An Introduction to Joint Pricing and Inventory Management under Stochastic Demand

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Abstract

In this paper, we review the models of joint pricing-inventory management under demand uncertainty. Our focus is on establishing a unified modeling framework that incorporates both pricing and inventory decisions. It applies to different problem settings in this area with proper adaptations. In particular, we employ our modeling framework to five different key problems: (1) single period problem; (2) multi period problem without setup costs, leadtimes and supply uncertainty; (3) multi period problem with setup costs; (4) multi period problem with positive leadtimes; and (5) multi period problem with supply uncertainty. We characterize the structure of the optimal policies in our key models and derive their properties. Other interesting extensions to our key models are also discussed and the paper is concluded with a few thoughts on promising future research opportunities.

Key words: joint pricing and inventory management; review; unified modeling framework

1 Introduction

This paper reviews an important area in inventory management: the joint pricing and inventory replenishment control in the face of uncertain demand.

Classical inventory and supply chain management models assume that the customer demand follows exogenous distributions (see, e.g., Ziplin 2000 and Porteous 2002). Typically, the objective of this stream of research is to understand operational efficiency and minimize expected cost, whereas the revenues are assumed constant and known to the decision maker.

Therefore, another assumption that the sales price of the product(s) is invariant also has to be imposed. In practice, however, the demand process can be controlled by changing the prevailing prices. The inventory replenishment/production strategy controls the supply side of the system whereas the pricing policy controls the demand side. Pricing and inventory replenishment policies complement each other to mitigate the risk of mismatch in supply and demand and enhance the profitability of the business. We will illustrate this point through reviewing several papers in the literature on joint pricing and inventory management under demand uncertainty.

Dynamic pricing strategy has been widely studied along with the fast development of revenue management in business areas like airline, hotel room, hospitality and car rental. The dynamic pricing literature usually makes the assumption that except for the initial procurement at the beginning of the planning horizon, no subsequent inventory replenishment can occur (see, e.g., Gallego and van Ryzin 1994 and 1997). Hence, this type of model is suitable for businesses with long production and delivery leadtimes, and relatively short selling seasons, so that it is very costly to readjust the inventory during the selling season. Thus, the only procurement cost is sunk during the planning horizon when the seller makes price adjustment decisions. The models reviewed in this paper, however, assume that the seller is capable of procuring and receiving additional inventory during the planning horizon so that two series of decisions, pricing and inventory replenishment, have to be made coordinately throughout the planning horizon to maximize the total profits.

The development of advanced information technologies facilitate the sellers to plan, implement and take advantage of the dynamic pricing strategies. Thanks to the technological breakthroughs, the cost of collecting and maintaining customer dada has dropped significantly in the recent decade. The IT decision support applications also enable the sellers to costless adjust and optimize sales prices based on complex optimization methods. Therefore, we have witnessed a fast growth of the employment of dynamic pricing strategy. For example, the sales prices of thousands of items in Amazon fluctuate everyday in order that the seller can best control demand and maximize profit.

The purpose of this paper is to review academic research papers on joint pricing and inventory replenishment control under demand uncertainty. Based on these papers, we introduce an unified analytic modeling framework that encompasses these two optimization decisions with self-contained notations. The major portion of this paper is devoted to the single-party optimization models that highlight the strategically complementary role of pricing and inventory decisions. We try to strike a balance between modeling and analysis: this paper presents the models and main results under different problem scenarios but the

methodology is only sketched so that readers interested in the detailed proofs are referred to the original papers. The historical roots of different pricing and inventory replenishment models are briefly discussed. Finally, we hope that the models are potentially useful for decision support in practice.

There are several other surveys of this topic, e.g., Eliashberg and Steinberg (1991), Elmaghraby and Keskinocak (2003), Yano and Gilbert (2003), Chan et al. (2004) and Chen and Simchi-Levi (2012). This paper differs from these reviews in the following aspects: (a) our focus is on introducing an unified modeling framework for the analysis of jointly pricing and inventory replenishment control problems; (b) We include a few most recent papers that are not reviewed in them. Despite the growing research on decentralized supply chain systems that incorporate combined pricing and inventory control in the recent decade, this paper only focuses on the single party optimization issues. For a summary of supply chain competition, coordination and cooperation literature with game theoretic flavor, interested readers are referred to the related section Chen and Simchi-Levi (2012).

The rest of the paper is organized as follows: in Section 2, we introduce and analyze the single period model. Section 3 discusses the multi-period model with several variations. Other extensions of the model are included in Section 4 and we conclude this paper with a summary and thoughts on potential research opportunities in Section 5.

2 Single Period Model

In this section, we introduce the building block of our models: the single period model. As an extension to the classical newsvendor model, the single period model assumes that an order quantity and selling price are decided and ordering cost incurs at the beginning of the period. During the period, the stochastic demand decreasing in the prevailing price realizes and revenue is collected. At the end of the period, holding cost for excessive inventory or backorder penalty for excessive demand applies. The seminal paper by Whitin (1955) is the first to analyze the single period combined pricing and inventory control problem. Mills (1959, 1962), Kalin and Carr (1962), Zabel (1970), and Polatoglu (1991) later reanalyze the newsvendor model under different demand specifications. This line of research is summarized in the comprehensive review by Petruzzi and Dada (1999) that synthesizes and extends the results in the newsvendor problem with pricing. We introduce a unified modeling approach for this single period model of joint pricing and inventory replenishment control.

As a price setter and inventory manager, the firm needs to jointly choose the stocking quantity x, incurring per-unit cost c, and sales price p before the stochastic price dependent

demand $d(p, \epsilon)$ realizes, where $d(p, \epsilon)$ is a decreasing function of p for any ϵ, with ϵ being a random perturbation. The set of feasible price levels is confined to the finite interval: $[\underline{p}, \bar{p}]$, where $\bar{p}$ is the maximum allowable price and $\underline{p}$ is the minimum allowable price. If the order quantity $x > d(p, \epsilon)$, a per-unit holding cost h is charged whereas if $x < d(p, \epsilon)$, a per-unit backorder penalty b is charged and the seller also has to pay αc $(0 \leq \alpha \leq 1)$ per-unit to fulfill the backlogged demand. Here we use the discount factor α to maintain the consistency with multi-period models. Without loss of generality, assume that $b > (1 - \alpha)c$, i.e., the backlogging penalty is higher than the saving from delaying an order.

