^{1-\xi_s}}\right)$ and $\hat{q}_j(r,u_j)$ is suboptimal to Problem (45e), the optimal objective value $\mathcal{Y}^h(E)$ from Problem (45e) is weakly higher than the optimal objective value of following optimization problem

$$\begin{aligned}
\max_{\boldsymbol{w},r}\ & \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\left(r, \frac{(w_j)^{\frac{1}{1-\xi_b}}}{(k_j^b)^{\frac{1}{1-\xi_b}}}\right) & \\
\text{s.t.}\ & \sum_{j\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}}, & \forall \widetilde{\mathcal{B}} \subseteq \mathcal{B}, \\
& w_j \ge 0, & \forall j \in \mathcal{B}, \\
& r \le \overline{v}_b. &
\end{aligned}$$

For any $r \in (-\infty, \overline{v}^b]$ and $\epsilon \in (0,1)$, we observe that $(w_j)^{\frac{1}{1-\xi_b}} = (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$ is feasible in the optimization problem above given that $w_j \ge 0$ for any $j \in \mathcal{B}$ and for any $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$,

$$\sum_{\in\widetilde{\mathcal{B}}} w_j^{\frac{1}{1-\xi_b}} = \sum_{j\in\widetilde{\mathcal{B}}} (k_j^b)^{\frac{1}{1-\xi_b}}(1-\epsilon)\frac{\sum_{i'\in\mathcal{S}}(k_{i'}^s)^{\frac{1}{1-\xi_s}}}{\sum_{j'\in\mathcal{B}}(k_{j'}^b)^{\frac{1}{1-\xi_b}}} \le \sum_{i\in N_E(\widetilde{\mathcal{B}})} (k_i^s)^{\frac{1}{1-\xi_s}},$$

where the inequality follows directly from the condition in the theorem statement. By letting $\overline{u} := \frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}$, we have that

$$\mathcal{Y}^h(E) \ \ge\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\left(r, (1-\epsilon)\frac{\sum_{i\in\mathcal{S}}(k_i^s)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}}\right) \ =\ \max_{r\le\overline{v}_b} \sum_{j\in\mathcal{B}} (k_j^b)^{\frac{1}{1-\xi_b}} r\hat{q}_j\big(r, (1-\epsilon)\overline{u}\big).$$

Step 2.2: Reformulate the problem for $\overline{\mathcal{Y}}(\overline{E})$. We first show that given the graph set to the complete graph $G(\mathcal{S}\cup\mathcal{B},\overline{E})$, the optimal solution to Problem (41) satisfies $(w^*_{j'})^{\frac{1}{1-\xi_b}} = (k^b_{j'})^{\frac{1}{1-\xi_b}} \frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Given the definition of $(\mathcal{S}_\tau,\mathcal{B}_\tau)$ in (11), in a complete graph, we have that $\mathcal{B}_1=\mathcal{B}$, as for any $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$, we have that

$$\frac{\sum_{i\in N_E(\widetilde{\mathcal{B}})}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k^b_j)^{\frac{1}{1-\xi_b}}} \overset{(a)}{=} \frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\widetilde{\mathcal{B}}}(k^b_j)^{\frac{1}{1-\xi_b}}} \overset{(b)}{\geq} \frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in N_E(\mathcal{B})}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}},$$

where Step (a) follows from the fact that network $G(\mathcal{S}\cup\mathcal{B},\overline{E})$ is complete; Step (b) follows from the condition that $\widetilde{\mathcal{B}}\subseteq\mathcal{B}$. By Lemma 13, we have $\frac{(w^*_{j'})^{\frac{1}{1-\xi_b}}}{(k^b_{j'})^{\frac{1}{1-\xi_b}}} = \frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}$ for any $j'\in\mathcal{B}$. Therefore, we can obtain that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}\, h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right).$$

Similar to Step 2.1, given definition of $h(.)$ in (40), we could reformulate $h(.)$ by defining that

$$\overline{q} := \underset{\widetilde{q}\in[0,\min\{1,\overline{u}\}]}{\arg\max} \left(F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right)\right)\widetilde{q}, \tag{47}$$

where we recall that we have set $\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}} = \overline{u}$ in Step 2.1 above. By letting $\overline{r} := F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}})$, given definition of $h(.)$ in (40), we have that

$$\overline{\mathcal{Y}}(\overline{E}) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}\, h\left(\frac{\sum_{i\in\mathcal{S}}(k^s_i)^{\frac{1}{1-\xi_s}}}{\sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}}\right) \;=\; \sum_{j\in\mathcal{B}}(k^b_j)^{\frac{1}{1-\xi_b}}\,\overline{r}\overline{q}.$$

Step 2.3: Establish that $\mathcal{Y}^h(E)\geq(1-\epsilon)\overline{\mathcal{Y}}(\overline{E})$. To establish the claim, for any $j\in\mathcal{B}$, we want to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$.