We assume that the stochastic demand function is of the following additive form:

$$d(p, \epsilon) = d(p) + \epsilon,$$

where $d(p)$ is the mean demand, a strictly decreasing function with $p(d)$ as its inverse and $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$, with mean normalized to zero. The additive demand model has been employed considerably since its introduction by Mills (1959) (see, e.g., Petruzzi and Dada 1999, Chen and Simchi-Levi 2004a, Li and Zheng 2006, Huh and Janakiraman 2008 and Pang et al. 2012) mainly because of its technical convenience. Throughout this paper, we also assume the demand is of additive form and specify whether and how the analytical results can be generalized under other demand forms in different problem scenarios. We change the decision variable from the sales price p to the mean demand $d \in [\underline{d}, \bar{d}]$, where $\underline{d} = d(\bar{p})$ and $\bar{d} = d(\underline{p})$. We assume the following:

Assumption **1** The expected revenue $R(d) := dp(d)$ is concave in expected demand d.

The concavity of $R(d)$ implies the decreasing marginal revenue with respect to expected demand d, which is a common assumption in the literature (see, e.g., Chen and Simchi-Levi 2004a, Li and Zheng 2006 and Pang et al. 2012). For a more comprehensive discussion on decreasing marginal revenue assumptions, see Ziya et al.(2004). Now we can formulate the single period joint pricing and ordering problem as follows:

$\pi(x, d) =$ the expected profit of the seller if he orders x units of inventory and sets the price at $p(d)$.

Then,

$$\pi(x, d) = \mathbb{E}\{p(d)(d+\epsilon) - cx - h(x-d-\epsilon)^+ - (b+\alpha c)(x-d-\epsilon)^-\} = R(d) - (b+\alpha c)d + (b-(1-\alpha)c)x + L(x-d),$$

where $y^+ := \max\{y, 0\}$, $y^- := \max\{-y, 0\}$, $\mathbb{E}\{\cdot\}$ is the taking expectation operator, and $L(y) := \mathbb{E}\{-(h + b + \alpha c)(y - \epsilon)^+\}$. The following theorem summarizes the optimal pricing and inventory ordering policy in the single period model:

THEOREM **2.1** (a) $\pi(x, d)$ is jointly concave and supermodular in (x, d).

(b) Let $(x^*, d^*) := \text{argmax}_{(x,d)}\pi(x, d)$, then $x^* = d^* + F^{-1}(\frac{b-(1-\alpha)c}{h+b+\alpha c})$.

(c) If $R(\cdot)$ is continuously differentiable, $d^* = \underline{d}$, $\bar{d}$, or $R'(d^*) = c$.

The supermodularity of $\pi(x, d)$ suggests that inventory and price are strategic substitutes. This is intuitive since if a higher price is set, lower demand will be induced and a smaller order should be made to meet the demand. Part (b) of Theorem 2.1 generalizes the well-established critical fractile solution to the newsvendor problem: d^* is the optimal price induced expected demand and $F^{-1}(\frac{b-(1-\alpha)c}{h+b+\alpha c})$ is the newsvendor critical fractile solution. When the demand is of more complex form, the critical fractile formula still holds but to find the optimal sales price requires more effort due to the lack of concavity or even quasi-concavity. Interested readers are referred to Chen and Simchi-Levi (2012) for a comprehensive survey of the single period models under different demand assumptions. As an endnote to this section, we remark that replenishment and pricing strategies are traditionally determined by separate units of a firm, the former by production/operations and the latter by marketing. On the other hand, however, the objective of the firm is to maximize the total expected profit. An interesting question naturally arises: how to align the incentives of different departments. Li and Atkins (2002) addresses this question using a linear transfer price.

3 Multi Period Model

In section, we study how to coordinate inventory replenishment and price adjustments in a multi-period setting. Unlike the classical inventory management literature (e.g., Karlin and Carr 1962) which assumes that the sales price remains constant, or the typical revenue management literature (e.g., Gallego and van Ryzin 1994 and 1997) that assumes the inventory is non-replenishable throughout the planning horizon, the models introduced in this section focus on the scenario where price and inventory can be adjusted coordinately. It has been established in this line of research that the "base-stock/list-price" policy is optimal (see, e.g., Zabel 1972, Thowsen 1975 and Federgruen and Heching 1999). i.e., there exists two time dependent values (I_t^*, p_t^*), such that seller should raise the inventory level to I_t^* and charge a sales price p_t^* if the inventory is below I_t^*; and order nothing and charge a price decreasing in the initial inventory level. We present the base multi-period model of joint pricing and inventory replenishment policy (Federgruen and Heching 1999) as well as its

extensions to the scenarios with fixed ordering cost (Chen and Simchi-levi 2004a), positive leadtimes (Pang et al. 2012) and supply uncertainty (Li and Zheng 2006).

3.1 Base Model

The dynamic combined pricing and inventory strategies under uncertain demand has received limited attention since its introduction in Zabel (1972) and Thowsen (1975) until the influential paper by Federgruen and Heching (1999) which provides a general treatment of this problem. Since Federgruen and Heching (1999), however, this line of literature has grown rapidly. Interested readers are refereed to Chen and Simchi-Levi (2012) for the development of this stream of research. In this subsection, we present a slightly simplified, yet easily generalizable, version of the model in Federgruen and Heching (1999).

Consider a firm facing random demand and periodically makes pricing and ordering decisions in a $T-$period planning horizon labeled backwards as $\{T, T-1, \cdots, 1\}$. As in the single period model, the seller needs to jointly choose the order-up-to level x_t and sales price p_t before the stochastic price dependent demand $d(p_t, \epsilon_t) = d_t + \epsilon_t$ realizes at the beginning of each decision epoch, with d_t being the expected demand and $\{\epsilon_t\}$ being $i.i.d$ random perturbations with mean 0. The sales price is determined by the expected demand the seller wants to induce: $p_t = p(d_t)$. As in the traditional inventory literature, excessive inventory incurs holding cost while excessive demand is fully backlogged and incurs backorder penalty. We employ the same set of notations as in the single period model to denote the cost parameters. The model parameters and demand are assumed to be stationary throughout the that they are constant. Let α $(0 \leq \alpha \leq 1)$ be the discount factor of revenues and costs in future periods, with $b > (1-\alpha)c$. The mean demand is confined to $[\underline{d}, \bar{d}]$.

To formulate the planning problem as a dynamic programming, let

$V_t(I_t) =$ the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with a net inventory level I_t, where I_t equals the end inventory level of period $t+1$.

The value function satisfies the following:

$$V_t(I_t) = cI_t + \max_{x_t \geq I_t, \underline{d} \leq d_t \leq \bar{d}} J_t(x_t, d_t),$$

where

$$\begin{aligned} J_t(x_t, d_t) =& \mathbb{E}\{p(d_t)(d_t + \epsilon_t) - cx_t - h(x_t - d_t - \epsilon_t)^+ - b(x_t - d_t - \epsilon_t)^- \\ &+ \alpha V_{t-1}(x_t - d_t - \epsilon_t)\} \\ =& R(d_t) - (b + \alpha c)d_t + (b - (1-\alpha)c)x_t + L_t(x_t - d_t), \end{aligned}$$

with $L_t(y) := \mathbb{E}[-(h+b)(y-\epsilon_t)^+ + \alpha(V_{t-1}(y-\epsilon_t)-cy)]$ and $V_0(I_0) \equiv 0$. As in the single-period model, we assume that $R(\cdot)$ is concave, as depicted in Assumption 1.

THEOREM **3.1** (Theorem 1 and 2 in Federgruen and Heching 1999) For any $t = T, T-1, \cdots, 1$,

(a) $J_t(x_t, d_t)$ is jointly concave and supermodular in (x_t, d_t) and $V_t(I_t)$ is concavely decreasing in I_t.

(b) Let $(x_t^*, d_t^*) = \text{argmax}_{(x_t,d_t)} J_t(x_t, d_t)$ and $d_t^*(x_t)$ be the maximizer of $J_t(x_t, d_t)$ for any fixed x_t. The optimal policy is to order $x_t^* - I_t$ and set the expected demand level d_t at $d_t^*(x_t^*)$ if $I_t \leq x_t^*$ while to order nothing and set $d_t = d_t^*(I_t)$ otherwise, where $d_t^*(\cdot)$ is an increasing function of I_t. i.e., the basestock list-price policy is optimal with basestock level x_t^* and list-price $p_t^* = p(d_t^*)$.

Theorem 3.1 is a generalization of Theorem 2.1 to the multi-period model. In particular, we show that the optimality of the basestock list-price policy in the finite-horizon model. Similar to Theorem 2.1, $J_t(x_t, d_t)$ is supermodular so that the sales price and order-up-to level are substitutive to each other. As a result, the optimal sales price charged is decreasing in the starting inventory level because the higher the initial inventory level, the more incentive the firm has to set a low price to turn it over. Federgruen and Heching (1999) also consider the model with markdowns (i.e., it is required that $p_t \leq p_{t-1}$) and the infinite horizon models with both discounted profit and long-run average profit. The basestock list-price policy remains optimal under these problem scenarios. It has been widely observed that the cost adjustments in a multi-period system are sometimes costly. Chen et al. (2011) develop the general model for the model with costly price adjustments and characterize the optimal policies for two special scenarios: (1) a model with inventory carryover and no fixed price-change costs and (2) a model with fixed price-change costs and no inventory carryover. An intuitive heuristic policy to tackle the general system whose optimal policy is expected to be very complicated was also proposed in their paper.

3.2 Model with Fixed Ordering Cost

In Chen and Simchi-Levi (2004a), an important extension to the multi-period base model is discussed: the model with fixed ordering cost. They show that, under additive demand form, the objective function is $k-$concave and an (s, S, p) policy is optimal while, under a more general demand form, the objective function is only symmetric $k-$concave and the (s, S, p) policy may not be optimal. Thomas (1974) first studies the periodic review finite

horizon model of combined pricing and inventory replenishment strategy with fixed ordering cost postulating first, without proving the optimality of the (s, S, p) policy. Polatoglu and Sahin (2000) identifies the sufficient condition that the (s, S, p) policy is optimal, but they do not justify whether there exists any demand function satisfying these conditions.

The formulation of the model with fixed ordering cost is almost the same as the base model except that the cost of ordering the inventory from I_t up-to x_t is $k\delta(x_t - I_t) + c(x_t - I_t)$, where $\delta(y) = \begin{cases} 0 & \text{if } x_t = I_t; \\ 1 & \text{if } x_t > I_t. \end{cases}$ We use $V_t^f(I_t)$ to denote the the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with a net inventory level I_t, where the superscript 'f' refers to fixed ordering cost. $V_t^f(I_t)$ satisfies the following recursive scheme:

$$V_t^f(I_t) = cI_t + \max_{x_t \geq I_t, \underline{d} \leq d_t \leq \bar{d}} \{-k\delta(x_t - I_t) + J_t^f(x_t, d_t)\},$$

where

$$\begin{aligned} J_t^f(x_t, d_t) =& \mathbb{E}\{p(d_t)(d_t + \epsilon_t) - cx_t - h(x_t - d_t - \epsilon_t)^+ - b(x_t - d_t - \epsilon_t)^- \\ &+ \alpha V_{t-1}^f(x_t - d_t - \epsilon_t)\} \\ =& R(d_t) - (b + \alpha c)d_t + (b - (1-\alpha)c)x_t + L_t^f(x_t - d_t), \end{aligned}$$

with $L_t^f(y) := \mathbb{E}[-(h+b)(y - \epsilon_t)^+ + \alpha(V_{t-1}^f(y - \epsilon_t) - cy)]$ and $V_0^f(I_0) = -cI_0^- + sI_0^+$. The optimal pricing and replenishment policy is then characterized as the following Theorem:

THEOREM **3.2** (Lemma 2 and Theorem 3.1 in Chen and Simchi-Levi 2004a) For any $t = T, T-1, \cdots, 1$,

(a) $J_t^f(x_t, d_t)$ is jointly continuous in (x_t, d_t). Let $d_t^{f*}(x_t)$ be the maximizer of $J_t^f(x_t, d_t)$ for any fixed x_t. We have that $x_t - d_t^{f*}(x_t)$ is increasing in x_t.

(b) $J_t^f(x_t, d_t^{f*}(x_t))$ and $V_t^f(I_t)$ are $k-$concave.