By the definition of $\hat{q}_j(r,u)$ in (46), we have that for any $j\in\mathcal{B}$,

$$\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) := \max\left\{\widetilde{q}: \overline{r}\leq F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\widetilde{q}^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right), 0\leq\widetilde{q}\leq\min\left\{1,(1-\epsilon)\overline{u}\right\}\right\}.$$

For simplicity of notations, we use $\hat{q}_j$ to denote $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})$. Since $F_b^{-1}(1-\widetilde{q}^{1-\xi_b}) - F_s^{-1}(\frac{(\widetilde{q})^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ decreases in $\widetilde{q}\in[0,\min\{1,(1-\epsilon)\overline{u}\}]$, we have that either $\overline{r} = F_b^{-1}(1-(\hat{q}_j)^{1-\xi_b}) - F_s^{-1}(\frac{(\hat{q}_j)^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$ or $\hat{q}_j = \min\{1,(1-\epsilon)\overline{u}\}$.

For any $j\in\mathcal{B}$, to show that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u})\geq(1-\epsilon)\overline{q}$, we consider the following two cases:

(1) if $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}$, then $\hat{q}_j=\min\{1,(1-\epsilon)\overline{u}\}\geq(1-\epsilon)\min\{1,\overline{u}\}=(1-\epsilon)\overline{q}$, where the last equality follows from the constraint in Problem (47);

(2) if $\overline{r} = F_b^{-1}(1-\hat{q}_j^{1-\xi_b}) - F_s^{-1}(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}})$, then based on the definition that $\overline{r} = F_b^{-1}(1-\overline{q}^{1-\xi_b}) - F_s^{-1}(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}})$ in Step 2.2, we have that

$$F_b^{-1}(1-\overline{q}^{1-\xi_b}) - F_s^{-1}\left(\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\right) = F_b^{-1}(1-\hat{q}_j^{1-\xi_b}) - F_s^{-1}\left(\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}\right).$$

Note that $F_b^{-1}(1-q^{1-\xi_b}) - F_s^{-1}(\frac{q^{1-\xi_s}}{u^{1-\xi_s}})$ strictly increases in $u\geq q\geq0$ and strictly decreases in $q\in[0,\min\{1,u\}]$. With the equation above, given that $0<(1-\epsilon)\overline{u}\leq\overline{u}$, we have that $\overline{q}\geq\hat{q}_j$, which further implies that $\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}}\leq\frac{\hat{q}_j^{1-\xi_s}}{((1-\epsilon)\overline{u})^{1-\xi_s}}$. This allows us to establish that $\hat{q}_j^{1-\xi_s}\geq((1-\epsilon)\overline{u})^{1-\xi_s}\frac{\overline{q}^{1-\xi_s}}{\overline{u}^{1-\xi_s}} = (\overline{q})^{1-\xi_s}(1-\epsilon)^{1-\xi_s}$. Therefore, we have $\hat{q}_j\geq(1-\epsilon)\overline{q}$.

Summarizing the two cases above, we can establish that

$$\mathcal{Y}^h(E) \overset{(a)}{\geq} \sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}\overline{r}\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \overset{(b)}{=} \sum_{j\in\mathcal{B}}(k_j^b)^{\frac{1}{1-\xi_b}}\overline{r}(1-\epsilon)\overline{q} \overset{(c)}{=} (1-\epsilon)\overline{\mathcal{Y}}(\overline{E}),$$

where (a) follows from Step 2.1 and $\overline{r} = F_b^{-1}(1-(\overline{q})^{1-\xi_b}) - F_s^{-1}(\frac{(\overline{q})^{1-\xi_s}}{(\overline{u})^{1-\xi_s}}) \leq F_b^{-1}(1) = \overline{v}_b$; (b) follows from the observation that $\hat{q}_j(\overline{r},(1-\epsilon)\overline{u}) \geq (1-\epsilon)\overline{q}$ for any $j \in \mathcal{B}$; (c) follows directly from the reformulation in Step 2.2. ■