(c) There exists (s_t, S_t) with $s_t \leq S_t$ such that it is optimal to order $S_t - I_t$ and set the expected demand level $d_t = d_t^{f*}(S_t)$ if $I_t < s_t$ and not not order anything and set $d_t = d_t^{f*}(I_t)$ otherwise. i.e., the $(s_t, S_t, p(d_t^{f*}(S_t)))$ policy is optimal.

Chen and Simchi-Levi (2004a) borrow from Scarf (1960) the idea of $k-$concavity to prove the optimality of the (s, S, p) policy. Unlike the base model, however, the optimal sales price $p_t^{f*}(I_t)$ is not necessarily decreasing in the initial inventory level I_t for which Chen and Simchi-Levi (2004a) find a counter example. When the demand form is more general (e.g.,

$d_t\epsilon_t^m + \epsilon_t^a$, where $\{(\epsilon_t^a, \epsilon_t^m)\}_{t=1}^T$ are *i.i.d.* random vectors with $\mathbb{E}\{\epsilon_t^a\} = 0$ and $\mathbb{E}\{\epsilon_t^m\} = 1$.), $J_t^f(x_t, d_t^{f*}(x_t))$ and $V_t^f(I_t)$ are not necessarily $k-$concave and the (s, S, p) policy may not be optimal. The counter-examples are given in Chen and Simchi-Levi (2004a). To find the optimal policy under general demand functions, Chen and Simchi-Levi (2004a) introduced a definition called symmetric $k-$convexity[1] and studied the properties of a sys-$k-$convex function. Theorem 4.1 in Chen and Simchi-Levi (2004a) proves that the optimal policy is an (s, S, A, p) policy, i.e., there exist $s_t \leq S_t$ and a set $A_t \subset [s_t, (s_t + S_t)/2]$ possibly empty, when the initial inventory level at the beginning of period t I_t is less than s_t or $I_t \in A_t$, an order of size $S_t - I_t$ is made (i.e., order up-to S_t). Otherwise, no order is placed.

A lot of follow-up papers on the joint pricing and inventory control problem with fixed ordering cost emerge after Chen and Simchi-Levi (2004a). e.g., Chen and Simchi-Levi (2004b, 2006) Chen et al. (2006) and Yin and Rajaram (2007), respectively, show the optimality of (s, S, p) policy in the infinite horizon, continuous review, periodic review with lost sales, and Markovian modulated demand cases, respectively; Huh and Janakiraman (2008) and Song et al. (2009) study the lost-sales system and characterize the optimal policy as (s, S) type (slight variation of the standard (s, S, p) policy); and Feng and Chen (2011) develop an efficient algorithm to determine the stationary (s, S, p) policy under long-run average profit criteria.

3.3 Model with Positive Leadtimes

Most of the dynamic joint pricing-inventory control models assume that the procurement leadtime is zero. To integrate positive leadtimes into the stochastic multi-period model is both important and challenging. Standard technique developed in Scarf (1960) to transfer an inventory model with positive leadtime to one with zero leadtime fails in the joint price and inventory control model with positive leadtimes since the price and inventory decisions in the same period will affect the performance of the system at different times. See Chen and Simchi-Levi and Pang et al. (2012) for more discussion on this issue. To resolve the intractable standard dynamic programming formulation, Federgruen and Heching (1999, 2002) propose a heuristic that helps reduce the dimensionality of the problem. Pang et al. (2012), however, make the first attempt to understand the structure of optimal policies in a multi-period joint inventory-pricing control system with positive leadtimes using analytical approach. The model presented in this section is based on Pang et al. (2012). They show that, under a proper transformation, the value and objective functions are $L^\natural$-concave (to

[1] A function f is called sym-$k-$convex for $k \geq 0$ if for any x_0, x_1 and $\lambda \in [0, 1]$, $f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}k$. A function f is sys-$k-$concave if $-f$ is sys-$k-$convex.

be defined shortly) and that the optimal order quantity and sales price are decreasing in the size of each outstanding order with limited sensitivity. We remark that positive leadtimes can be addressed under continuous review models with exponential or Erlangian leadtimes and Poisson demand (e.g., Chen et al. 2006 and Pang and Chen 2011).

The technical development is based on two key notions: $L^\natural - convcavity$ and *multimodularity*, the definition of which is given as follows:

Definition **3.1** 1. Assume that $\mathcal{V} \subset \mathbb{R}^L$ is a polyhedron that forms a lattice and the set $\mathcal{T} := \{(\mathbf{v}, \zeta) | \mathbf{v} \in \mathcal{V}, \zeta \in \mathbb{R}_+, \mathbf{v} - \zeta\mathbf{e} \in \mathcal{V}\}$ is also a lattice, where $\mathbf{e}$ is an L vector of all ones. A function $f : \mathcal{V} \to \mathbb{R}$ is $L^\natural$-concave if $\psi(\mathbf{v}, \zeta) := f(\mathbf{v} - \zeta\mathbf{e})$ is supermodular on $\mathcal{T}$.

2. A function $\hat{f} : \mathbb{R}^L \to \mathbb{R}$ is multimodular if

$$\hat{\psi}(\mathbf{v}, \zeta) := \hat{f}(v_0 - \zeta, v_1 - v_0, \ldots, v_{L-1} - v_{L-2}),\ \zeta \in \mathbb{R},\ \mathbf{z} \in \mathbb{R}^L$$

is supermodular.

$L^\natural - concavity$ implies the usual concavity, supermodularity, diagonal-dominance (Zipkin 2008), and, hence, a local maximum to be global maximum. It is also showed by Zipkin (2008) that $L^\natural - concavity$ is preserved under maximization. Interested readers are referred to Murota (2003, 2005 and 2009) for more comprehensive studies of $L^\natural - concavity$.

Consider a generalization of the model developed in Section 3.1 where the ordering leadtime is L (a positive integer), i.e., the order placed in period t arrives at the beginning of period $t + L$. In period t, the order due in this period arrives; the firm then makes a simultaneous ordering-pricing decision; and demand realizes in response to the list price p_t. Let q_t be the order quantity in period t and $\mathbf{I}_t = (I_{0,t}, I_{1,t}, \ldots, I_{L-1,t})$ be the outstanding inventory vector in period t, where $I_{0,t}$ is the inventory level at the beginning of period t after the order due in t arrives and $I_{i,t} = q_{t-L+i}$, for $i = 1, 2, \ldots, L - 1$.

The state of the system is given by the $L-$vector $\mathbf{I}_t$ and the starting state of next period is:

$$\mathbf{I}_t^+ = (I_{0,t} - d_t - \epsilon_t + I_{1,t}, I_{2,t}, \ldots, I_{L-1,t}, q_t),$$

if the firm orders q_t and controls the expected demand to d_t in period t. We use $\hat{V}_t^l(\mathbf{I}_t)$ to denote the maximum expected discounted profit in periods $t, t-1, \ldots, 1$, with initial value $\hat{V}_0^l(\mathbf{I}_0) = sI_{0,t}^+ - cI_{0,t}^-$, where the superscript 'l' stands for leadtime. We also assume that there is no ordering in the last L periods in the planning horizon, i.e., $q_t = 0$ for $t = L, L-1, \ldots, 1$. We can formulate the problem as the following dynamic programming:

$$\hat{V}_t^l(\mathbf{I}_t) = \max_{(d_t, q_t)} \{\hat{J}_t^l(\mathbf{I}_t, q_t, d_t)\},$$

$$\text{where } \hat{J}_t^l(\mathbf{I}_t, q_t, d_t) = R(d_t) - cq_t - \mathcal{L}^l(I_{0,t} - d_t) + \alpha\mathbb{E}[\hat{V}_{t-1}^l(\mathbf{I}_t^+)],$$

with $\mathcal{L}^l(y) := \mathbb{E}[h(y-\epsilon_t)^+ + b(y-\epsilon_t)^-]$.

To characterize the structure of the optimal pricing and ordering policy, we apply the following state transformation: let

$$v_{j,t} = \sum_{i=0}^{j} x_{i,t}, \text{ for } j = 0, 1, \ldots, L-1.$$

The system is then reformulated to an $L-$stage serial inventory system with unit transition times between consecutive stages. Stage $L-1$ is the stage that replenishes from outside supplier and stage 0 is the stage that faces customer demand. All but stage 0 do not hold inventory. $v_{j,t}$ is analogous to the echelon inventory position of stage j in period t. Suppose the "inventory position" vector of each stage in period after receiving delivery due in period t is $\mathbf{v}_t = (v_{0,t}, v_{1,t}, \ldots, v_{L-1,t})$, where $v_{l,t} \leq v_{l+1,t}$. Let $x_t = v_{L-1,t} + q_t$ be the order-up-to level of the inventory position in stage $L-1$. The starting state of next period is:

$$\begin{aligned}\mathbf{v}_t^+ &= (v_{1,t} - d_t - \epsilon_t, v_{2,t} - d_t - \epsilon_t, \ldots, v_{L-1,t} - d_t - \epsilon_t, y_t - d_t - \epsilon_t) \\ &= (v_{1,t}, v_{2,t}, \ldots, v_{L-1,t}, x_t) - (d_t + \epsilon_t)\mathbf{e}.\end{aligned}$$

The dynamic programming can be rewritten as follows:

$$V_t^l(\mathbf{v}_t) = \max_{(d_t, x_t)} \{J_t^l(\mathbf{v}_t, x_t, d_t)\},$$

$$\text{where } J_t^l(\mathbf{v}_t, x_t, d_t) = R(d_t) - c(x_t - v_{L-1,t}) - \mathcal{L}^l(v_{0,t} - d_t) + \alpha\mathbb{E}[V_{t-1}^l(\mathbf{v}_t^+)].$$

We define the optimal decisions in period t as follows:

$$(d_t^{l*}(\mathbf{v}_t), x_t^{l*}(\mathbf{v}_t)) = \text{argmax}_{(d_t, x_t)} J_t^l(\mathbf{v}_t, x_t, d_t),$$

and $(\hat{d}_t^{l*}(\mathbf{I}_t), \hat{q}_t^{l*}(\mathbf{I}_t))$ as the corresponding optimal policy in the original DP. We have the following structural results:

THEOREM **3.3** (Theorem 1, 2 and Lemma 1 in Pang et al. 2012) For any $t = T, T-1, \ldots, 1$ and $\omega > 0$,

(a) $V_t^l(\mathbf{v}_t)$ and $J_t^l(\mathbf{v}_t, y_t, d_t)$ are $L^\natural$-concave.

(b) $x_t^{l*}(\mathbf{v}_t)$ is increasing in $\mathbf{v}_t$, but $x_t^{l*}(\mathbf{v}_t + \omega\mathbf{e}) \leq x_t^{l*}(\mathbf{v}_t) + \omega$, i.e., the optimal order quantity $q_t^{l*}(\mathbf{v}_t) := y_t^{l*}(\mathbf{v}_t) - v_{L-1,t}$ is increasing in $v_{l,t}$, $l = 0, 1, \ldots, L-2$, but

$$q_t^{l*}(\mathbf{v}_t) - \omega \leq q_t^{l*}(\mathbf{v} + \omega\mathbf{e}_{L-1}) \leq q_t^{l*}(\mathbf{v} + \omega\mathbf{e}) \leq q_t^{l*}(\mathbf{v}_t),$$

where $\mathbf{e}_l$ is the $(l+1)-$th unit vector of dimension L.

(c) $d_t^{l*}(\mathbf{v}_t)$ is nondecreasing in $\mathbf{v}_t$, with the following inequalities:

$$d_t^{l*}(\mathbf{v}_t) \leq d_t^{l*}(\mathbf{v}_t + \omega\mathbf{e}) \leq d_t^{l*}(\mathbf{v}_t) + \omega.$$

(d) In terms of the original outstanding inventory vector $\mathbf{I}_t$, we have the following inequalities:

$$-\omega \leq \hat{q}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_{L-1})-\hat{q}_t^{l*}(\mathbf{I}_t) \leq \cdots \leq \hat{q}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_1)-\hat{q}_t^{l*}(\mathbf{I}_t) \leq \hat{q}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_0)-\hat{q}_t^{l*}(\mathbf{I}_t) \leq 0,$$

$$0 \leq \hat{d}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_{L-1})-\hat{d}_t^{l*}(\mathbf{I}_t) \leq \cdots \leq \hat{d}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_1)-\hat{d}_t^{l*}(\mathbf{I}_t) \leq \hat{d}_t^{l*}(\mathbf{I}_t+\omega\mathbf{e}_0)-\hat{d}_t^{l*}(\mathbf{I}_t) \leq \omega.$$

Part (a) of Theorem 3.3 establishes the the structure of the problem: both objective and value functions are $L^\natural$-concave. The advantage of conducting the state transformation so that the objective functions enjoy the $L^\natural$-concave property is multi-facet: (1) $L^\natural-$concavity guarantees that a local optimum is also a global optimum, thus greatly reducing the computational effort to solve this DP; and (2) $L^\natural-$concavity implies supermodularity and, hence, the complementary relationships between different components of the state vector enables us to do several sensitivity analysis, thus delivering several interesting insights from the model. Parts (b) - (d) of Theorem 4 summarize how the optimal policies change with the outstanding order and on-hand inventory. In particular, it shows that the optimal order quantity is decreasing in the size of each out standing order $\mathbf{I}_{l,t}$ for $l = 0, 1, 2, \ldots L-1$, with limited sensitivity, and it is more sensitive to more recently placed orders than those placed earlier and current current inventory level. However, the optimal expected demand is increasing in the outstanding orders and on hand inventory level, and it is more sensitive to older orders than newer ones, with limited sensitivity as well. These structural properties can also be generalized to the multi-stage model with constant or non-crossing stochastic leadtimes. See the Pang et al. (2012) for details.

Although the model developed in Pang et al. (2012) makes an exciting progress in the joint pricing-inventory management problem with positive leadtimes, there are two fundamental challenges that need further investigation: (1) we still do not know the exact form of the optimal policy though the well known basestock/list-price policy is no longer optimal; and (2) the dimensionality of the problem is not reduced and, hence, the computation of the optimal policy still suffers from the "curse of dimensionality".

3.4 Model with Supply Uncertainty

In quite a lot of industries like in the production of, to name a few, vaccines, semi-conductor chips and refined oil, the system planner often faces the supply uncertainty, random yield

in particular, issues. There are a lot of papers on the periodic review inventory model with random yield, see Lee and Yano (1995) for a comprehensive review. Li and Zheng (2006) first addresses the joint pricing and inventory control problem in the presence of random yield. They show that, with additive demand form, the optimal policy is of threshold type, i.e., it is optimal to produce/procure if and only if the starting inventory level is below a threshold and both order quantity and that both optimal price and order quantity are decreasing in the starting inventory. In addition, their model suggests that yield uncertainty gives rise to profit reduction and drives the firm to set a higher ordering threshold and charger a higher price. These structural properties still hold in the infinite horizon setting.

The multi-period model with random yield is a direct extension to the base model, which incorporates yield uncertainty. Following Li and Zheng (2006), we employ the stochastically proportional yield model, i.e., the firm can receive $q_t\xi_t$ units of inventory if he orders q_t units, where $\{\xi_t\}_{t=1}^T$ are *i.i.d.* nonnegative random variables supported on $[0,1]$ independent of ϵ_t's. Let $V_t^r(I_t)$ be the maximum profit of the firm in period $t, t-1, \ldots, 1$ when the starting inventory of the firm is I_t in period t, where the superscript 'r' refers to random yield.

$$V_t^r(I_t) = \max_{(q_t,d_t)} J_t^r(q_t, d_t, I_t),$$

where

$$J_t^r(q_t, d_t, I_t) = R(d_t) - cq_t - \mathbb{E}\{h(I_t+q_t\xi_t-d_t-\epsilon_t)^+ + b(I_t+q_t\xi_t-d_t-\epsilon_t)^-\} + \alpha\mathbb{E}\{V_{t-1}^r(I_t+q_t\xi_t-d_t-\epsilon_t)\},$$

and $V_0^r(I_0) = -cI_0^- + sI_0^+$. We have the following structural results:

THEOREM **3.4** (Lemma 3.1, Theorem 3.6, 3.7 and 4.5 in Li and Zheng 2006) For any $t = T, T-1, \ldots, 1$,

(a) $J_t^r(q_t, d_t, I_t)$ is jointly concave in (q_t, d_t, I_t) and $V_t^r(I_t)$ is concave in I_t.

(b) The optimal order quantity $q_t^{r*}(I_t)$ is decreasing in I_t and optimal expected demand $d_t^{r*}(I_t)$ is increasing in I_t, i.e., both optimal order quantity and optimal sales price is decreasing in starting inventory level.

(c) Let $I_t^r = \inf\{I_t : q_t^{r*}(I_t) = 0\}$. I_t^r is the threshold value for replenishment: $q_t^{r*}(I_t) = 0$ if $I_t \geq I_t^r$ and $q_t^{r*}(I_t) > 0$ if $I_t < I_t^r$.

(d) Assume that $V_t^c(I_t)$, (I_t^c and $d_t^{c*}(I_t)$, respectively) is the optimal value function (replenishment threshold and optimal expected demand, respectively) for the system without supply uncertainty (i.e., $\xi_t = 1$ *a.s.*) and with procurement cost $c/\mathbb{E}[\xi_t]$, i.e., both systems have the same expected unit procurement cost. We have:

(i) $V_t^r(I_t) \leq V_t^c(I_t)$ for any I_t, i.e., the system with yield uncertainty is always less profitable.

(ii) $I_t^r \geq I_t^c$, i.e., the system with yield uncertainty has higher replenishment threshold.

(iii) $d_t^{r*}(I_t) \leq d_t^{c*}(I_t)$, i.e., the system with yield uncertainty will charge a higher price.

Theorem 3.4 generalizes Theorem 3.1 to the model with random yield. The comparison between the base model and the model with random yield is also interesting, which shows that yield uncertainty harms the system performance and the customers have to pay a higher price and, thus, indirectly suffer from yield uncertainty. Feng (2010) studies a similar model that captures supply uncertainty in the form of random capacity. Under capacity uncertainty, the optimal policy is also of threshold type and this form of supply uncertainty also results in higher replenishment threshold and sales price.

4 Other Extensions

So far we have established a unified modeling framework to address a few most important variations of the joint pricing-inventory control problem in both single and multi period settings. There are, however, quite a few other important extensions to our basic single/multi period models. In this section, we briefly summarize the main results on these extensions in the existing literature without digging into the modeling details. All of the problems described in this section can be analyzed using our unified modeling framework with proper adaptations.

Few papers in the literature try to study the distribution inventory systems due to their technical intractability. Federgruen and Heching (2002) study a distribution system with geographically dispersed retailers who face price-dependent stochastic demand. The system planner needs to dynamically adjust retailer prices and inventories. The exact problem is prohibitively intractable and the authors develop an approximate model in which simple optimal policy exists.

A few papers analyze multiple product models in the stochastic setting. Zhu and Thonemann (2009) extends the model in Federgruen and Heching (1999) by studying two substitutable products. They show that a base-stock like policy for the two products is optimal. Aydin and Porteous (2008) investigate the optimal inventory levels and prices of multiple products in a given assortment in a newsvendor model. They provide the sufficient

conditions under which the first order condition guarantees global optimality, though the objective function is, in general, not jointly concave. For other papers on the multi-product joint pricing and inventory management model, we refer interested readers to Chen and Simchi-Levi (2012) for a review.

As is discussed in the previous sections, the pricing decision is usually determined by the marketing department of a firm (see, e.g., Federgruen and Heching 1999, Li and Atkins 2002 for more detailed discussions), it is natural to study the inventory/marketing interface, with consumer behavior into consideration in particular, in the joint pricing-inventory control framework. Eliashberg and Steinberg (1991) review the literature on the interface of operations and marketing with an emphasis on integrated inventory and pricing models. A few papers incorporate reference price (see, e.g., Mazumdar et al. 2005 for a comprehensive review of the reference price models in marketing literature)to the joint pricing-inventory model under stochastic demand. Integrating reference in the state space significantly complicates the analysis of the optimal policy, because the expected profit is not concave even in the single period model. Gimpl-Heersink (2008) [Chen and Zhang (2009)] studies a single [multi] period joint pricing-inventory control model with reference price, and shows that a reference price dependent base-stock policy is optimal. Some other papers also consider the joint pricing-inventory control model under inventory-dependent demand. The first kind of dependence of demand on inventory is studied by Dana and Petruzzi (2001) and Blakrishnan et al. (2008), in which the authors assume that the inventory level available to customers has a positive impact on the demand, since the product availability usually acts as a promotional factor among customers. They show that the promotional effect of inventory will drive the firm to set a higher inventory level and charge a higher sales price in single period models. Yang and Zhang (2013) considers the other kind of dependence of demand on inventory: leftover inventory is a negative indicator of the freshness and popularity of the product and, hence, negatively impacts future demand. They prove that an inventory-dependent order-up-to/dispose-down-to list-price policy is optimal and demonstrate that inventory dependent demand derives the firm to order less/dispose more and charge a lower price to compensate for the demand loss caused by leftover inventory.

5 Concluding Remarks

As surveyed above, the joint pricing-inventory management under stochastic demand has received considerable attention in the literature, with several important problems solved. In this paper, we review the important progresses in this stream of research based on which we

establish a unified modeling framework that encompasses pricing and inventory decisions with the hope that it will be useful for developing decision support tools in practice. In particular, Using our modeling framework, we in-depth analyze four key problems in the area of joint pricing-inventory management under stochastic demand: the single period problem (based on Petruzzi and Dada 1999), the multi period basic problem without setup costs, leadtimes and supply uncertainty (based on Federgruen and Heching 1999), the multi period problem with setup cost (based on Chen and Simchi-Levi 2004a), the multi-period model with positive leadtimes (based on Pang et al. 2012), and the multi period model with supply uncertainty (based on Li and Zheng 2006). We characterize the structure and property of the optimal policies in all our key models and discussed the historical roots of the papers on these problems. Finally, we review papers in the literature that discuss other important extensions to our basic models, such as multi-location and multi-product models as well as the interface between marketing and inventory management under joint pricing-inventory control framework. Our survey is by no means exhaustive, yet providing a guideline to a unified modeling approach to analyze the joint pricing-inventory management under demand uncertainty problems.

Despite the fruitful stream of research on the joint pricing-inventory control, there are many important problems still unsolved as promising future research directions. To avoid repetition, we only present the potential research opportunities not adequately discussed in other literature surveys, but refer interested readers to Chen and Simchi-Levi (2012) for a more thorough coverage of topics that remain to be explored.

First, we believe the joint pricing-inventory control model under fluctuating procurement costs (e.g., commodity prices, see Zhang (2012) for a review of literature on inventory management under fluctuating costs) is worth studying. All models in the existing literature require that the procurement cost in each decision epoch, though time-dependent, is known to the firm throughout the planning horizon, which severely restrict their application to the market where the firm faces sharp procurement cost fluctuations. Incorporating cost fluctuation into the joint pricing-inventory control model involves the following interesting issues: (1) The pricing control acts as a leverage to control demand to the most profitable level in accordance to the current cost; (2) The firm can apply price adjustment to pass the procurement cost risk to its customers; (3) The current procurement cost summarizes the information regarding costs of the product in later periods so that the firm should deal with both the trade off between understock and overstock and the risk and opportunity of fluctuating prices. In particular, we want to understand how the interplay between the uncertainty at the demand (stochastic demand) and supply (fluctuating costs) sides influence

the optimal pricing and ordering policies.

Second, few words have been stated on the optimal policy of the multi-stage/multi-echelon dynamic problem with positive leadtimes, with the only exception Pang et al. (2012). As discussed in Chen and Simchi-Levi (2012), the echelon inventory position technique developed in Scarf (1960) fails if we incorporate the pricing decision into the model, since the pricing and inventory decisions come to effect in different periods when we have positive leadtimes. To be more specific, we still do not know the answers to the following three fundamental questions: (1) What is the structure of the optimal policy? (2) Can we reduce the dimensionality of the state space of the DP? And (3) How can we efficiently compute the optimal policy? Clearly, the solution to the third question is based on those to the first two.

Last but not least, the joint pricing-inventory management under cash flow constraint has yet been studied. As argued in Li et al. (2011), start-up and growing firms usually face capital shortage problems and hence, the pricing and ordering policies of a firm with cash flow constraint highly depends on the available capital it has to procure. The pricing flexibility improves the firm's capability to match supply with demand while the constrained capital dampens such potential. It's interesting to investigate how the pricing control helps the firm mitigate the capital shortage risk.

References

[1] Aydin, G., E. Porteus. 2008. Joint inventory and pricing decisions for an assortment. *Oper. Res.* **56**(5) 1247-1255.

[2] Chan, L. M. A., Z.-J. M. Shen, D. Simchi-Levi, J. L. Swann. 2004. Coordination of pricing and inventory decisions: A survey and classification. D. Simchi-Levi, S. D. Wu, Z.-J. M. Shen, eds. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era.* Springer Science + Business Media, New York, N.Y., 335-392.

[3] Chen, F., S. Ray, Y. Song. 2006. Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. *Naval Res. Logist.* **53**(2) 117-136.

[4] Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Oper. Res.* **52**(6) 887-896.

[5] Chen, X., D. Simchi-Levi. 2012. Pricing and inventory management. R. Philips O. Ozalp, eds. *The Oxford Handbook of Pricing Management.* Oxford University Press, USA, 784-824.

[6] Chen, X., S. X. Zhou, Y. Chen. 2011. Integration of inventory and pricing decisions with costly price adjustments. *Oper. Res.* **59**(5) 1144-1158.

[7] Eliashberg, J., R. Steinberg. 1991. Marketing-production joint decision making. J. Eliashberg, J. D. Lilien, eds. *Handbooks of Operations Research and Management, Volume 5: Management Science in Marketing.* Elsevier, North-Holland, 827-880.

[8] Elmaghraby, W. P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Sci.* **49**(10) 1287-1309.

[9] Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Oper. Res.* **47**(3) 454-475.

[10] Federgruen, A., A. Heching. 2002. Multilocation combined pricing and inventory control. *Service Oper. Management* **4**(4) 275-295.

[11] Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40**(8) 999-1020.

[12] Gallego, G., G. van Ryzin. 1997. A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* **45**(1) 24-41.

[13] Huh, W, G. Janakiraman. 2008. (s, S) optimality in joint inventory-pricing control: An alternative approach. *Oper. Res.* **56**(3) 783-790.

[14] Karlin, S., C. R. Carr. 1962. Prices and optimal inventory policy. K. J. Arrow, S. Kalin, H. Scarf, eds. *Studies in Applied Probability and Management Science.* Stanford University Press, Stanford, 159-172.

[15] Li, J., J. Chen, S. Wang. 2011. *Risk Management of Supply and Cash Flows in Supply Chains.* Springer-Verlag, New York.

[16] Li, Q., D. Atkins. 2002. Coordinating replenishment and pricing in a firm. *Manufacturing Service Oper. Management* **4**(4) 241-257.

[17] Li, Q., S. Zheng. 2006. Joint inventory replenish and pricing control for systems with uncertain yield and demand. *Oper. Res.* **54**(4) 696-705.

[18] Mazumdar, T., S. P. Raj, I. Sinha. 2005. Reference price research: Review and propositions. *J. Marketing* **69**(4) 84-102.

[19] Mills, E. S. 1959. Uncertainty and price theory. *Quar. J. Econom.* **73**(1) 116-130.

[20] Mills, E. S. 1962. *Price, Output and Inventory Policy.* John Wiley, New York.

[21] Murota, K. 2003. *Discrete Convex Analysis. SIAM Monographs on Discrete Mathematics and Applications.* SIAM, Philadelphia.

[22] Murota, K. 2005. Note on multimodularity and $L-$convexity. *Math. Oper. Res.* **30**(3) 658-661.

[23] Murota, K. 2006 Recent development in discrete convex analysis. W. Cook, L. Lowasz, J. Vygen, eds. *Research Trends in Combinatorial Optimization, Bonn 2008.* Springer-Verlag, Berlin, 219-260.

[24] Pang, Z., F. Chen. 2011. Dynamic inventory-pricing control in a continuous review system with batch ordering and random leadtimes. Working paper, Lancaster University, UK.

[25] Pang, Z., F. Y. Chen, Y. Feng. 2012. A note on the structure of joint inventory-pricing control with leadtimes. *Oper. Res.* **60**(3) 581-587.

[26] Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: a review with extensions. *Oper. Res.* **47**(2) 183-194.

[27] Polatoglu, L. H. 1991. Optimal order quantity and pricing decisions in single period inventory systems. *Internat. J. Production Econom.* **23** 175-185.

[28] Porteous, E. L. 2002. *Foundations of Stochastic Inventory Theory.* Stanford Business Books, Stanford.

[29] Scarf, H. 1960. The optimality of (s, S) policies for the dynamic inventory problem. *Proc. of the first Stanford Symposium on Mathematical Methods in Social Science* (June).

[30] Thowsen, G. T. 1975. A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Res. Logist. Quant.* **22**(3) 461-476.

[31] Whitin, T. M. 1955. Inventory control and price theory. *Management Sci.* **2**(1) 61-68.

[32] Yang, N., R. Zhang. 2013. Dynamic pricing and inventory management under inventory-dependent demand. Working paper. Washington University in St. Louis, MO.

[33] Yano, C. A., S. M. Gilbert. 2003. Cordinated pricing and production/procurement decisions: A review. A. Chakravarty, J. Eliashberg, eds. *Managing Business Interfaces: Marketing, Engineering and Manufacturing Perspectives.* Kluwer Academic, Boston, M.A., 65-103.

[34] Yano, C., H. Lee. 1995. Lot-sizing with random yields.: A review. *Oper. Res.* **43**(2) 311-334.

[35] Yin, R., K. Rajaram. 2007. Joint pricing and inventory control with Markovian demand model. *Euro. J. Oper. Res.* **182**(1) 113-126.

[36] Zabel, E. 1970. Monopoly and Uncertainty. *Rev. Econom. Studies* **37**(2) 205-219.

[37] Zabel, E. 1972. Multiperiod Monopoly under uncertainty. *J. Econom. Theory* **5**(3) 524-536.

[38] Zhang, R. 2012. Pricing and Inventory Management under Fluctuating Costs. Unpublished manuscript. Washington University in St. Louis, MO.

[39] Zhu, K., U. Thonemann. 2009. Coordination of pricing and inventory control across products. *Naval Res. Logist.* **56**(2) 175-190.

[40] Zipkin, P. H. 2000. *Foundations of Inventory Management.* McGraw-Hill, Columbus.

[41] Zipkin, P. H. 2008. On the structure of lost-sales inventory models. *Oper. Res.* **56**(4) 937-944.

[42] Ziya, S., H. Ayhan, R. D. Foley. 2004. Relationships among three assumptions in revenue management. *Oper. Res.* **52**(5) 804-809.